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ON HYBRID CURVES

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Abstract

In this paper, we first define the vector product in a special analog Minkowski Geometry $(R^3, \langle,\rangle)$, which is identified with the space of spatial hybrids. Next, we derive the Frenet-Serret frame formulae for a three dimensional non-parabolic curve by using the spatial hybrids and the vector product. However, we present the Frenet-Serret frame formulae of a non-lightlike hybrid curve in R^4 and an illustrative example for all theorems of the paper with MATLAB 2016a codes.

Keywords: Hybrid numbers, Curvatures, Frenet-Serret frame formulae

1. Introduction

In literature, there are many papers for the curve theory. Especially, the properties of the curve also can be calculated with quaternions and octonions. Bharati et. al. in [1] present the Frenet-Serret formulae for a quaternionic curve. Çöken et. al. in [2] obtain the Frenet-Serret formulae of the curve which is identified by quaternions in $\mathbb{E}_2^4$. Moreover, Dağdeviren et. al. in [3] calculate the Frenet-Serret formulae for dual quaternionic curves. Ohashi in [4] calculates the properties of the spatial octonionic curves.

In literature, hybrid numbers have mostly studied like quaternions and octonions in many different areas. Özdemir in [5] defines the hybrid numbers $H = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h},$ where $a,b,c,d \in R, \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1.$ The set of all hybrid numbers is denoted by $\mathbb{H}.$ The hybrid numbers are widely studied by mathematicians or physicists, such as in [6-8]. Özdemir in [6] gives a new method for finding n^{th} roots of a 2×2 real matrix with the help of hybrid numbers. He finds the De Moivre's formula according to type and character of the 2×2 real matrix. Öztürk in [7] examines the concept of similarity for hybrid numbers by using the solutions of some linear equations. Akbıyık et.al. in [8] calculate the Euler's and De Moivre's formulas for the 4×4 matrices associated with hybrid numbers by using trigonometric identities. Hybrid numbers are also studied with some special numbers such as Horadam numbers in [9], Fibonacci numbers in [10].

In this study, we firstly define the vector product in a special analog Minkowski Geometry $(R^3, \langle,\rangle)$, which is identified with the space of spatial hybrid numbers. Secondly, we obtain Frenet-Serret frame formula for non-parabolic spatial hybrid curve. Thirdly, we examine Frenet-Serret frame for the curve in R_4 which is identified with non-lightlike hybrid. Finally, we define some Matlab function codes. With these Matlab codes, we present an example for all results.

2. Preliminaries

In this article, we will present some fundamental definitions and properties about hybrid numbers. The scalar part of the hybrid number $H = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} \in \mathbb{H}$ is denoted by S_H and is defined as $S_H = a.$ Similarly, the vector part of the hybrid number

$$H = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} \in \mathbb{H}$$

is denoted $\mathbf{V}_H$ and it is defined by $\mathbf{V}_H = b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h},$ [5]. The hybridian product of two hybrid numbers H_1 and H_2 is defined

$$\begin{aligned} H_1 \times H_2 &= (a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}) \times (a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}) \\ &= a_1a_2 - b_1b_2 + b_1c_2 + c_1b_2 + d_1d_2 \\ &\quad + a_1(b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}) \\ &\quad + a_2(b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}) \\ &\quad + (b_1d_2 - d_1b_2)\mathbf{i} \\ &\quad + (b_1d_2 - c_1d_2 - d_1b_2 + d_1c_2)\boldsymbol{\varepsilon} \\ &\quad + (c_1b_2 - b_1c_2)\mathbf{h}. \end{aligned}$$

Therefore, the hybrid numbers is non-commutative but associative, [5]. The conjugate of a hybrid number H is $\overline{H} = S_H - \mathbf{V}_H = a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}.$ Moreover, by using hybridian product, we have $H \times \overline{H} = \overline{H} \times H.$ Additionally, $char(H) = H \times \overline{H} = \overline{H} \times H = a^2 + (b-c)^2 - c^2 - d^2$ is called the character of the hybrid number of H, [5]. In addition, we can give the following Matlab function code for calculating the conjugate.

```
function p=coo(H)
p=[H(1),-H(2),-H(3),-H(4)]
%Here we consider the hybrid number as a vector in R^4
```

The hybrid number H can be classified as spacelike, timelike or lightlike hybrid number if $char(H) < 0, char(H) > 0$ or $char(H) = 0,$ respectively. The norm of the hybrid number H, denoted by $\| H \|,$ is calculated by $\sqrt{|char(H)|},$ [5].

