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Some New Hilbert Sequence Spaces

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ABSTRACT: The main purpose of the present paper is to study of some new Hilbert sequence spaces h_∞, h_c and h_0. New Hilbert sequence spaces h_∞, h_c and h_0 consisting all the sequences whose H- transforms are in the spaces l_∞, c and c_0, respectively. The new Hilbert sequence spaces h_∞, h_c and h_0 that are BK- spaces and prove that the spaces h_∞, h_c and h_0 are linearly isomorphic to the spaces l_∞, c and c_0, respectively. Afterward the bases and α, β and γ duals of these spaces will be given. Finally, matrix the classes $(h_c : l_p)$ and $(h_c : c)$ have been characterized.

Keywords: Hilbert sequence spaces; α, β and γ duals and bases of sequence; Matrix mappings.

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Yeni Hilbert Dizi Uzayları

ÖZET: Bu çalışmadaki amacımız h_∞, h_c ve h_0 ile gösterdiğimiz; sınırlı, yakınsak ve sıfıra yakınsak Hilbert dizi uzaylarını oluşturarak, Hilbert matrisi ile oluşturulan bu yeni h_∞, h_c ve h_0 Hilbert dizi uzaylarının birer BK-uzayları oldukları sırasıyla; l_∞, c ve c_0 dizi uzaylarını kapsadığını ve lineer olarak izomorf olduklarını gösterdikten sonra, α-, β- ve γ- duallerini hesaplayarak, $(h_c : l_p)$ ve $(h_c : c)$ matris dönüşümlerini yapmaktır.

Anahtar Kelimeler: Hilbert dizi uzayları, α-, β- ve γ- dualleri, Dizilerin tabanları, Matris dönüşümleri.

INTRODUCTION

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called as a sequence space. We write l_∞, c and c_0, for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, l_1 and l_p, we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely summable series, respectively; where $1 \le p < \infty$.

Let X, Y be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n, $k \in N$. Then, the matrix A defines a transformation from X into Y and we denote it by $A : X \to Y$, if for every sequence $x=(x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in Y, where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1.1}$$

for each $n \in N$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X : Y)$, we denote the class of all matrices A such that $A : X \to Y$. Thus $A \in (X : Y)$ if and only if the series on the right side of (1.1) converges for each $n \in N$ and every $x \in X$, and we have $Ax = \{(Ax)_n\} \in Y$ for all $x \in X$.

A sequence space λ with a linear topology is called an K-space provided of the maps $p_i : \lambda \to C$ defined by $p_i(x) = x_i$ is continuous for all $i \in N$; where C

denotes the set of complex number and $N=\{0,1,2,...\}$. An K- space λ is called an FK - space provided λ is a complete linear metric space. An FK- space provided whose topology is normable is called a BK- space. An FK- space provided whose topology is normable is called a BK- space [1].

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A=\{x=(x_k)\in w\ :\ Ax\in X\} \qquad (1.2)$$

which is a sequence space.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method was used by authors [2]-[8]. They introduced the sequence spaces $(c_0)_{T^r}=t_0^r$ and $(c)_{T^r}=t_c^r$ in [2], $(c_0)_{E^r}=e_0^r$ and $(c)_{E^r}=e_c^r$ in [3], $(c_0)_C=\bar{c}_0$ and $c_C=\bar{c}$ in [4], $(l_p)_{E^r}=e_p^r$ in [5], $(l_\infty)_{R^t}=r_\infty^t$, $c_{R^t}=r_c^t$ and $(c_0)_{R^t}=r_0^r$ in [6], $(l_p)_C=X_p$ in [7] and $(l_p)_{N_q}$ in [8] where T^r, E^r, C, R^t and N_q denote the Taylor, Euler, Cesaro, Riesz and Nörlund means, respectively. Following [2] - [8], this way, the purpose of this paper is to introduce the new Hilbert sequence spaces h_∞, h_c and h_0.

