Celal Bayar University Journal of Science

On 7-Dimensional Nilpotent Leibniz Algebras with 1-Dimensional Leib Ideal

İsmail Demir[1]*

[1] Usak University, Faculty of Engineering and Natural Sciences Department of Mathematics, Uşak, Türkiye
*ismail.demir@usak.edu.tr
*Orcid: 0000-0002-8070-6489

Received: 8 August 2023
Accepted: 18 December 2023
DOI: 10.18466/cbayarfbe.1339702

Abstract

Leibniz algebras are nonanticommutative versions of Lie algebras. Lie algebras have many applications in many scientific areas as well as mathematical areas. Scientists from different disciplines have used specific examples of Lie algebras according to their needs. However, we mathematicians are more interested in generality than in obtaining a few examples. The classification problem for Leibniz algebras has an intrinsically wild nature as in Lie algebras. In this article, the approach of congruence classes of bilinear forms is extended to classify certain subclasses of seven-dimensional nilpotent Leibniz algebras over complex numbers. Certain cases of seven-dimensional complex nilpotent Leibniz algebras of those with one-dimensional Leib ideal and derived algebra of codimension two are classified.

Keywords: Bilinear forms, Classification, Leibniz algebra, Nilpotency.

1. Introduction

Although first considered by Bloh in 1965 [1], Leibniz algebras as nonantisymmetric (nonanticommutative) generalization of Lie algebras were presented by Loday [2]. A vector space L over $\mathbb{C}$ with a bilinear product $[\ ,\ ]: L \times L \rightarrow L$ satisfying the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in L$ is said to be a Leibniz algebra. The lower central series of a Leibniz algebra L can be defined as $L \supseteq L^2 \supseteq L^3 \supseteq \cdots$ where $L^1 = L$ and $L^k = [L, L^{k-1}]$ for integers $k \geq 2$. If $L^{c+1} = 0$ whenever $L^c \neq 0$ for some $c > 0$, then L is a nilpotent Leibniz algebra of class c. L is called odd-nilpotent if its all nontrivial ideals of the lower central series are odd-dimensional. Leibniz algebra L of dimension n is called filiform Leibniz algebra if $dim(L^j) = n - j$ for $2 \leq j \leq n$. $Leib(L)$ generated by the squares, $[a, a]$, for all $a \in L$ is an ideal of L, is of the utmost importance while studying structure theory of Leibniz algebras. The center of a Leibniz algebra L can be defined by $Z(L) = \{b \in L \mid [a,b] = 0 = [b,a] \ for\ all\ a \in L\}$. Non-split Leibniz algebras are those that cannot be expressed as the direct sum of nontrivial ideals. Throughout this paper, we assume Leibniz algebras are non-split non-Lie vector spaces over $\mathbb{C}$.

It is an important but nontrivial task to classify any kind of nonassociative algebras. Because the classification of nilpotent Lie algebras is regarded as wild, the classification of nilpotent Leibniz algebras is also wild. In fact, the problem is more complicated for Leibniz algebras due to a lack of anticommutativity. Many researchers have provided numerous results on the classification of nilpotent Leibniz algebras over $\mathbb{C}$ until now (see [2-14]); however, the problem has still not been completed. Seven-dimensional odd-nilpotent Leibniz algebras have been classified in [14] with the congruence classes of the bilinear forms approach. The main aim of this paper is to apply the same technique to give the classification of some subcases seven dimensional nilpotent Leibniz algebras with one-dimensional Leib ideal. The isomorphism test between the classes can be done by using Algorithm 2.6 proposed in [5].

2. Preliminaries

We include the following useful Lemmas from [12].

Lemma 2.1. $L^c \subseteq Z(L)$ if L is a class c nilpotent Leibniz algebra.

Lemma 2.2. Let L be a non-split Leibniz algebra. Then, $Z(L) \subseteq L^2$.

Lemma 2.3. Any nilpotent Leibniz algebra L satisfies $Leib(L) \subseteq Z(L)$.

Lemma 2.4. For any n-dimensional nilpotent Leibniz algebra L; $dim(Z(L)) = n - i$ and $dim(Leib(L)) = 1$ imply $dim(L^2) \leq \frac{i^2-i+2}{2}$.

Lemma 2.5. For any n-dimensional nilpotent Leibniz algebra L; $dim(L^2) = n - i, dim(Leib(L)) = 1$, and $dim(L^3) = j$ imply the inequality $n \leq j + \frac{i^2+i+2}{2}$. Furthermore, if $Leib(L) \subseteq L^3$, then $n \leq j + \frac{i^2+i}{2}$.

