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Research Article

General solution to a difference equation and the long-term behavior of some of its solutions

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Abstract

Closed-from formulas for the general solution to a difference equation are given, generalizing some special cases in the literature. We also analyze and give some comments on the results on the long-term behaviour of some solutions of the special cases.

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Keywords. difference equation, general solution, closed-form formula

1. Introduction and preliminaries

We use the standard notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ for the sets of natural, whole and real numbers, respectively. If $l \in \mathbb{Z}$, then $\mathbb{N}_l := \{n \in \mathbb{Z} : n \geq l\}$. If $s, t \in \mathbb{Z}$, $s \leq t$, then $i = \overline{s,t}$, means that i takes the values of $\mathbb{Z}$ such that $s \leq i \leq t$. We understand that $\prod_{i=m}^{m-1} b_i = 1$, for any $m \in \mathbb{Z}$.

Many formulas for solutions to difference equations and systems in closed form can be found in quite old literature [9, 12, 13, 21, 22]. For some old presentations of the methods for finding them see [19, 20]. One can also consult, e.g., [10, 15, 23–25, 27, 28]. Some recent formulas and tricks for finding solutions to nonlinear difference equations and systems can be found, e.g., in [14, 35, 37, 49–60]. The formulas are usually useful in dealing with the solutions to the equations and systems. However, it is a rare situation that an equation or system is solvable. Even if the equation or system is solvable, there is a possibility that obtained formulas are not so useful for investigation of the long-term behavior of their solutions. Therefore, one can try to find some other type of relations, for instance, their invariants [30, 31, 33, 39, 40].

The bilinear difference equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b, c, d, x_0 \in \mathbb{R}$, is one of the first nonlinear equations for which was shown its solvability [19, 20]. In [52], among other things, we presented some historical facts about the difference equation. Many other facts and connections with other difference equations and systems can be found, for instance, in [1, 2, 10, 11, 18–20, 23, 26–28, 51, 52, 56, 57, 60].

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Eq.(1.1) can be solved by transforming it to a linear difference equation of second order. This fact, among other things, is also employed in our studies presented here. Generally speaking, solvability of many difference equations and systems is shown by transforming them to some known solvable ones by some suitable transformations [14,35,49–55,57–60].

The following result was essentially proved in [9] and [12], and can be found in many books and papers (see, e.g., [18,51,61]).

Lemma 1.1. *Consider the equation*

$$x_{n+2} + a_1 x_{n+1} + a_0 x_n = 0, \quad n \in \mathbb{N}_0,$$

where $a_1, x_0, x_1 \in \mathbb{R}$ *and* $a_0 \in \mathbb{R} \setminus \{0\}$. *Then, the following statements hold.*

(1) *If* $a_1^2 \neq 4a_0$, *then*

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n - (x_1 - \lambda_1 x_0)\lambda_2^n}{\lambda_1 - \lambda_2}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}.$$

(2) *If* $a_1^2 = 4a_0$, *then*

$$x_n = ((x_1 - \lambda x_0)n + \lambda x_0)\lambda^{n-1}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $\lambda = -a_1/2$.

The equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-4}}{cx_{n-3} + dx_{n-4}}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $a, b, c, d \in \mathbb{R}$, $x_{-j} \in \mathbb{R}$, $j = \overline{0,4}$, was studied recently in [38], where some formulas for solutions in the cases:

(1) $a = b = c = d = 1$;
(2) $a = b = c = 1$, $d = -1$;
(3) $a = c = 1$, $b = d = -1$;
(4) $a = c = d = 1$, $b = -1$,

were presented. Besides, [38] gives some claims on the long-term behavior of solutions to Eq.(1.4).

The purpose of the paper is to show that Eq.(1.4) is a special case of a solvable difference equation, from which the solvability in the cases (1)-(4) follows. We also analyze the claims on the long-term behavior of solutions to Eq.(1.4) formulated in [38] and show that they are false.