The Hilbert Matrix Of Inverse Formula And Hilbert Sequence Spaces

The $n\times n$ matrix $H=[h_{ij}]=[\frac{1}{i+j-1}]_{i,j=1}^n$ is a Hilbert matrix [9]. The inverse of Hilbert's Matrix H^{-1} [10] is given by

$$h_{ij}^{-1}=(-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2. \qquad (2.1)$$

We introduce all bounded, convergent and null of the Hilbert sequence spaces, respectively.

$$h_\infty=\left\{x=(x_k)\in w\ :\ \sup_m\left|\sum_{k=1}^m\frac{1}{n+k-1}x_k\right|<\infty\right\}$$

$$h_c=\left\{x=(x_k)\in w\ :\ \lim_{m\to\infty}\sum_{k=1}^m\frac{1}{n+k-1}x_k \text{ exists}\right\}$$

and

$$h_0=\left\{x=(x_k)\in w\ :\ \lim_{m\to\infty}\sum_{k=1}^m\frac{1}{n+k-1}x_k=0\right\}.$$

With the notation of (1.2), we may redefine the spaces h_∞, h_c and h_0 as follows:

$$h_0=(c_0)_H,\ h_c=(c)_H \text{ and } h_\infty=(l_\infty)_H. \qquad (2.2)$$

If λ is an normed or paranormed sequence space, then matrix domain λ_H is called an Hilbert sequence space. We define the sequence $y=(y_m)$ which will be frequently used, as the H - transform of a sequence $x=(x_m)$ i.e.,

$$y_m=\sum_{k=1}^m\frac{1}{n+k-1}x_k,\quad m,n\in N. \qquad (2.3)$$

It can be easily shown that h_∞, h_c and h_0 are linear and normed spaces by the following norm:

$$\|x\|_{h_0}=\|x\|_{h_c}=\|x\|_{h_\infty}=\|Hx\|_{l_\infty}. \qquad (2.4)$$

Theorem 1. The sequence spaces h_∞, h_c and h_0 endowed with the norm (2.4) are Banach spaces.

Proof. Let sequence $(x^p)=\left(x_0^{(p)},x_1^{(p)},x_2^{(p)},...\right)$ at h_∞ a Cauchy sequence for all $p\in N$. Then, there exists $n_0=n_0(\varepsilon)$ for every $\varepsilon>0$ such that $\|x^p-x^r\|_\infty<\varepsilon$ for all p, $r>n_0$. Hence, $|H(x^p-x^r)|<\varepsilon$ for all p, $r>n_0$ and for each $k\in N$. Therefore, $(Hx_k^p)=((Hx^0)_k,(Hx^1)_k,(Hx^2)_k,...)$ is a Cauchy sequence in the set of complex numbers C. Since C is complete, it is convergent say $\lim_{p\to\infty}(Hx^p)_k=(Hx)_k$ and $\lim_{m\to\infty}(Hx^m)_k=(Hx)_k$ for each $k\in N$.

Hence, we have
$\lim_{n\to\infty}|Hx_k^p-x_k^m|=|H(x_k^p-x_k)-H(x_k^m-x_k)|\le\varepsilon$
for all $n\ge n_0$. This implies that $\|x^p-x^m\|\to\infty$ for p, $m\to\infty$. Now, we should that $x\in h_\infty$. We have

$$\|x\|_\infty = \|Hx\|_\infty = \sup_m \left|\sum_{k=1}^{m} \frac{1}{n+k-1} x_k\right| = \sup_m \left|\sum_{k=1}^{m} \frac{1}{n+k-1} (x_k - x_k^p + x_k^p\right|$$
$$\leq \left\|x^p - x\right\|_\infty + \left|Hx_k^p\right| < \infty$$

for p, $k \in N$. This implies that $x = (x_k) \in h_\infty$. Thus, h_∞ the space is a Banach space with the norm (2.4).

It can be shown that h_0 and h_c are closed subspaces of h_∞ which leads us to the consequence that the spaces and are also the Banach spaces with the norm (2.4). Furthermore, since h_∞ is a Banach space with continuous coordinates, i.e., $\left\|H\left(x_k^p - x\right)\right\|_\infty \to \infty$ imples $\left|H\left(x_k^p - x_k\right)\right| \to \infty$ for all $k \in N$, it is also a BK - space.

Theorem 2. The sequence spaces h_∞, h_c and h_0 are linearly isomorphic to the spaces l_∞, c and c_0, respectively, i.e $h_\infty \cong l_\infty$, $h_c \cong c$ and $h_0 \cong c_0$.