The vector $\varepsilon_H = ((b-c), c, d)$ is called the hybrid vector of the number $H = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$. The character of the vector ε_H is given by $char_\varepsilon(H) = -(b-c)^2 + c^2 + d^2$, [5]. In this case, the hybrid number H can be also classified as hyperbolic, elliptic or parabolic hybrid number, if $char_\varepsilon(H) > 0, char_\varepsilon(H) < 0$ or $char_\varepsilon(H) = 0$, respectively. The types of the hybrid number H are identified by using these conditions. Furthermore, the norm of the hybrid vector of H, denoted by $N(H)$, is calculated by $\sqrt{|char_\varepsilon(H)|}$, [5].
The scalar product of two hybrid numbers H_1 and H_2 is defined

$$
\begin{aligned}
g(H_1, H_2) &= \frac{1}{2}(H_1 \times \bar{H}_2 + H_2 \times \bar{H}_1) \\
&= a_1 a_2 + b_1 b_2 - b_1 c_2 - b_2 c_1 - d_1 d_2,
\end{aligned}
$$

where $H_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$ and $H_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$, [5]. The vector product of two hybrid numbers H_1 and H_2 is defined

$$
H_1 * H_2 = \frac{1}{2}(H_1 \times \bar{H}_2 - H_2 \times \bar{H}_1), \tag{1}
$$

where $H_1 = a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h}$ and $H_2 = a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon} + d_2\mathbf{h}$, [5].

Let $\mathbb{H}^p = \{A \in \mathbb{H} \mid A + \bar{A} = 0\}$ be the spatial hybrid numbers space and let $(R^3, \langle,\rangle)$ be the $3-$ dimensional vector space with a inner product (a non-degenerate, symmetric bilinear form)

$$
\langle \mathbf{v}, \mathbf{w} \rangle = b_1 b_2 - b_1 c_2 - c_1 b_2 - d_1 d_2, \tag{2}
$$

where $\mathbf{v} = (b_1, c_1, d_1)$ and $\mathbf{w} = (b_2, c_2, d_2) \in R^3$. The inner space $(R^3, \langle,\rangle)$ is identified with the space of spatial hybrid numbers $\mathbb{H}^p$. Also, we can give the following Matlab function code for calculating the inner product of two spatial hybrids.

```
function s=innerr(v,w)
s=v(1)*w(1)-v(1)*w(2)-v(2)*w(1)-v(3)*w(3)
end
```

Moreover, we can define the vector product of any two vectors $\mathbf{v} = (b_1, c_1, d_1)$ and $\mathbf{w} = (b_2, c_2, d_2)$ in the inner product space $(R^3, \langle,\rangle)$ as follow

$$
\mathbf{v} \wedge \mathbf{w} = \begin{bmatrix} -d_1 & 0 & b_1 \\ -d_1 & d_1 & b_1 - c_1 \\ c_1 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} b_2 \\ c_2 \\ d_2 \end{bmatrix}. \tag{3}
$$

This vector product is equal to the negative sign of the vector product (1). In addition, we can give the following Matlab function code for calculating the vector product.

```
function s=vectorproduct(v,w)
b1=v(1);
c1=v(2);
d1=v(3);

b2=w(1);
c2=w(2);
d2=w(3);

A=[-d1 0 b1;
    -d1 d1 (b1-c1);
    c1 -b1 0];
B=[b2;
c2; d2];
s=A*B
end
```

The vector product has the following properties:

$$\begin{aligned} &\mathbf{v}\wedge\mathbf{v}=0\\ &\mathbf{v}\wedge\mathbf{w}=-\mathbf{w}\wedge\mathbf{v}\\ &\mathbf{v}\wedge(\mathbf{v}\wedge\mathbf{w})=\langle\mathbf{v},\mathbf{w}\rangle\mathbf{v}-\langle\mathbf{v},\mathbf{v}\rangle\mathbf{w}. \end{aligned}$$