2. Solvability of a generalization of Eq. (1.4)

Here we show that Eq.(1.4) is a special case of a solvable difference equation. Before we state our first result note that Eq.(1.4) can be rewritten in the form

$$x_{n+1} = x_n \frac{acx_{n-3} + (ad + b)x_{n-4}}{cx_{n-3} + dx_{n-4}}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

The form suggests studying a natural generalization of Eq.(1.4) (see Eq.(2.2) below).

Theorem 2.1. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha^2+\beta^2 \neq 0 \neq \gamma^2+\delta^2$, and Ψ be a homeomorphism of $\mathbb{R}$ such that $\Psi(0)=0$. Then, the equation*

$$x_{n+1} = \Psi^{-1}\left(\Psi(x_n)\frac{\alpha\Psi(x_{n-3})+\beta\Psi(x_{n-4})}{\gamma\Psi(x_{n-3})+\delta\Psi(x_{n-4})}\right), \quad n \in \mathbb{N}_0, \tag{2.2}$$

is solvable in closed form.

Proof. If there is $n_0 \in \mathbb{N}_0$ such that $x_{n_0}=0$, then if x_{n_0+1} is defined it must be equal to zero. But then x_{n_0+5} is not defined. Hence, from now on for a solution $(x_n)_{n\in\mathbb{N}_{-4}}$ to Eq.(2.2) we suppose $x_n \neq 0$ for $n \in \mathbb{N}_{-4}$, which implies

$$\Psi(x_n) \neq 0, \quad \text{for} \quad n \in \mathbb{N}_{-4}. \tag{2.3}$$

Eq.(2.2) along with the conditions posed on Ψ imply

$$\Psi(x_{n+1}) = \Psi(x_n)\frac{\alpha\Psi(x_{n-3})+\beta\Psi(x_{n-4})}{\gamma\Psi(x_{n-3})+\delta\Psi(x_{n-4})}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

Using the transformation

$$y_n = \frac{\Psi(x_n)}{\Psi(x_{n-1})}, \quad n \in \mathbb{N}_{-3}. \tag{2.5}$$

in (2.4) we obtain the equation

$$y_{n+1} = \frac{\alpha y_{n-3}+\beta}{\gamma y_{n-3}+\delta}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Let

$$z_m^{(j)} = y_{4m-j}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}. \tag{2.7}$$

Then

$$z_{m+1}^{(j)} = \frac{\alpha z_m^{(j)}+\beta}{\gamma z_m^{(j)}+\delta}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}. \tag{2.8}$$

Using the change of variables

$$z_m^{(j)} = \frac{u_{m+1}^{(j)}}{u_m^{(j)}} - \frac{\delta}{\gamma}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}, \tag{2.9}$$

where $\gamma \neq 0$, in (2.8) we obtain

$$\gamma^2 u_{m+2}^{(j)} - \gamma(\alpha+\delta)u_{m+1}^{(j)} + (\alpha\delta-\beta\gamma)u_m^{(j)} = 0, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}. \tag{2.10}$$

There are several cases to be considered.

Case $\alpha\delta \neq \beta\gamma$, $\gamma \neq 0$. Under the assumptions, there are several subcases to be considered.

Case $(\alpha+\delta)^2 \neq 4(\alpha\delta-\beta\gamma)$. Employing (1.2) we obtain

$$u_m^{(j)} = \frac{(u_1^{(j)}-\lambda_2 u_0^{(j)})\lambda_1^m - (u_1^{(j)}-\lambda_1 u_0^{(j)})\lambda_2^m}{\lambda_1-\lambda_2}, \tag{2.11}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$, where

$$\lambda_1 = \frac{\alpha+\delta+\sqrt{(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)}}{2\gamma} \tag{2.12}$$

and

$$\lambda_2 = \frac{\alpha+\delta-\sqrt{(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)}}{2\gamma}. \tag{2.13}$$

Using (2.11) in (2.9) we have

$$z_m^{(j)} = \frac{(z_0^{(j)} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(z_0^{(j)} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$, that is

$$y_{4m-j} = \frac{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma}, \tag{2.14}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$.