Lemma 2.6. For any n-dimensional nilpotent Leibniz algebra L; $dim(L^2) = n - i$ and $L^4 \neq 0$ imply $dim(Z(L)) < n - i - 1$.

The following matrices are the canonical forms for the congruence classes of matrices associated with a bilinear form on a complex vector space. Denoting

$$X \backslash Y := \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}$$

Theorem 2.1. [15] Any complex square matrix is congruent to a direct sum of the following canonical forms of matrices:

$$A_{2n+1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \backslash \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{2n}(\alpha) = \begin{bmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 1 \\ 0 & ⋰ & ⋰ & 0 \\ \alpha & 1 & 0 & 0 \end{bmatrix} \backslash \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \alpha \\ 0 & ⋰ & ⋰ & 0 \\ 1 & \alpha & 0 & 0 \end{bmatrix}, \alpha \neq \pm 1$$

$$C_{2n+1} = \begin{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 0 & ⋰ & ⋰ & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & ⋰ & ⋰ & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D_{2n} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & ⋰ & ⋰ & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \backslash \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & ⋰ & ⋰ & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, (n\ even)$$

$$E_{2n} = \begin{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 0 & ⋰ & ⋰ & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & ⋰ & ⋰ & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F_{2n} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & ⋰ & ⋰ & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \backslash \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & ⋰ & ⋰ & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, (n\ odd)$$

3. Classification

Let L be 7-dimensional nilpotent Leibniz algebra with 1-dimensional Leib ideal. Some subclasses of 7-dimensional odd-nilpotent Leibniz algebras have been classified in [14]. For the sake of simplicity, we will consider Leibniz algebras with the derived algebra of codimension two, because employing the congruence classes of bilinear forms technique is easier in that situation.

Choose $dim(L^2) = n - 2$ and $Leib(L) = span\{v_n\}$. Extending it to a basis $\{v_3, v_4, \dots, v_{n-1}, v_n\}$ for L^2 and take a subspace V in L so that $L = L^2 \oplus V$. Therefore, $[u, v] = \beta_3 v_3 + \beta_4 v_4 + \beta_{n-1} v_{n-1} + \beta_n v_n$ for $3 \leq k \leq n, \beta_k \in \mathbb{C}$, for each $u, v \in V$. The bilinear form $f(\ ,\ ): V \times V \to \mathbb{C}$ provided by $f(u, v) = \beta_n$ for each $u, v \in V$. Let $\{v_1, v_2\}$ be a basis for V, and using Theorem 2.1, we can easily determine that the possible matrices of the bilinear form above are the following:

$$F_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_1 \oplus C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$C_1 \oplus C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$$

where $\alpha \neq \pm 1$. We only consider non-Lie Leibniz algebras so that we can eliminate the matrix F_2.In addition, it is sufficient to focus only the matrices $A_1 \oplus C_1$ and $C_1 \oplus C_1$ since the other two matrices yield algebras that are always isomorphic to algebras obtained by these two matrices, as proved in Lemma 2.1 in [10].

Denote the following invariant
$\chi(L) = (dim(L), dim(L^2), dim(L^3), \dots, dim(L^c))$
where c is the class of nilpotency. Then, take a 7-dimensional nilpotent Leibniz algebra L with $dim(Leib(L)) = 1$ where $dim(L^2) = 5$. Notice that there is no Leibniz algebra for the cases $dim(L^3) = 0, 1, 2,$ as Lemma 2.5 suggests. Hence, we have $dim(L^3) = 3$. Odd-nilpotent subclasses of this case are already classified in [14]. The remaining cases are listed below:

i. $\chi(L) = (7, 5, 3, 2)$
ii. $\chi(L) = (7, 5, 3, 2, 1)$

Theorem 3.1. There does not exist any Leibniz algebra with Leib ideal of dimension one in the case $\chi(L) = (7, 5, 3, 2)$.

Proof. Take a nilpotent Leibniz algebra L with $\chi(L) = (7, 5, 3, 2)$ and $dim(Leib(L)) = 1$. We see that $Leib(L) \subseteq Z(L)$ by using Lemma 2.3. Besides, from Lemma 2.5, we obtain $Leib(L) \nsubseteq L^3$. Lemma 2.2 implies $L^4 \subseteq Z(L) \subset L^2$. Then, by using Lemma 2.6, we deduce $1 < dim(Z(L)) < 4$. But $dim(Z(L))$ cannot be 2, because otherwise $L^4 = Z(L)$ implies that $Leib(L) \subseteq L^4 \subset L^3$ which contradicts with $Leib(L) \nsubseteq L^3$. Hence, suppose $dim(Z(L)) = 3$. Taking a complementary subspace W to L^3 in L^2. Since $L^4 \neq 0$, we have $L^3 \neq Z(L)$. Moreover, from $L^4 \subseteq Z(L)$, we can see that the only possibility is $dim(L^3 \cap Z(L)) = 2$. Using $Leib(L) \nsubseteq L^3, L^4$, choose $Leib(L) = span\{w_7\}, L^4 = span\{w_5, w_6\}$ and, $L^3 = span\{w_4, w_5, w_6\}$. Then, $Z(L) = span\{w_5, w_6, w_7\}$ and $L^2 = span\{w_3, w_4, w_5, w_6, w_7\}$. Later, take $A = span\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$. Then, the nonzero products in L are given as follows:

$[w_1, w_1] = \alpha_1 w_7, [w_2, w_2] = \alpha_2 w_7,\ [w_1, w_2] = \alpha_3 w_3 + \alpha_4 w_4 + \alpha_5 w_5 + \alpha_6 w_6 + \alpha_7 w_7, [w_2, w_1] = -\alpha_3 w_3 - \alpha_4 w_4 - \alpha_5 w_5 - \alpha_6 w_6 + \alpha_8 w_7, [w_1, w_3] = \beta_1 w_4 + \beta_2 w_5 + \beta_3 w_6 = -[w_3, w_1], [w_2, w_3] = \beta_4 w_4 + \beta_5 w_5 + \beta_6 w_6 = -[w_3, w_2], [w_1, w_4] = \gamma_1 w_5 + \gamma_2 w_6 = -[w_4, w_1], [w_2, w_4] = \gamma_3 w_5 + \gamma_4 w_6 = -[w_4, w_2], [w_3, w_4] = \gamma_5 w_5 + \gamma_6 w_6 = -[w_4, w_3]$.

We obtain the following equations using Leibniz identity:

$$\gamma_5 = 0 = \gamma_6$$
$$\beta_4\gamma_1 - \beta_1\gamma_3 = 0 \qquad \textbf{(3.1)}$$
$$\beta_4\gamma_2 - \beta_1\gamma_4 = 0 \qquad \textbf{(3.2)}$$

Assume $\gamma_3 = 0$. Then, $\gamma_1 \neq 0$ and from Equation 3.1, we have $\beta_4 = 0$. But $dim(L^3) = 3$ with Equation 3.2 implies $\gamma_4 = 0$ which contradicts with the fact that $dim(L^4) = 2$. Suppose $\gamma_3 \neq 0$. Later, with the change-of-basis $x_1 = \gamma_3 w_1 - \gamma_1 w_2, x_2 = w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = w_7$, we can force $\gamma_1 = 0$. Additionally, from Equation 3.1, we get $\beta_1 = 0$. But $dim(L^3) = 3$ with Equation 3.2 implies $\gamma_2 = 0$ which contradicts with the fact that $dim(L^4) = 2$. Therefore, there is no Leibniz algebra in the case $\chi(L) = (7, 5, 3, 2)$ and $dim(Leib(L)) = 1$. The proof is completed. ■

Suppose $\chi(L) = (7, 5, 3, 2, 1)$ and $dim(Leib(L)) = 1$. We have $Leib(L) \subseteq Z(L)$ due to Lemma 2.3. Besides, from Lemma 2.5, we obtain $Leib(L) \nsubseteq L^3$. Lemma 2.2 implies $L^5 \subseteq Z(L) \subset L^2$. Then, by using Lemma 2.6, we deduce $1 \leq dim(Z(L)) < 4$. If $dim(Z(L)) = 1$, then $Leib(L) = Z(L) = L^5 \subset L^3$, we arrive a contradiction. Hence, $dim(Z(L)) = 2$ or $dim(Z(L)) = 3$. We will first consider the case $dim(Z(L)) = 3$.

Theorem 3.2. Let $\chi(L) = (7, 5, 3, 2, 1)$, $dim(Leib(L)) = 1$ and $dim(Z(L)) = 3$. Then, L is isomorphic to one of the following algebras with nontrivial multiplications ($i^2 = -1$):

L1 $[\zeta_1, \zeta_1] = \zeta_7, [\zeta_1, \zeta_2] = \zeta_3 = -[\zeta_2, \zeta_1], [\zeta_1, \zeta_3] = \zeta_5 = -[\zeta_3, \zeta_1],\ [\zeta_1, \zeta_5] = \zeta_6 = -[\zeta_5, \zeta_1], [\zeta_2, \zeta_3] = \zeta_4 = -[\zeta_3, \zeta_2]$