On the other hand, by using the inner product (2) and the vector product (3) in $(R^3,\langle,\rangle)$, the hybridian product of two hybrid numbers H_1 and H_2 can be expressed as

$$H_1\times H_2=S_{H_1}S_{H_1}-\langle\mathbf{V}_{H_1},\mathbf{V}_{H_2}\rangle+S_{H_1}\mathbf{V}_{H_2}+S_{H_2}\mathbf{V}_{H_1}+\mathbf{V}_{H_1}\wedge\mathbf{V}_{H_2}.$$

In addition, we can give the following Matlab function code for calculating the hybridian product.

```
function p=hproduct(a,b)
sa=a(1);
sb=b(1);
va=[a(2) a(3) a(4)];
vb=[b(2) b(3) b(4)];
v1=sa*vb;
v2=sb*va;
v3=transpose(vectorproduct(va,vb));
vi=innerr(va,vb);
v=v1+v2+v3;
p=[sa*sb-vi v(1) v(2) v(3)];
end
% Note that here a=H_1, b=H_2.
```

If $A,B\in\mathbb{H}^p$, then the hybridian product of two spatial hybrid number

$$A\times B=-\langle A,B\rangle+A\wedge B$$

is obtained.

3. Frenet-Serret Formula for a Spatial Hybrid Curve

The differentiable curve $\alpha : I \subset R \to \mathbb{H}^p$ defined by $\alpha(s) = \alpha_1(s)\mathbf{i} + \alpha_2(s)\boldsymbol{\varepsilon} + \alpha_3(s)\mathbf{h}$ is called a spatial hybrid curve for $I \subset R$ and $s \in I$.

Definition 3.1 Let $\alpha : I \subset R \to \mathbb{H}^p$ be a spatial hybrid curve. If for every $s \in I$

$$g(\alpha'(s), \alpha'(s)) = \frac{1}{2}(\alpha'(s) \times \overline{\alpha'(s)} + \alpha'(s) \times \overline{\alpha'(s)}) = \alpha'(s) \times \overline{\alpha'(s)} = \epsilon_0,$$

where $\epsilon_0 \in \{1, -1\}$, then the curve is called unit speed spatial hybrid curve. If $g(\alpha'(s), \alpha'(s)) = 1$ is called hyperbolic, $g(\alpha'(s), \alpha'(s)) = -1$ elliptic and $g(\alpha'(s), \alpha'(s)) = 0$ parabolic curve.
In this paper we only consider frames which do not contain parabolic and lightlike vectors.

Theorem 3.2 Let $\alpha(s) = \alpha_1(s)\mathbf{i} + \alpha_2(s)\boldsymbol{\varepsilon} + \alpha_3(s)\mathbf{h}$ be the smooth unit speed non-parabolic spatial hybrid curve so that the tangent $\mathfrak{t}(s) = \alpha'(s)$ has unit length $g(t(s), t(s)) = \epsilon_0$ for all $s \in I$. Then

i. $\mathfrak{t}'$ is $g-$ orthogonal to $\mathfrak{t}$.
ii. $\mathfrak{t}' \times \bar{\mathfrak{t}}$ is a spatial hybrid.

Proof. Let $\alpha(s) = \alpha_1(s)\mathbf{i} + \alpha_2(s)\boldsymbol{\varepsilon} + \alpha_3(s)\mathbf{h}$ be a non-parabolic unit speed hybrid curve. Then $\mathfrak{t}(s) = \alpha_1'(s)\mathbf{i} + \alpha_2'(s)\boldsymbol{\varepsilon} + \alpha_3'(s)\mathbf{h}$ and $\bar{\mathfrak{t}} = -\alpha_1'(s)\mathbf{i} - \alpha_2'(s)\varepsilon - \alpha_3'(s)\mathbf{h}$. It can be verify that $\overline{\mathfrak{t}'} = \bar{\mathfrak{t}}'$. The condition $g(\mathfrak{t}(s), \mathfrak{t}(s)) = \epsilon_0$ implies that $\mathfrak{t} \times \bar{\mathfrak{t}} = \epsilon_0$ and

$$\mathfrak{t}' \times \bar{\mathfrak{t}} + \mathfrak{t} \times \bar{\mathfrak{t}}' = 0. \tag{4}$$

i. By using equation (4) and $g(\mathfrak{t}(s), \mathfrak{t}(s)) = \epsilon_0$, we find
$g(\mathfrak{t}, \mathfrak{t}') = \frac{1}{2}(\mathfrak{t} \times \overline{\mathfrak{t}'} + \mathfrak{t}' \times \bar{\mathfrak{t}}) = \frac{1}{2}(\mathfrak{t}' \times \bar{\mathfrak{t}} + \mathfrak{t} \times \bar{\mathfrak{t}}') = 0,$ which proves the condition i.