Combining (2.5) and (2.14) we have

$$\Psi(x_{4m-j}) = \left(\frac{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \Psi(x_{4m-j-1}),$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$, as well as

$$\Psi(x_{4m-j}) = y_{4m-j}y_{4m-j-1}y_{4m-j-2}y_{4m-j-3}\Psi(x_{4m-j-4}), \tag{2.15}$$

for $m \in \mathbb{N}$, $j = \overline{1,4}$.

Thus

$$\begin{aligned}
\Psi(x_{4m}) &= \Psi(x_{-4}) \prod_{i=0}^{m} y_{4i}y_{4i-1}y_{4i-2}y_{4i-3}, \\
\Psi(x_{4m+1}) &= \Psi(x_{-3}) \prod_{i=0}^{m} y_{4i+1}y_{4i}y_{4i-1}y_{4i-2}, \\
\Psi(x_{4m+2}) &= \Psi(x_{-2}) \prod_{i=0}^{m} y_{4i+2}y_{4i+1}y_{4i}y_{4i-1}, \\
\Psi(x_{4m+3}) &= \Psi(x_{-1}) \prod_{i=0}^{m} y_{4i+3}y_{4i+2}y_{4i+1}y_{4i},
\end{aligned}$$

for $m \in \mathbb{N}_0$, which implies

$$x_{4m} = \Psi^{-1}\left(\Psi(x_{-4}) \prod_{i=0}^{m} y_{4i}y_{4i-1}y_{4i-2}y_{4i-3} \right), \tag{2.16}$$

$$x_{4m+1} = \Psi^{-1}\left(\Psi(x_{-3}) \prod_{i=0}^{m} y_{4i+1}y_{4i}y_{4i-1}y_{4i-2} \right), \tag{2.17}$$

$$x_{4m+2} = \Psi^{-1}\left(\Psi(x_{-2}) \prod_{i=0}^{m} y_{4i+2}y_{4i+1}y_{4i}y_{4i-1} \right), \tag{2.18}$$

$$x_{4m+3} = \Psi^{-1}\left(\Psi(x_{-1}) \prod_{i=0}^{m} y_{4i+3}y_{4i+2}y_{4i+1}y_{4i} \right), \tag{2.19}$$

for $m \in \mathbb{N}_0$, where

$$
\begin{aligned}
&y_{4m}y_{4m-1}y_{4m-2}y_{4m-3}\\
&=\left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
&y_{4m+1}y_{4m}y_{4m-1}y_{4m-2}\\
&=\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
&y_{4m+2}y_{4m+1}y_{4m}y_{4m-1}\\
&=\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
&y_{4m+3}y_{4m+2}y_{4m+1}y_{4m}\\
&=\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{2.23}
$$

for $m\in\mathbb{N}_0$. Hence, (2.16)-(2.23) present the general solution to Eq.(2.2) in this case.

Case $(\alpha+\delta)^2=4(\alpha\delta-\beta\gamma)$. From (1.3) we have

$$u_m^{(j)}=((u_1^{(j)}-\lambda u_0^{(j)})m+\lambda u_0^{(j)})\lambda^{m-1},\tag{2.24}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$, where

$$\lambda=\frac{\alpha+\delta}{2\gamma}\neq 0.$$

Using (2.24) in (2.9) we obtain

$$z_m^{(j)}=\frac{((z_0^{(j)}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(z_0^{(j)}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma},$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$, that is,

$$y_{4m-j}=\frac{((y_{-j}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(y_{-j}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma},\tag{2.25}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$.

Relations (2.5) and (2.25) yield

$$\Psi(x_{4m-j})=\left(\frac{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\Psi(x_{4m-j-1}),\tag{2.26}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$.