L2 $[\zeta_1, \zeta_1] = \zeta_7, [\zeta_1, \zeta_2] = \zeta_3 = -[\zeta_2, \zeta_1], [\zeta_1, \zeta_3] = \zeta_4 = -[\zeta_3, \zeta_1],\ [\zeta_2, \zeta_3] = \zeta_5 = -[\zeta_3, \zeta_2], [\zeta_2, \zeta_5] = \zeta_6 = -[\zeta_5, \zeta_2]$

L3 $[\zeta_1, \zeta_1] = \zeta_7, [\zeta_1, \zeta_2] = \zeta_3 = -[\zeta_2, \zeta_1], [\zeta_1, \zeta_3] = \zeta_5 = -[\zeta_3, \zeta_1],\ [\zeta_1, \zeta_5] = \zeta_6 = -[\zeta_5, \zeta_1], [\zeta_2, \zeta_2] = \zeta_7, [\zeta_2, \zeta_3] = \zeta_4 = -[\zeta_3, \zeta_2]$

L4 $[\zeta_1, \zeta_1] = \zeta_7, [\zeta_1, \zeta_2] = \zeta_3 = -[\zeta_2, \zeta_1], [\zeta_1, \zeta_3] = \zeta_5 = -[\zeta_3, \zeta_1],\ [\zeta_1, \zeta_5] = i\zeta_6 = -[\zeta_5, \zeta_1], [\zeta_2, \zeta_2] = \zeta_7, [\zeta_2, \zeta_3] = \zeta_4 = -[\zeta_3, \zeta_2], [\zeta_2, \zeta_5] = \zeta_6 = -[\zeta_5, \zeta_2]$

Proof. Take a complementary subspace W to L^3 in L^2. Since $L^4 \neq 0$, we have $L^3 \neq Z(L)$. We have $L^3 \cap Z(L) \neq \emptyset$, since $L^5 \subseteq Z(L)$. Furthermore, $dim(L^3 \cap Z(L)) = 1$ implies $W \subseteq Z(L)$ and since

$$L^3 = [L, L^2] = [L, L^3 \oplus W] = L^4$$

we arrive at a contradiction. Therefore, $dim(L^3 \cap Z(L)) = 2$. Using $Leib(L) \nsubseteq L^3, L^4, L^5$, choose $Leib(L) = span\{w_7\}, L^5 = span\{w_6\}, L^4 = span\{w_5, w_6\}$, and $L^3 = span\{w_4, w_5, w_6\}$. Then, $Z(L) = span\{w_4, w_6, w_7\}$ and $L^2 = span\{w_3, w_4, w_5, w_6, w_7\}$. Later, take $V = span\{w_1, w_2\}$.

Case 1. If the bilinear form matrix is $A_1 \oplus C_1$, then the nonzero products in L can be given as:

$[w_1, w_1] = w_7, [w_1, w_2] = \alpha_1 w_3 + \alpha_2 w_4 + \alpha_3 w_5 + \alpha_4 w_6 = -[w_2, w_1], [w_1, w_3] = \beta_1 w_4 + \beta_2 w_5 + \beta_3 w_6 = -[w_3, w_1], [w_2, w_3] = \beta_4 w_4 + \beta_5 w_5 + \beta_6 w_6 = -[w_3, w_2], [w_1, w_5] = \gamma_1 w_6 = -[w_5, w_1], [w_2, w_5] = \gamma_2 w_6 = -[w_5, w_2], [w_3, w_5] = \gamma_3 w_6 = -[w_5, w_3]$.

From Leibniz identity, we get the following equations:

$$\gamma_3 = 0$$
$$\beta_5\gamma_1 - \beta_2\gamma_2 = 0 \qquad \textbf{(3.3)}$$

First, suppose $\gamma_2 = 0$. Then, $\gamma_1 \neq 0$ and from Equation 3.3, we have $\beta_5 = 0$. Using $dim(L^3) = 3$, we can see that $\beta_2, \beta_4 \neq 0$. Then, the change of basis $\zeta_1 = w_1, \zeta_2 = w_2, \zeta_3 = \alpha_1 w_3 + \alpha_2 w_4 + \alpha_3 w_5 + \alpha_4 w_6, \zeta_4 = \alpha_1(\beta_4 w_4 + \beta_6 w_6), \zeta_5 = \alpha_1(\beta_1 w_4 + \beta_2 w_5 + \beta_3 w_6) + \alpha_3\gamma_1 w_6, \zeta_6 = \alpha_1\beta_2\gamma_1 w_6, \zeta_7 = w_7$ shows L is isomorphic to $L1$. Next, suppose $\gamma_2 \neq 0$. Applying the change of basis $x_1 = \gamma_2 w_1 - \gamma_1 w_2, x_2 = w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = \gamma_2^2 w_7$, we can force $\gamma_1 = 0$. Then, from Equation 3.3, we get $\beta_2 = 0$. Therefore, $\beta_1, \beta_5 \neq 0$ since $dim(L^3) = 3$. The change of basis $\zeta_1 = w_1, \zeta_2 = w_2, \zeta_3 = \alpha_1 w_3 + \alpha_2 w_4 + \alpha_3 w_5 + \alpha_4 w_6, \zeta_4 = \alpha_1(\beta_1 w_4 + \beta_3 w_6), \zeta_5 = \alpha_1(\beta_4 w_4 + \beta_5 w_5 + \beta_6 w_6) + \alpha_3\gamma_2 w_6, \zeta_6 = \alpha_1\beta_5\gamma_2 w_6, \zeta_7 = w_7$ shows L is isomorphic to $L2$.