ii. By using equation (4) and the properties of conjugate, we find
$\mathfrak{t}' \times \bar{\mathfrak{t}} + \overline{\mathfrak{t}' \times \bar{\mathfrak{t}}} = \mathfrak{t}' \times \bar{\mathfrak{t}} + \bar{\bar{\mathfrak{t}}} \times \overline{\mathfrak{t}'} = \mathfrak{t}' \times \bar{\mathfrak{t}} + \mathfrak{t} \times \bar{\mathfrak{t}}' = 0,$ which proves ii.

Now, we can define the unit spatial hybrid n_1 and the non-negative scalar function κ_1 by the equations

$$\mathfrak{t}' = \epsilon_1 \kappa_1 n_1, g(n_1, n_1) = \epsilon_1, \epsilon_1 \in \{1, -1\} \tag{5}$$

and $\kappa_1 = \| \mathfrak{t}' \|$ since $\mathfrak{t}$ is a spatial hybrid. From the property i of the Theorem 3.2, we find that $g(n_1, \mathfrak{t}) = \dfrac{1}{\epsilon_1 \kappa_1} g(\mathfrak{t}', \mathfrak{t}) = 0$ which implies that n_1 is $g-$orthogonal to $\mathfrak{t}$. Differentiating the equation $g(\mathfrak{t}, n_1) = 0$, we find

$$g(t', n_1) + g(t, n_1') = 0. \tag{6}$$

From the equation (5), we get $g(t', n_1) = g(\epsilon_1\kappa_1 n_1, n_1) = \kappa_1$ and substituting this equation into equation (6), we obtain

$$g(n_1', t) = -\kappa_1. \tag{7}$$

Next, we put $n_2 = t \times n_1$. By the property ii of the Theorem 3.2, we find that n_2 is a spatial hybrid and

$$t \times n_2 = -\langle t, n_2\rangle - t \wedge n_2 = -t \wedge n_2 = -t \wedge (t \wedge n_1) = \langle t, n_1\rangle t - \langle t, t\rangle n_1 = -\epsilon_0 n_1,$$
$$n_1 \times n_2 = n_1 \wedge n_2 = -n_1 \wedge (n_1 \wedge t) = -(\langle n_1, t\rangle n_1 - \langle n_1, n_1\rangle t) = \epsilon_1 t.$$

We also find that $g(n_2, t) = 0$ and $g(n_2, n_1) = 0$. So t, n_1, n_2 are mutually orthogonal spatial hybrid and $\epsilon_2 = g(n_2, n_2) = g(t \times n_1, n_2) = g(t, n_1 \times n_2) = g(t, \epsilon_1 t) = \epsilon_0\epsilon_1$, where $\epsilon_2 \in \{-1, 1\}$. Thus, we can construct the following multiplication table for the orthonormal set of spatial hybrids $\{t, n_1, n_2,\}$.

Table 1. Products of $\{t, n_1, n_2,\}$

$\times$	t	n_1	n_2
t	$-\epsilon_0$	n_2	$-\epsilon_0 n_1$
n_1	$-n_2$	$-\epsilon_1$	$\epsilon_1 t$
n_2	$\epsilon_0 n_1$	$-\epsilon_1 t$	$-\epsilon_2$