We also have

$$
\begin{aligned}
y_{4m}y_{4m-1}y_{4m-2}y_{4m-3}&=\left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{2.27}
$$

$$y_{4m+1}y_{4m}y_{4m-1}y_{4m-2} = \left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+2)+\lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})m+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})m+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})m+\lambda} - \frac{\delta}{\gamma}\right), \tag{2.28}$$

$$y_{4m+2}y_{4m+1}y_{4m}y_{4m-1} = \left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+2)+\lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+2)+\lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})m+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})m+\lambda} - \frac{\delta}{\gamma}\right), \tag{2.29}$$

$$y_{4m+3}y_{4m+2}y_{4m+1}y_{4m} = \left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+2)+\lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+2)+\lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+2)+\lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})m+\lambda} - \frac{\delta}{\gamma}\right), \tag{2.30}$$

for $m \in \mathbb{N}_0$. Hence, (2.16)-(2.19), (2.27)-(2.30) present the general solution to Eq.(2.2) in this case.

Case $\gamma = 0$. Under the condition we have $\delta \neq 0$ and

$$y_{n+1} = \frac{\alpha}{\delta}y_{n-3} + \frac{\beta}{\delta}, \quad n \in \mathbb{N}_0, \tag{2.31}$$

that is,

$$z_{m+1}^{(j)} = \frac{\alpha}{\delta}z_m^{(j)} + \frac{\beta}{\delta}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}. \tag{2.32}$$

Case $\alpha = \delta$. We have

$$z_m^{(j)} = \frac{\beta}{\delta}m + z_0^{(j)}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}.$$

that is,

$$y_{4m-j} = \frac{\beta}{\delta}m + y_{-j} = \frac{\beta}{\delta}m + \frac{\Psi(x_{-j})}{\Psi(x_{-j-1})}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}.$$

This relation, (2.5) and (2.15) imply

$$\begin{aligned}
\Psi(x_{4m}) &= \left(\frac{\beta}{\delta}m + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}m + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right) \\
&\quad \times \left(\frac{\beta}{\delta}m + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}m + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\Psi(x_{4m-4}), \\
\Psi(x_{4m+1}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}m + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right) \\
&\quad \times \left(\frac{\beta}{\delta}m + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}m + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\Psi(x_{4m-3}), \\
\Psi(x_{4m+2}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}(m+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right) \\
&\quad \times \left(\frac{\beta}{\delta}m + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}m + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\Psi(x_{4m-2}), \\
\Psi(x_{4m+3}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}(m+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right) \\
&\quad \times \left(\frac{\beta}{\delta}(m+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}m + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\Psi(x_{4m-1})
\end{aligned}$$

for $m \in \mathbb{N}_0$, so that

$$\begin{aligned}
\Psi(x_{4m}) =& \Psi(x_{-4})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right) \\
&\times \left(\frac{\beta}{\delta}j + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}j + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right), \\
\Psi(x_{4m+1}) =& \Psi(x_{-3})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right) \\
&\times \left(\frac{\beta}{\delta}j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}j + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right), \\
\Psi(x_{4m+2}) =& \Psi(x_{-2})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right) \\
&\times \left(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right), \\
\Psi(x_{4m+3}) =& \Psi(x_{-1})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right) \\
&\times \left(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\right),
\end{aligned}$$

for $m \in \mathbb{N}_0$, and finally

$$x_{4m} = \Psi^{-1}\Big(\Psi(x_{-4}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big) \times \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big)\Big), \tag{2.33}$$

$$x_{4m+1} = \Psi^{-1}\Big(\Psi(x_{-3}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big)\Big), \tag{2.34}$$

$$x_{4m+2} = \Psi^{-1}\Big(\Psi(x_{-2}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big) \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big)\Big), \tag{2.35}$$

$$x_{4m+3} = \Psi^{-1}\Big(\Psi(x_{-1}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big) \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big) \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big) \Big(\frac{\beta}{\delta} j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big)\Big), \tag{2.36}$$

for $m \in \mathbb{N}_0$. Hence, (2.33)-(2.36) present the general solution to Eq.(2.2) in this case.