Proof. To prove the fact $h_0 \cong c_0$, we should show the existence of a linear bijection between the spaces h_0 and c_0. Consider the transformation T defined, with the notation (2.3), from h_0 to c_0. The linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Let $y \in c_0$. We define the sequence $x = (x_n)$ as follows:

$$x_n = \sum_{i=1}^{n} (-1)^{i+j} (i+j-1) \binom{n+i-1}{n-j} \binom{n+j-1}{n-i} \binom{i+j-2}{i-1}^2 y_k.$$

Then

$$\lim_{m\to\infty} (Hx)_m = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{n+k-1} x_k = \lim_{m\to\infty} y_m = 0.$$

Thus, we have that $x \in h_0$. In addition, note that

$$\|x\|_{h_0} = \sup_{m\in\mathrm{N}} \left|\sum_{k=1}^{m} \frac{1}{n+k-1} x_k\right| = \sup_{m\in\mathrm{N}} |y_m| = \|y\|_{c_0} < \infty.$$

Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says us that the spaces h_0 to c_0 are linearly isomorphic. In the same way, it can be shown that h_c and h_∞ are linearly isomorphic to c and l_∞, respectively, and so we omit the detail.

Theorem 3. The sequence space h_∞, h_c and h_0 includes the sequence spaces l_∞, c and c_0, respectively, i.e. $l_\infty \subset h_\infty$, $c \subset h_c$ and $c_0 \subset h_0$.

Proof. We only prove the conclusion $l_\infty \subset h_\infty$ and the rest follows in a similar way. Let $x \in l_\infty$. Then, using (2.3) and (2.4), we obtain

$$\|x\|_\infty = \|Hx\|_\infty = \sup_{m\in\mathrm{N}} \left|\sum_{k=1}^{m} \frac{1}{n+k-1} x_k\right| \leq \sup_n |x_k| \sup_n |H| = \|x\|_{h_\infty}$$

which means that $x \in h_\infty$.

The Bases Of The Spaces h_c And h_0

First we define the Schauder bases. A sequence $(b_n)_{n\in\mathrm{N}}$ in a normed sequence space λ is called a Schauder basis (or briefly bases) [11], if for every $x \in \lambda$ there is a unique sequence (α_n) of scalars such that

$\lim_{n\to\infty} \left\|x - \left(\alpha_0 x_0 + \alpha_1 x_1 + ... + \alpha_n x_n\right)\right\| = 0$. In this section, we shall give the Schauder bases of the spaces h_c and h_0.

Theorem 4. Let $k \in \mathrm{N}$ a fixed natural number and $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathrm{N}}$ where

$$b_n^{(k)} = (-1)^{n+k} (n+k-1) \binom{m+n-1}{m-k} \binom{m+k-1}{m-n} \binom{n+k-1}{n-1}^2.$$

Then the following assertions are true:

i. The sequence $\left\{b_n^{(k)}\right\}$ is a basis for the space h_0 and every $x \in h_0$ has a unique representation of the form $x = \sum_k \lambda_k b^{(k)}$ where $\lambda_k = (Hx)_k$ for all $k \in \mathrm{N}$.

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ii. The set $\{e,b^{(0)},b^{(1)},...,b^{(k)},...\}$ is a basis for the space h_c and every $x\in h_c$ has a unique representation of the form $x=le+\sum_k(\lambda_k-l)b^{(k)}$ where $l=\lim_{k\to\infty}(Hx)_k$ and $\lambda_k=(Hx)_k$ for all $k\in\mathrm{N}$.

The $\alpha-$, $\beta-$ and $\gamma-$duals of the spaecs h_∞, h_c and h_0

For the sequence spaces λ and μ define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k)\in w\ :\ xz=(x_kz_k)\in\mu \text{ for all } x\in\lambda\}.$$

The α-, β- and γ- duals of the sequence spaces λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by Garling [12], by $\lambda^\alpha=S(\lambda,l_1)$, $\lambda^\beta=S(\lambda,cs)$ and $\lambda^\gamma=S(\lambda,bs)$. We shall begin with the lemmas due to Stieglitz and Tietz [13], which are needed in the proof of the theorems 5-7. We denote by K and F finite subsets of N.

Lemma 1. $A\in(c_0\ :\ l_1)=(c\ :\ l_1)$ if and only if, for $(a_{nk})\in R$

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty \tag{4.1}$$

Lemma 2. $A\in(c_0\ :\ c)$ if and only if

$$\sup_n\sum_k|a_{nk}|<\infty, \tag{4.2}$$

$$\lim_{n\to\infty}a_{nk}=\alpha_k,\ (k\in\mathrm{N}). \tag{4.3}$$

Lemma 3. $A\in(c_0\ :\ l_\infty)$ if and only if (4.2) holds.