By taking the g -inner product with t and n_1, we find that the vector

$$n_1' - \epsilon_0 g(n_{1'}, t)t - \epsilon_1 g(n_1', n_1)n_1$$

is perpendicular to t and n_1, so it is parallel to n_2. Thus for some scalar κ_2

$$\epsilon_2\kappa_2 n_2 = n_1' - \epsilon_0 g(n_1', t)t - \epsilon_1 g(n_1', n_1)n_1,$$

where $g(n_2, n_2) = \epsilon_2$. By using equation (7), we obtain

$$n_2 = \frac{\epsilon_2}{\kappa_2}(n_1' + \epsilon_0\kappa_1 t),$$

or

$$n_1' = -\epsilon_0\kappa_1 t + \epsilon_2\kappa_2 n_2 \tag{8}$$

and $\kappa_2 = \| n_{1'} + \epsilon_0\kappa_1 t \|$. Let differentiate to the both side of $n_2 = t \times n_1$, then we have

$$n_2' = \mathfrak{t}' \times n_1 + \mathfrak{t} \times n_1'. \;\; n_2' = \mathfrak{t}' \times n_1 + \mathfrak{t} \times n_1'.$$

By using the equations (5) and (8), we have

$$n_2' = \kappa_1 n_1 \times n_1 + \mathfrak{t} \times (-\epsilon_0 \epsilon_1 \kappa_1 \mathfrak{t} + \epsilon_2 \kappa_2 n_2) = -\epsilon_0 \epsilon_2 \kappa_2 n_1.$$

Now we can give the following theorem without proof.

Theorem 3.3 Let $\alpha : I \subset R \to \mathbb{H}^p$ be a non-parabolic spatial hybrid curve with $\kappa_1, \kappa_2 > 0$. The associated Frenet-Serret frame field $(\mathfrak{t}, n_1, n_2)$ along α satisfies

$$\begin{bmatrix} \mathfrak{t}' \\ n_1' \\ n_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_1 \kappa_1 & 0 \\ -\epsilon_0 \kappa_1 & 0 & \epsilon_2 \kappa_2 \\ 0 & -\epsilon_0 \epsilon_2 \kappa_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n_1 \\ n_2 \end{bmatrix}, \tag{9}$$

where $\epsilon_0 = g(\mathfrak{t}, \mathfrak{t}), \epsilon_1 = g(n_1, n_1)$ and $\epsilon_2 = g(n_2, n_2)$.

4. Frenet-Serret Formulae for a Hybrid Curve

Let $\beta(s) = \beta_0(s) + \beta_1(s)\mathbf{i} + \beta_2(s)\boldsymbol{\varepsilon} + \beta_3(s)\mathbf{h}$ be a curve in $\mathbb{H}$ over the interval I such that the tangent $T^\beta(s) = \beta'(s)$. If $g(\beta'(s), \beta'(s)) = 1$ is called spacelike, $g(\beta'(s), \beta'(s)) = -1$ timelike and $g(\beta'(s), \beta'(s)) = 0$ lightlike curve.

From now on, we will study the non-lightlike curve. So, we consider $T^\beta(s) = \beta'(s)$ with a unit magnitude $\| T^\beta \| = 1$.

Let us write

$$T^{\beta'} = \mu_1 K_1 N_1^\beta, K_1 = \| T^{\beta'} \|, \| N_1^\beta \| = 1, \tag{10}$$

where $\mu_1 = g(N_1^\beta, N_1^\beta)$. If we calculate the derivative of $\| T^\beta \| = 1$ with respect to s and by using the equation (10), we find $N_1^\beta \times \overline{T^\beta} + T^\beta \times \overline{N_1^\beta} = 0$. This gives that N_1^β is $g-$ orthogonal to T^β and $N_1^\beta \times \overline{T^\beta}$ is consider as a spatial hybrid denoted by $\mathfrak{t}$. Because of the fact that T^β and N_1^β are unit hybrids, $\mathfrak{t}$ is a unit hybrid, too. And by using some algebraic operation, we have $\mathfrak{t} \times T^\beta = (N_1^\beta \times \overline{T^\beta}) \times T = (N_1^\beta \times T^{\beta - 1}) \times T^\beta = \mu_0 N_1^\beta$, where $\mu_0 = g(T^\beta, T^\beta)$. So, we can also define N_1^β as

$$\mu_0 N_1^\beta = \mathfrak{t} \times T^\beta, \tag{11}$$

where $\mathfrak{t} = N_1^{\beta} \times \overline{T^{\beta}}$ along α. Note that $\mathfrak{t}$ is determined uniquely as a smooth unit spatial hybrid by the equation (11). By differentiating the equation (11) with respect to s and using the equations (9), (10), (11), we obtain the derivative of the vector N_1^{β} as