Case $\alpha \neq \delta$. Eq.(2.32) implies

$$z_m^{(j)} = \frac{\beta}{\alpha - \delta}\Big(\Big(\frac{\alpha}{\delta}\Big)^m - 1\Big) + \Big(\frac{\alpha}{\delta}\Big)^m z_0^{(j)},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$, that is,

$$y_{4m-j} = \beta \frac{(\alpha/\delta)^m - 1}{\alpha - \delta} + \Big(\frac{\alpha}{\delta}\Big)^m y_{-j}. \tag{2.37}$$

Relations (2.5), (2.15) and (2.37) imply

$$\begin{aligned}
\Psi(x_{4m}) &= \left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\Psi(x_{4m-4}),\\
\Psi(x_{4m+1}) &= \left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\Psi(x_{4m-3}),\\
\Psi(x_{4m+2}) &= \left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\Psi(x_{4m-2}),\\
\Psi(x_{4m+3}) &= \left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\Psi(x_{4m-1}),
\end{aligned}$$

for $m\in\mathbb{N}_0$.

Hence

$$\begin{aligned}
\Psi(x_{4m}) = \Psi(x_{-4}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_0)}{\Psi(x_{-1})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_{-1})}{\Psi(x_{-2})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_{-2})}{\Psi(x_{-3})} \right), \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_{-3})}{\Psi(x_{-4})} \right), \\
\Psi(x_{4m+1}) = \Psi(x_{-3}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j+1} - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-3})}{\Psi(x_{-4})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_0)}{\Psi(x_{-1})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_{-1})}{\Psi(x_{-2})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_{-2})}{\Psi(x_{-3})} \right), \\
\Psi(x_{4m+2}) = \Psi(x_{-2}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j+1} - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-2})}{\Psi(x_{-3})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j+1} - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-3})}{\Psi(x_{-4})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_0)}{\Psi(x_{-1})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_{-1})}{\Psi(x_{-2})} \right), \\
\Psi(x_{4m+3}) = \Psi(x_{-1}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j+1} - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-1})}{\Psi(x_{-2})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j+1} - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-2})}{\Psi(x_{-3})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j+1} - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-3})}{\Psi(x_{-4})} \right) \\
& \times \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{\Psi(x_0)}{\Psi(x_{-1})} \right),
\end{aligned}$$

for $m \in \mathbb{N}_0$, so that

$$x_{4m} = \Psi^{-1}\bigg(\Psi(x_{-4})\prod_{j=0}^{m}\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\bigg), \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\bigg)\bigg), \tag{2.38}$$

$$x_{4m+1} = \Psi^{-1}\bigg(\Psi(x_{-3})\prod_{j=0}^{m}\bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\bigg)\bigg), \tag{2.39}$$

$$x_{4m+2} = \Psi^{-1}\bigg(\Psi(x_{-2})\prod_{j=0}^{m}\bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\bigg)\bigg), \tag{2.40}$$

$$x_{4m+3} = \Psi^{-1}\bigg(\Psi(x_{-1})\prod_{j=0}^{m}\bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\bigg) \times\bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\bigg)\bigg), \tag{2.41}$$

for $m \in \mathbb{N}_0$. Hence, (2.38)-(2.41) present the general solution to Eq.(2.2) in this case.

Case $\alpha\delta = \beta\gamma$, $\alpha = 0$. We have $\beta \neq 0$, $\gamma = 0$ and $\delta \neq 0$. Thus

$$x_{n+1} = \Psi^{-1}\bigg(\frac{\beta}{\delta}\Psi(x_n)\bigg), \quad n \in \mathbb{N}_0, \tag{2.42}$$

which yields

$$x_n = \Psi^{-1}\left(\left(\frac{\beta}{\delta}\right)^n \Psi(x_0)\right), \quad n \in \mathbb{N}_0. \tag{2.43}$$