Case 2. If the bilinear form matrix is $C_1 \oplus C_1$, then the nonzero products in L can be given as:

Celal Bayar University Journal of Science
Volume 19, Issue 4, 2023, p 343-349
Doi: 10.18466/cbayarfbe.1339702

$[w_1,w_1]=w_7,[w_1,w_2]=\alpha_1w_3+\alpha_2w_4+\alpha_3w_5+\alpha_4w_6=-[w_2,w_1],\ [w_2,w_2]=w_7,[w_1,w_3]=\beta_1w_4+\beta_2w_5+\beta_3w_6=-[w_3,w_1],[w_2,w_3]=\beta_4w_4+\beta_5w_5+\beta_6w_6=-[w_3,w_2],[w_1,w_5]=\gamma_1w_6=-[w_5,w_1],[w_2,w_5]=\gamma_2w_6=-[w_5,w_2],[w_3,w_5]=\gamma_3w_6=-[w_5,w_3]$.

Again, Leibniz identity yields same equations as in Case 1. Let $\gamma_2=0$. Then, $\gamma_1\neq 0$ since $dim(Z(L))=3$. From Equation 3.3, we have $\beta_5=0$. Using $dim(L^3)=3$, we obtain $\beta_2,\beta_4\neq 0$. Then, the change of basis $\zeta_1=w_1,\zeta_2=w_2,\zeta_3=\alpha_1w_3+\alpha_2w_4+\alpha_3w_5+\alpha_4w_6,\zeta_4=\alpha_1(\beta_4w_4+\beta_6w_6),\zeta_5=\alpha_1(\beta_1w_4+\beta_2w_5+\beta_3w_6)+\alpha_3\gamma_1w_6,\zeta_6=\alpha_1\beta_2\gamma_1w_6,\zeta_7=w_7$ shows L is isomorphic to $L3$. Further, take $\gamma_2\neq 0$. If $\gamma_1=0$, then $x_1=w_2,x_2=w_1,x_3=w_3,x_4=w_4,x_5=w_5,x_6=w_6,x_7=w_7$ is the change of basis forces $\gamma_2=0$. Therefore, L is isomorphic to $L3$. Let $\gamma_1\neq 0$. Assume that $\gamma_1^2+\gamma_2^2\neq 0$. Then $x_1=\gamma_1w_1+\gamma_2w_2,x_2=\gamma_2\,w_1-\gamma_1w_2,x_3=w_3,x_4=w_4,x_5=w_5,x_6=w_6,x_7=(\gamma_1^2+\gamma_2^2)w_7$ is the change of basis that forces $\gamma_2=0$ and consequently L is isomorphic to $L3$. Take $\gamma_1^2+\gamma_2^2=0$. Then, from Equation 3.3, we obtain $\beta_2^2+\beta_5^2=0$. Notice that $\beta_1^2+\beta_4^2\neq 0$ due to $dim(L^3)=3$. When $\beta_1^2+\beta_4^2\neq 0$, applying the change of basis $x_1=\beta_4w_1-\beta_1w_2,x_2=\beta_1w_1+\beta_4w_2,x_3=w_3,x_4=w_4,x_5=w_5,x_6=w_6,x_7=(\beta_1^2+\beta_4^2)w_7$ forces $\beta_1=0$. Thus, without loss of generality, we can take $\beta_1=0$. Finally, the change of basis $\zeta_1=w_1,\zeta_2=w_2,\zeta_3=\alpha_1w_3+\alpha_2w_4+\alpha_3w_5+\alpha_4w_6,\zeta_4=\alpha_1(\beta_4w_4+\beta_5w_5+\beta_6w_6)+\alpha_3\gamma_2w_6,\zeta_5=\alpha_1\beta_2\,w_5+(\alpha_1\beta_3+\alpha_3\gamma_1)w_6,\zeta_6=\alpha_1\beta_2\gamma_2w_6,\zeta_7=w_7$ shows L is isomorphic to $L4$. ∎

We obtain 4 single algebras. Similarly, the classification of the case $\chi(L)=(7,5,3,2,1), dim\big(Leib(L)\big)=1$ and $dim(Z(L))=2$ can be obtained by applying the aforementioned technique above.