$$N_1^{\beta\prime} = -\mu_0\mu_1 K_1 \epsilon_0 T^{\beta} + \mu_0 \epsilon_1 \kappa_1 N_2^{\beta},$$

where $N_2^{\beta} = n_1 \times T^{\beta}$. Moreover, $N_2^{\beta}(s)$ is a smooth hybrid function of s and the norm of the vector N_2^{β} is $\| N_2^{\beta} \| = 1$. Also, T^{β}, N_1^{β} and N_2^{β} are $g-$orthogonal to each other. Because $\mathfrak{t}$ and n_1 are $g-$ orthogonal. Now, if we calculate the derivative of N_2^{β} with respect to s and by using the equations (9), (10) and (11), we obtain the derivative of the vector N_2^{β} as

$$N_2^{\beta\prime} = -\epsilon_0 \kappa_1 N_1^{\beta} + (\epsilon_2 \kappa_2 - \mu_1 K_1) N_3^{\beta},$$

where $N_3^{\beta} = n_2 \times T^{\beta}$. In addition to this, $N_3^{\beta}(s)$ is a smooth hybrid function of s with $\| N_3^{\beta} \| = 1$. Also, $T^{\beta}, N_1^{\beta}, N_2^{\beta}$ and N_3^{β} are $g-$ orthogonal to each other. Because $\mathfrak{t}, n_1$ and n_2 are $g-$ orthogonal. If we calculate the derivative of N_3^{β} with respect to s and by using the equations (9), (10) and (11), we find similarly the derivative of the vector N_3^{β} as

$$N_3^{\beta\prime} = \epsilon_0 (\mu_1 K_1 - \epsilon_2 \kappa_2) N_2^{\beta}.$$

Thus, we can give following theorem:

Theorem 4.4 Let $\beta : I \subset R \to \mathbb{H}$ be a non-lightlike hybrid curve with $\kappa_1, \kappa_2 > 0$ and $K_1 > 0.$ The associated non-lightlike Frenet-Serret frame field $(T^{\beta}, N_1^{\beta}, N_2^{\beta}, N_3^{\beta})$ along β satisfies

$$\begin{bmatrix} T^{\beta\prime} \\ N_{1'}^{\beta} \\ N_{2'}^{\beta} \\ N_{3'}^{\beta} \end{bmatrix} = \begin{bmatrix} 0 & \mu_1 K_1 & 0 & 0 \\ -\mu_0\mu_1\epsilon_0 K_1 & 0 & \mu_0\epsilon_1\kappa_1 & 0 \\ 0 & -\epsilon_0\kappa_1 & 0 & \epsilon_2\kappa_2 - \mu_1 K_1 \\ 0 & 0 & \epsilon_0(\mu_1 K_1 - \epsilon_2\kappa_2) & 0 \end{bmatrix} \begin{bmatrix} T^{\beta} \\ N_1^{\beta} \\ N_2^{\beta} \\ N_3^{\beta} \end{bmatrix}.$$

5. Application

Let's given the curve

$$\alpha(s) = \frac{\sqrt{2}}{2}(cosh(s) - sinh(s))\mathbf{i} - \frac{\sqrt{2}}{2} sinh(s)\boldsymbol{\varepsilon} - \frac{\sqrt{2}}{2} s\mathbf{h}$$

and

$$\beta(s) = \frac{\sqrt{2}}{2}\sinh(s) + s\mathbf{i} + 0\boldsymbol{\varepsilon} + \frac{\sqrt{2}}{2}\cosh s\mathbf{h}.$$

By using the following Matlab codes,

```
clc;
clear all;
syms s
alpha=[sqrt(2)/2*(-sinh(s)+cosh(s)),-sqrt(2)/2*sinh(s), -sqrt(2)/2*s]
%%%%%%%%%%t%%%%%%%%%%
V0=diff(alpha,s)
norma=sqrt(abs(innerr(V0,V0)))
e0=subs(innerr(V0,V0),s,0)
t=V0
t0=subs(V0,s,0)
%%%%%n1%%%
tu=diff(t,s)
k1=sqrt(abs(innerr(tu,tu)))
k10=subs(k1,s,0)
V11=1/k1*tu
e1=subs(innerr(V11,V11),s,0)
n1=e1*V11
n10=subs(n1,s,0)
%%%%%%n2%%%%%%%%%
n2=transpose(vectorproduct(t,n1))
n20=subs(n2,s,0)
e2=innerr(n20,n20)