Case $\alpha\delta = \beta\gamma$, $\alpha \neq 0$, $\beta = 0$. Under the conditions we have $\delta = 0$ and $\gamma \neq 0$. Thus

$$x_{n+1} = \Psi^{-1}\left(\frac{\alpha}{\gamma}\Psi(x_n)\right), \quad n \in \mathbb{N}_0, \tag{2.44}$$

which yields

$$x_n = \Psi^{-1}\left(\left(\frac{\alpha}{\gamma}\right)^n \Psi(x_0)\right), \quad n \in \mathbb{N}_0. \tag{2.45}$$

Case $\alpha\delta = \beta\gamma$, $\delta = 0$. Under the conditions we have $\gamma \neq 0$, $\beta = 0$ and $\alpha \neq 0$, and consequently obtain Eq.(2.44) whose solution is (2.45).

Case $\alpha\delta = \beta\gamma$, $\gamma = 0$. Under the conditions we have $\delta \neq 0$, $\alpha = 0$ and $\beta \neq 0$. and consequently obtain Eq.(2.42) whose solution is (2.43).

Case $\alpha\beta\gamma\delta \neq 0$. In this case we obtain Eq.(2.42), i.e., Eq.(2.44), whose solution is (2.43), i.e., (2.45). □

Remark 2.2. From Theorem 2.1, some calculation and a representation in [51] are obtained the closed-form formulas in [38] for solutions to the special cases (i)-(iv) of Eq.(1.4). We omit the standard problem.

3. On some results on the long-term behavior of solutions to Eq. (1.4)

One of the main problems in the theory of difference equations is describing the long-term behaviour of their solutions (see, e.g., [2, 3, 5–8, 16–18, 23, 26, 29, 34–36, 40–48, 50, 53–55, 58, 59] and the references therein).

Here we analyze the claims on the long-term behaviour of some solutions to the special cases of Eq.(1.4) considered in [38]. The paper starts with finding equilibria of Eq.(1.4) and, unfortunately, made a typical mistake. Namely, for an equilibrium $\bar{x}$ of Eq.(1.4) it has to hold the relation

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{(c+d)\bar{x}}. \tag{3.1}$$

It is wrongly claimed therein that (3.1) is equivalent to the relation

$$\bar{x}^2(1-a)(c+d) = b\bar{x}^2. \tag{3.2}$$

Then it is assumed $(1-a)(c+d) \neq b$, and from (3.2) concluded that $\bar{x} = 0$ is a unique equilibrium point of Eq.(1.4). But $\bar{x} = 0$ is not a solution to (3.1). As a consequence of the wrong claim, the following claim ([38, Theorem 5]) is wrong.

Claim 1. *Assume that* $b(d+3c) < (1-a)(c+d)^2$. *Then the equilibrium point of Eq.*(1.4) *is locally asymptotically stable.*

The following claim is Theorem 6 in [38].

Claim 2. *The equilibrium point* $\bar{x}$ *of Eq.*(1.4) *is global attractor if* $d(1-a) \neq b$.

From the same reason the claim is not well-formulated. Moreover, if we even ignore the problem with the choice of a wrong equilibrium, the claim is not correct. We show this by an example. Before presenting the example note that Eq.(2.1) is obtained from Eq.(2.2) for $\Psi(x) = x$, $\alpha = ac$, $\beta = ad + b$, $\gamma = c$ and $\delta = d$.

Example 3.1. Consider the equation

$$x_{n+1} = x_n \frac{\alpha x_{n-3} + \beta x_{n-4}}{\gamma x_{n-3} + \delta x_{n-4}}, \quad n \in \mathbb{N}_0. \tag{3.3}$$

where $\alpha, \beta, \gamma, \delta$ are positive numbers satisfying the conditions

$$\alpha + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} > \delta + 2\gamma, \tag{3.4}$$

$$\delta(\gamma - \alpha) \neq \beta\gamma - \delta\alpha, \tag{3.5}$$

whereas the initial values x_{-j}, $j = \overline{0,4}$, satisfy the conditions

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2 - \frac{\delta}{\gamma}, \quad i = \overline{0,3}, \tag{3.6}$$

where λ_2 is given in (2.13).