Theorem 5. Let $a=(a_k)\in w$ and the matrix

$$B=(-1)^{n+k}(n+k-1)\binom{m+n-1}{m-k}\binom{m+k-1}{m-n}\binom{n+k-1}{n-1}^2.$$

The α- dual of the sequence spaces h_∞, h_c and h_0 is the set

$$D=\left\{a=(a_k)\in w\ :\ \sup_{K\in F}\sum_n\left|\sum_{k\in K}h^{-1}_{nk}a_n\right|<\infty\right\}.$$

Wherein h^{-1}_{nk} is as defined (2.1).

Proof. Let $a=(a_n)\in w$ and consider the matrix B whose rows are the products of the rows of the matrix H^{-1} and sequence $a=(a_n)$. Bearing in mind the relation (2.3), we immediately derive that

$$a_nx_n=\sum_{k=1}^{n}h^{-1}_{nk}a_ny_k=(By)_n,\ n\in\mathrm{N}. \tag{4.4}$$

We therefore observe by (4.4) that $ax=(a_nx_n)\in l_1$ whenever $x\in h_\infty$, h_c and h_0 if and only if $By\in l_1$ whenever $y\in l_\infty$, c, and c_0. Then, by appliying Lemma 1 we get

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K}h^{-1}_{nk}a_n\right|<\infty$$

which yields the consequences that $\{h_\infty\}^\alpha=\{h_c\}^\alpha=\{h_0\}^\alpha=D$.

Theorem 6. Consider the sets D_1, D_2, D_3 and D_4 defined as follows:

$$D_1=\left\{a=(a_k)\in w\ :\ \sup_m\sum_{k=1}^{m}\left|\sum_{n=k}^{m}h^{-1}_{nk}a_n\right|<\infty\right\},$$

$$D_2=\left\{a=(a_k)\in w\ :\ \sum_{n=k}^{m}h^{-1}_{nk}a_n \text{ exists for each } k\in\mathrm{N}\right\},$$

$$D_3=\left\{a=(a_k)\in w : \lim_{m\to\infty}\sum_{k=1}^{m}\left|\sum_{n=k}^{m}h^{-1}_{nk}a_n\right|=\sum_{k=1}^{m}\left|\lim_{n\to\infty}\sum_{n=k}^{m}h^{-1}_{nk}a_n\right|\right\}$$

and

$$D_4=\left\{a=(a_k)\in w\ :\ \lim_{m\to\infty}\sum_{k=1}^{m}\sum_{n=k}^{m}h^{-1}_{nk}a_n \text{ exists}\right\}.$$

Wherein h^{-1}_{nk} is as defined (2.1). Then $\{h_0\}^\beta=D_1\cap D_2$ and $\{h_c\}^\beta=D_1\cap D_2\cap D_4$ and $\{h_\infty\}^\beta=D_2\cap D_3$.

Proof. We only give the proof space h_0. Since the proof may give by a similar way for the spaces h_c and h_∞, we omit it. Consider the equation

$$\sum_{k=1}^{m} a_k x_k = \sum_{k=1}^{m}\left[\sum_{k=1}^{m} h_{nk}^{-1} y_k\right] a_k = \sum_{k=1}^{m}\left[\sum_{k=n}^{m} h_{nk}^{-1} a_k\right] y_n = (Dy)_n,$$

where $D=\left[\sum_{k=n}^{m} h_{nk}^{-1} a_k\right]$. Thus, we deduce from Lemma 2 with (4.4) that $ax=(a_k x_k)\in cs$ whenever $x=(x_k)\in h_0$ if and only if $Dy \in c$ whenever $y = (y_k) \in c_0$. Therefore, using relations (4.3) and (4.4), we conclude that $\lim_{n\to\infty} h_{nk}^{-1} a_k$ exists for each $k \in \mathrm{N}$ and $\sup_{n\in \mathrm{N}} \sum_{k=1}^{n}\left|h_{nk}^{-1} a_k\right| < \infty$ which shows that $\{h_0\}^{\beta} = D_1 \cap D_2$.

Theorem 7. The $\gamma-$ dual of the sequence spaces h_∞, h_c and h_0 are

$$D_5 = \left\{ a=(a_k)\in w \ : \ \sup_n \sum_{k=0}^{n} h_{nk}^{-1} a_k < \infty \right\}.$$

Wherein h_{nk}^{-1} is as defined (2.1).