```

```
%%%%%%%%%%%tu%%%%
ttu=subs(diff(t,s),s,0) %We check the correctness!!
ttuc=e1*k10*n10
%%%%%%%%%%n1u
n1u=vpa(subs(diff(n1,s),s,0),6)
k2d=vpa(n1u+e0*k10*t0,6)
k2=vpa(sqrt(abs(innerr(k2d,k2d))),6)
n1uc=vpa(-e0*k10*t0+e2*k2*n20,6)


%%%%%%%%%%n2u
n2u=vpa(subs(diff(n2,s),s,0),6)
n2uc=vpa(-e0*e2*k2*n10,6)

%%%%%%%%%%%Example 2 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%

beta=1/sqrt(2)*[sinh(s),s,0, cosh(s)]
%%%%%%%%%%t%%%%%%%%%%
V0=diff(beta,s)
mu0=subs(innerh(V0,V0),s,0)
T=V0;
T0=subs(V0,s,0)
%%%%%N1%%%
Tu=diff(T,s);
K1=sqrt(abs(innerh(Tu,Tu)));
K10=vpa(subs(K1,s,0),6)
```

```
N11=vpa(1/K1*Tu,6);
mu1=vpa(subs(innerh(N11,N11),s,0),6)
N1=mu1*N11;
N10=vpa(subs(N1,s,0),6)

%%%%%%t=N1*CONJ(T)%%%%%%%%%
tt=vpa(hproduct(N1,coo(T)),6);
tt0=vpa(subs(tt,s,0),6)

N10CHECKK=vpa(mu0*hproduct(tt0,T0),6);
N10;
n1=[0, n1(1),n1(2), n1(3)];
n2=[0, n2(1),n2(2), n2(3)];
N2=vpa(hproduct(n1,T),6);
N20=vpa(subs(N2,s,0),6)
N3=vpa(hproduct(n2,T),6);
N30=vpa(subs(N3,s,0),6)
%%%% Curvatures and Derivative Formulae
%%%%%%%%N1
N1u=diff(N1,s);
N1d=-mu0*mu1*K1*e0*T+mu0*e1*k1*N2;
N1u0=vpa(subs(N1u,s,0),6);
N1d0=vpa(subs(N1d,s,0),6);
N1u0decimal=arrayfun(@(N1u0) sprintf('%.6f',N1u0), N1u0, 'uniform', 0)
N1d0decimal=arrayfun(@(N1d0) sprintf('%.6f',N1d0), N1d0, 'uniform', 0)
%%%%%%%%N2
N2u=diff(N2,s);
```

```
N2d=-e0*k1*N1+(e2*k2-mu1*K1)*N3;
N2u0=vpa(subs(N2u,s,0),6);
N2d0=vpa(subs(N2d,s,0),6);
N2u0decimal=arrayfun(@(N2u0) sprintf('%.6f',N2u0), N2u0, 'uniform', 0)
N2d0decimal=arrayfun(@(N2d0) sprintf('%.6f',N2d0), N2d0, 'uniform', 0)

%%%%%%%%N3
N3u=diff(N3,s);
N3d=e0*(mu1*K1-e2*k2)*N2;
N3u0=vpa(subs(N3u,s,0),6);
N3d0=vpa(subs(N3d,s,0),6);
N3u0decimal=arrayfun(@(N3u0) sprintf('%.6f',N3u0), N3u0, 'uniform', 0)
N3d0decimal=arrayfun(@(N3d0) sprintf('%.6f',N3d0), N3d0, 'uniform', 0)
```

we have the Frenet-Serret frame fields along α and β at $s=0$. Finally, all results for the curve α are listed from the output of the program above as follow:

$$
\begin{aligned}
\mathfrak{t}(0) &= -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\boldsymbol{\varepsilon} - \frac{\sqrt{2}}{2}\mathbf{h}, \\
\kappa_1(0) &= \frac{\sqrt{2}}{2} \equiv 0.707107, \\
n_1(0) &= 1\mathbf{i} + 0\boldsymbol{\varepsilon} + 0\mathbf{h}, \\
n_2(0) &= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\boldsymbol{\varepsilon} - \frac{\sqrt{2}}{2}\mathbf{h}, \\
\kappa_2(0) &= \frac{\sqrt{2}}{2} \equiv 0.707107,
\end{aligned}
$$