Theorem 3.3. Let $\chi(L)=(7,5,3,2,1)$, $dim(Leib(L))=1$ and $dim(Z(L))=2$. Then, L is isomorphic to one of the following algebras with nontrivial multiplications (here $i^2=-1$):

L1 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1]$

L2 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],\ [\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_3]=\zeta_6=-[\zeta_3,\zeta_2]$

L3 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],\ [\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],\ [\zeta_2,\zeta_3]=\zeta_5=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=\zeta_6=-[\zeta_4,\zeta_2]$

L4 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],\ [\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_3]=\zeta_5+\zeta_6=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=\zeta_6=-[\zeta_4,\zeta_2]$

L5 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],\ [\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_2,\zeta_3]=\zeta_5=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_2],[\zeta_3,\zeta_4]=-\zeta_6=-[\zeta_4,\zeta_3]$

L6 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],\ [\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_2,\zeta_3]=\zeta_5+\zeta_6=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_2],[\zeta_3,\zeta_4]=-\zeta_6=-[\zeta_4,\zeta_3]$

L7 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_5+\zeta_6=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_6=-[\zeta_4,\zeta_1],[\zeta_2,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_2],[\zeta_2,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_2]$

L8 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_5+\zeta_6=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_2],[\zeta_3,\zeta_4]=\zeta_6=-[\zeta_4,\zeta_3]$

L9 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_5+\alpha\zeta_6=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_2],[\zeta_2,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_2],[\zeta_3,\zeta_4]=\zeta_6=-[\zeta_4,\zeta_3]$

L10 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_2]=\zeta_7$

L11 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_2]=\zeta_7,[\zeta_2,\zeta_3]=\zeta_6=-[\zeta_3,\zeta_2]$

L12 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_2]=\zeta_7,[\zeta_2,\zeta_3]=\zeta_5+\alpha\zeta_6=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=\zeta_6=-[\zeta_4,\zeta_2]$

L13 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\beta\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_2]=\zeta_7,[\zeta_2,\zeta_3]=\zeta_5+\alpha\zeta_6=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_2],[\zeta_3,\zeta_4]=-\zeta_6=-[\zeta_4,\zeta_3]$

L14 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_2]=\zeta_7,[\zeta_2,\zeta_3]=i\zeta_4+\alpha\zeta_5+\beta\zeta_6=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=i\zeta_5+\gamma\zeta_6=-[\zeta_4,\zeta_2],[\zeta_2,\zeta_5]=i\zeta_6=-[\zeta_5,\zeta_2]$

L15 $[\zeta_1,\zeta_1]=\zeta_7,[\zeta_1,\zeta_2]=\zeta_3=-[\zeta_2,\zeta_1],[\zeta_1,\zeta_3]=\zeta_4=-[\zeta_3,\zeta_1],[\zeta_1,\zeta_4]=\zeta_5=-[\zeta_4,\zeta_1],[\zeta_1,\zeta_5]=\zeta_6=-[\zeta_5,\zeta_1],[\zeta_2,\zeta_2]=\zeta_7,[\zeta_2,\zeta_3]=i\zeta_4+\alpha\zeta_5+\beta\zeta_6=-[\zeta_3,\zeta_2],[\zeta_2,\zeta_4]=i\zeta_5+\gamma\zeta_6=-[\zeta_4,\zeta_2],[\zeta_2,\zeta_5]=\theta\zeta_6=-[\zeta_5,\zeta_2],[\zeta_3,\zeta_4]=\delta\zeta_6=-[\zeta_4,\zeta_3]$

Celal Bayar University Journal of Science
Volume 19, Issue 4, 2023, p 343-349
Doi: 10.18466/cbayarfbe.1339702

Proof. By Lemma 2.1 and Lemma 2.3, we see that $Leib(L), L^5 \subseteq Z(L)$. Then by using $Leib(L) \nsubseteq L^3, L^4, L^5$, choose $Leib(L) = span\{w_7\}, L^5 = span\{w_6\}, L^4 = span\{w_5, w_6\}$ and $L^3 = span\{w_4, w_5, w_6\}$. Therefore, $Z(L) = span\{w_6, w_7\}$ and $L^2 = span\{w_3, w_4, w_5, w_6, w_7\}$. Take $V = span\{w_1, w_2\}$.