Proof. We only give the proof space h_0. Consider the equality

$$\left|\sum_{k=1}^{m} a_k x_k\right| = \left|\sum_{n=1}^{m} a_n\left[\sum_{k=1}^{n} h_{nk}^{-1} y_k\right]\right| = \left|\sum_{k=1}^{m} h_{nk}^{-1} a_k y_k\right| \le \sum_{k=1}^{m}\left|h_{nk}^{-1} a_k\right|\left|y_k\right|.$$

Taking supremum over $m \in N$, we get

$$\sup_m\left|\sum_{k=1}^{m} a_k x_k\right| \le \sup_m\left(\sum_{k=1}^{m}\left|h_{nk}^{-1} a_k\right|\left|y_k\right|\right) \le \|y\|_{c_0} \sup_m\left(\sum_{k=1}^{m} h_{nk}^{-1} a_k\right) \le \infty.$$

This means that $a=(a_k)\in\{h_0\}^{\gamma}$. Hence,

$$D_5 \subset \{h_0\}^{\gamma}. \tag{4.5}$$

Conversely, let $a=(a_k)\in\{h_0\}^{\gamma}$ and $x \in h_0$. Then one can easily see that $\left(\sum_{k=1}^{m} h_{nk}^{-1} a_k y_k\right)\in l_\infty$ whenever $ax = (a_k x_k) \in bs$. This implies that matrix $\sum_{k=n}^{m} h_{nk}^{-1} a_k$ is in the class $(c_0 : l_\infty)$. Hence, the condition $\sup_m \sum_{k=1}^{m}\left|h_{nk}^{-1} a_k\right| < \infty$ is satisfied, which implies that $a = (a_k) \in D_5$.
In other words,

$$\{h_0\}^{\gamma} \subset D_1. \tag{4.6}$$

Therefore, by combining inclusions (4.5) and (4.6), we estahlish that the γ- dual of the sequence spaces h_0 is D_5, which completes the proof.

Some Matrix Mappings Related to Hilbert Sequence Spaces

In this section, we give the characterization of the classes $(h_c : l_p)$ and $(h_c : c)$. As the following theorems can be proved using standart methods, we omit the detail.

Lemma 4. [13, p. 57] The matrix mappings between BK- spaces are continuous.

Lemma 5. [13, p. 128] $A\in(c : l_p)$ if and only if

$$\sup_{K\in F}\sum_n\left|\sum_{k\in K} a_{nk}\right|^p < \infty, \quad 1\le p<\infty. \tag{5.1}$$

Theorem 8. $A \in (h_c : l_p)$ if and only if the following conditions are satisfied:

$$\sup_{K\in F}\sum_k\left|\sum_{k\in K}\sum_{n=k}^{m} h_{nk}^{-1} a_{kn}\right|^p < \infty, \tag{5.2}$$

$$\sum_{n=k}^{m} h_{nk}^{-1} a_{kn} \text{ exists for all } k, n \in \mathrm{N} \tag{5.3}$$

$$\sum_k\sum_{n=k}^{m} h_{nk}^{-1} a_{kn} \text{ converges for all } n \in \mathrm{N} \tag{5.4}$$

$$\sup_{m\in N}\sum_{k=1}^{m}\left|\sum_{n=k}^{m} h_{nk}^{-1} a_{kn}\right| < \infty, 1\le p<\infty \tag{5.5}$$

and for $p = \infty$, conditions (5.3) and (5.5) are satisfied and

$$\sup_{n\in \mathrm{N}}\sum_{k=0}^{n}\left|\sum_{n=k}^{m} h_{nk}^{-1} a_{kn}\right| < \infty. \tag{5.6}$$

Wherein h_{nk}^{-1} is as defined (2.1) for every m, n, $k \in N$.

Theorem 9. $A\in(h_c : c)$ if and only if conditions (5.3), (5.5) and (5.6) are satisfied,

$$\lim_{n\to\infty} g_{nk} = \alpha_k \quad \text{for all } k \in \mathrm{N} \tag{5.7}$$

and

$$\lim_{n\to\infty} g_{nk} = \alpha. \tag{5.8}$$

Where $g_{nk} = \sum_{n=k}^{m} h_{nk}^{-1} a_{kn}$

and

$$h_{nk}^{-1} = (-1)^{n+k}(n+k-1)\binom{m+n-1}{m-k}\binom{m+k-1}{m-n}\binom{n+k-1}{n-1}^{2}$$

for every m , n , $k \in N$.

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