From (3.5) we easily see that in this case $d(1-a) \neq b$, which is a condition in Claim 1.

From (2.16)-(2.23) it follows that for $m \in \mathbb{N}_0$ we have

$$x_{4m} = x_{-4} \prod_{i=0}^{m} y_{4i} y_{4i-1} y_{4i-2} y_{4i-3} \tag{3.7}$$

$$x_{4m+1} = x_{-3} \prod_{i=0}^{m} y_{4i+1} y_{4i} y_{4i-1} y_{4i-2}, \tag{3.8}$$

$$x_{4m+2} = x_{-2} \prod_{i=0}^{m} y_{4i+2} y_{4i+1} y_{4i} y_{4i-1}, \tag{3.9}$$

$$x_{4m+3} = x_{-1} \prod_{i=0}^{m} y_{4i+3} y_{4i+2} y_{4i+1} y_{4i}, \tag{3.10}$$

where

$$\begin{aligned}
&y_{4m} y_{4m-1} y_{4m-2} y_{4m-3} \\
&= \left(\frac{(\frac{x_0}{x_{-1}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_0}{x_{-1}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_0}{x_{-1}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\
&\quad \times \left(\frac{(\frac{x_{-1}}{x_{-2}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_{-1}}{x_{-2}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_{-1}}{x_{-2}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\
&\quad \times \left(\frac{(\frac{x_{-2}}{x_{-3}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_{-2}}{x_{-3}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_{-2}}{x_{-3}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\
&\quad \times \left(\frac{(\frac{x_{-3}}{x_{-4}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_{-3}}{x_{-4}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-3}}{x_{-4}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_{-3}}{x_{-4}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right),
\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&y_{4m+1}y_{4m}y_{4m-1}y_{4m-2}\\
&=\left(\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&y_{4m+2}y_{4m+1}y_{4m}y_{4m-1}\\
&=\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&y_{4m+3}y_{4m+2}y_{4m+1}y_{4m}\\
&=\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{3.14}
$$

for $m\in\mathbb{N}_0$, where λ_1 and λ_2 are given in (2.12) and (2.13), respectively.

Further, we have

$$\begin{aligned}
&\lim_{m\to+\infty}\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
&=\lim_{m\to+\infty}\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
&=\lim_{m\to+\infty}\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
&=\lim_{m\to+\infty}\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
&=\lambda_1-\frac{\delta}{\gamma}>1,
\end{aligned}$$

where first we have used the assumption (3.6), whereas in the last inequality we have used the condition in (3.4), from which along with (3.7)-(3.14) it follows that for such solutions we have $\lim_{n\to+\infty} x_n = +\infty$, that is, the solutions are even unbounded refuting Claim 2.

Remark 3.2. It is easy to find the parameters $\alpha, \beta, \gamma, \delta$ satisfying the conditions (3.4) and (3.5), as well as the positive initial values so that the relations in (3.6) hold. For example, we can chose $\alpha = \gamma = \delta = 1$ and $\beta = 2$, in which case we have that $a = b = c = d = 1$ in Eq.(1.4).

Remark 3.3. The only correct result on the behaviour of solutions to Eq.(1.4) in [38] is Theorem 7 therein, which states that every positive solution to the equation is bounded if $\min\{a, b, c, d\} > 0$ and $a + \frac{b}{d} < 1$. However, the result is trivial, since in this case from Eq.(1.4) it immediately follows that for each positive solution to the equation we have $0 < x_{n+1} \leq (a + \frac{b}{d})x_n < x_n$, $n \in \mathbb{N}_0$. Moreover, from the estimate and by a well-known theorem it follows that each positive solution to Eq.(1.4) converges [62], and that the limit is equal to zero, which was not noticed in [38]. For some results on the boundedness of solutions to various difference equations and systems see, for instance, [4–8, 29, 31, 32, 34, 36, 40–43, 45, 47] and the related references therein.

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