Case 1. If the bilinear form matrix is $A_1 \oplus C_1$, then the nonzero products in L can be given as:

$[w_1, w_1] = w_7, [w_1, w_2] = \alpha_1 w_3 + \alpha_2 w_4 + \alpha_3 w_5 + \alpha_4 w_6 = -[w_2, w_1], [w_1, w_3] = \beta_1 w_4 + \beta_2 w_5 + \beta_3 w_6 = -[w_3, w_1], [w_2, w_3] = \beta_4 w_4 + \beta_5 w_5 + \beta_6 w_6 = -[w_3, w_2], [w_1, w_4] = \gamma_1 w_5 + \gamma_2 w_6 = -[w_4, w_1], [w_2, w_4] = \gamma_3 w_5 + \gamma_4 w_6 = -[w_4, w_2], [w_3, w_4] = \gamma_5 w_5 + \gamma_6 w_6 = -[w_4, w_3], [w_1, w_5] = \theta_1 w_6 = -[w_5, w_1], [w_2, w_5] = \theta_2 w_6 = -[w_5, w_2], [w_3, w_5] = \theta_3 w_6 = -[w_5, w_3], [w_3, w_5] = \theta_4 w_6 = -[w_5, w_3]$.

From Leibniz identity, we get the following equations:

$$\theta_4 = \theta_3 = \gamma_5 = 0$$

$$\beta_4\gamma_1 - \beta_1\gamma_3 = 0 \quad \textbf{(3.4)}$$

$$\gamma_3\theta_1 - \gamma_1\theta_2 - \alpha_1\gamma_6 = 0 \quad \textbf{(3.5)}$$

$$\beta_4\gamma_2 + \beta_5(\theta_1 - \theta_2) - \beta_1\gamma_4 + \alpha_2\gamma_6 = 0 \quad \textbf{(3.6)}$$

First, suppose $\gamma_3 = 0$. Since $\gamma_1 \neq 0$ and from Equation 3.4, we have $\beta_4 = 0$. Using $dim(L^3) = 3$, we can see that $\beta_1 \neq 0$. Suppose $\theta_2 = 0$. Then, from Equations 3.5 and 3.6, we obtain $\gamma_6 = 0 = \beta_5\theta_1 - \beta_1\gamma_4$. With the following change of basis $x_1 = \gamma_4 w_1 - \gamma_1 w_2, x_2 = w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = \gamma_4^2 w_7$, we can force $\gamma_2 = 0$. The following is a transition matrix switching the basis $W = \{w_1, w_2, \dots, w_7\}$ to the basis $\zeta = \{\zeta_1, \zeta_2, \dots, \zeta_7\}$

$$\mathcal{B}_1 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_2 & a^2b\alpha_1\beta_1 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_3 & a^2b(\alpha_1\beta_2 + \alpha_2\gamma_1) & a^3b\alpha_1\beta_1\gamma_1 & 0 & 0 \\ 0 & 0 & ab\alpha_4 & a^2b(\alpha_1\beta_3 + \alpha_3\theta_1) & a^3b(\alpha_1\beta_2 + \alpha_2\gamma_1)\theta_1 & a^4b\alpha_1\beta_1\gamma_1\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2 \end{pmatrix}$$

- If $\gamma_4 = \beta_6 = 0$, then taking $a = b = 1$ in $\mathcal{B}_1$, we get the algebra $L1$.
- If $\gamma_4 = 0$ and $\beta_6 \neq 0$, then taking $a = 1, b = \frac{\beta_1\gamma_1\theta_1}{\beta_6}$ in $\mathcal{B}_1$, we obtain $L2$.
- If $\gamma_4 \neq 0$ and $\gamma_1\beta_6 - \beta_2\gamma_4 = 0$, then taking $a = 1, b = \frac{\gamma_1\theta_1}{\gamma_4}$ in $\mathcal{B}_1$, we get the algebra $L3$.
- If $\gamma_4 \neq 0$ and $\gamma_1\beta_6 - \beta_2\gamma_4 \neq 0$, then taking $a = \frac{\gamma_1\beta_6 - \beta_2\gamma_4}{\beta_1\gamma_1^2\theta_1^2}, b = \frac{(\gamma_1\beta_6 - \beta_2\gamma_4)^2}{\beta_1^2\gamma_1^3\theta_1^3\gamma_4}$ in $\mathcal{B}_1$, we arrive the algebra $L4$.