and

$$
\begin{bmatrix} \mathfrak{t}' \\ n_1' \\ n_2' \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{t} \\ n_1 \\ n_2 \end{bmatrix},
$$

where $\epsilon_0 = -1, \epsilon_1 = 1$ and $\epsilon_2 = -1.$ Similarly, if running the program continue, we have following results:

$$
\begin{aligned}
T^{\beta}(0) &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathbf{i} + 0\boldsymbol{\varepsilon} + 0\mathbf{h}, \\
K_1(0) &= 0.707107, \\
N_1^{\beta}(0) &= 0 + 0\mathbf{i} + 0\boldsymbol{\varepsilon} - \mathbf{h}, \\
N_2^{\beta}(0) &= -0.707107 + 0.707107\mathbf{i} + 0\boldsymbol{\varepsilon} + 0\mathbf{h}, \\
N_3^{\beta}(0) &= 0 + 1\mathbf{i} + 1\boldsymbol{\varepsilon} + 0\mathbf{h}
\end{aligned}
$$

and

$$
\begin{bmatrix} T^{\beta\prime} \\ N_1^{\beta\prime} \\ N_2^{\beta\prime} \\ N_3^{\beta\prime} \end{bmatrix} = \begin{bmatrix} 0 & -0.707107 & 0 & 0 \\ -0.707107 & 0 & 0.707107 & 0 \\ 0 & 0.707107 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T^{\beta} \\ N_1^{\beta} \\ N_2^{\beta} \\ N_3^{\beta} \end{bmatrix},
$$

where $\mu_0 = 1$ and $\mu_1 = -1.$ Note that we use the properties of the first curve α when calculating the properties of the second curve $\beta.$ In addition, the above Matlab codes also check the correctness of the derivative formulas for these examples.

4. Conclusion

Spatial hybrid-valued functions of a single real variable determine a curve in a special analog Minkowski Geometry. The Frenet-Serret frame formulae for non-parabolic curve, is derived with the help of the vector product which we define in the special Minkowski space and the spatial hybrids. Also, we presents Frenet-Serret frame for the non-lightlike hybrid curve with the help of the Frenet formula for non-parabolic spatial hybrid curves.

As a result of this article, the beginning of the curve theory for hybrid numbers are constructed. Thus, the special curves with hybrid numbers can be studied by the light of this article in the future.

References

[1] Bharathi, K., Nagaraj, M., "Quaternion valued function of a real variable Serret–Frenet formulae", Indian J. Pure Appl. Math. 16 (1985) : 741–756.

[2] Coken, A.C., Tuna, A., "On the quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}_2^4$ ", Appl. Math. Comput. 155 (2004) : 373-389.

[3] Dağdeviren, A., Yüce, S., "Dual quaternions and dual quaternionic curves", Filomat 33(4) (2019) : 1037–1046.

[4] Ohashi, M., "G 2-Congruence theorem for curves in purely imaginary octonions and its application", Geom Dedicata 163 (2013) : 1–17.

[5] Özdemir, M., "Introduction to Hybrid Numbers", Adv. Appl. Clifford Algebras 28(11) (2018).

[6] Özdemir, M., "Finding n^{th} Roots of a 2×2 real matrix using De Moivre's formula", Advances in Applied Clifford Algebras 29(2) (2019).

[7] Öztürk, İ., Özdemir, M., "Similarity of hybrid numbers", Mathematical Methods in Applied Sciences 43(15) (2020): 8867-8881.

[8] Akbıyık, M., S. Yamaç Akbıyık, E. Karaca, F. Yılmaz, "De Moivre's and Euler Formulas for matrices of hybrid numbers", Axioms 2021, 10, 213.

[9] Szynal-Liana, A., "The Horadam Hybrid Numbers", Discussiones Mathematicae General Algebra and Applications 38 (2018) : 91–98.

[10] Kızılateş, C., "A new generalization of Fibonacci hybrid and Lucas hybrid numbers", Chaos, Solitons & Fractals 130 (2020) : 109449.