Take $\theta_2 \neq 0$. The following change of basis $x_1 = \theta_2 w_1 - \theta_1 w_2, x_2 = w_2, x_3 = w_3, x_4 = \theta_2 w_4 - \gamma_4 w_5, x_5 = w_5, x_6 = w_6, x_7 = \theta_2^2 w_7$ forces $\gamma_4 = \theta_1 = 0$. Notice that Equation 3.5 do not allow γ_6 to be zero. Then, the change of basis $x_1 = \gamma_6 w_1 - \gamma_2 w_3, x_2 = w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = \gamma_6^2 w_7$ forces $\gamma_2 = 0$. Choose the transition matrix switching the basis $W = \{w_1, w_2, \dots, w_7\}$ to the basis $\zeta = \{\zeta_1, \zeta_2, \dots, \zeta_7\}$ as

$$\mathcal{B}_2 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_2 & a^2b\alpha_1\beta_1 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_3 & a^2b(\alpha_1\beta_2 + \alpha_2\gamma_1) & a^3b\alpha_1\beta_1\gamma_1 & 0 & 0 \\ 0 & 0 & ab\alpha_4 & a^2b(\alpha_1\beta_3 + \alpha_3\theta_1) & 0 & a^3b^2\alpha_1\beta_1\gamma_1\theta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2 \end{pmatrix}$$

- If $\alpha_1\beta_6 + \alpha_3\theta_2 = 0$, then taking $a = 1, b = \frac{\beta_1\gamma_1}{\beta_5}$ in $\mathcal{B}_2$, we get the algebra $L5$.
- If $\alpha_1\beta_6 + \alpha_3\theta_2 \neq 0$, then taking $a = \sqrt{\frac{\alpha_1\beta_6 + \alpha_3\theta_2}{\alpha_1\beta_1\gamma_1\theta_2}}, b = \frac{\alpha_1\beta_6 + \alpha_3\theta_2}{\alpha_1\beta_5\theta_2}$ in $\mathcal{B}_2$, we get the algebra $L6$.

Later, suppose $\gamma_3 \neq 0$. With the change of basis $x_1 = \gamma_3 w_1 - \gamma_1 w_2, x_2 = w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = \gamma_3^2 w_7$, we can force $\gamma_1 = 0$. Then, by Equation 3.4, we have $\beta_1 = 0$. Choose the transition matrix switching the basis $W = \{w_1, w_2, \dots, w_7\}$ to the basis $\zeta = \{\zeta_1, \zeta_2, \dots, \zeta_7\}$ as the following

Celal Bayar University Journal of Science
Volume 19, Issue 4, 2023, p 343-349
Doi: 10.18466/cbayarfbe.1339702

$$\mathcal{B}_3 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_2 & ab^2\alpha_1\beta_4 & 0 & 0 & 0 \\ 0 & 0 & ab\alpha_3 & ab^2(\alpha_1\beta_5 + \alpha_2\gamma_3) & ab^3\alpha_1\beta_4\gamma_3 & 0 & 0 \\ 0 & 0 & ab\alpha_4 & ab^2(\alpha_1\beta_6 + \alpha_2\gamma_4 + \alpha_3\theta_2) & ab^3(\alpha_1\beta_4\gamma_4 + \alpha_1\beta_5\theta_2 + \alpha_2\gamma_3\theta_2)\theta_1 & ab^4\alpha_1\beta_4\gamma_3\theta_2(or\ a^2b^3\alpha_1\beta_4\gamma_3\theta_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2 \end{pmatrix}$$

If $\gamma_6 = 0$, then Equation 3.5 imply $\theta_1 = 0$. Then, taking $a = b^2 \frac{\gamma_3\theta_2}{\gamma_2}$ while $b = \sqrt{\frac{(\alpha_1\beta_3+\alpha_2\gamma_2)\,\gamma_3^2\theta_2^2-(\alpha_1\beta_4\gamma_4+\alpha_1\beta_5\theta_2+\alpha_2\gamma_3\theta_2)\gamma_2\gamma_3\theta_2}{\alpha_1\beta_4\gamma_2\,\gamma_3^2\theta_2^2}}$ in $\mathcal{B}_3$, we arrive the algebra $L7$. Let $\gamma_6 \neq 0$. Then, with the changes of bases $x_1 = \gamma_6 w_1 - \gamma_2 w_3, x_2 = w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = \gamma_6^2 w_7$ and $x_1 = w_1, x_2 = \gamma_6 w_2 - \gamma_4 w_3, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = \gamma_3^2 w_7$ make $\gamma_2 = 0$ and $\gamma_4 = 0$, respectively.

- If $\theta_2 = 0$, then letting $a = b^2 \frac{\beta_4\gamma_3}{\beta_2}$ while $b = \sqrt{\frac{(\alpha_1\beta_3+\alpha_3\theta_1)\beta_4^2\,\gamma_3^2-(\alpha_1\beta_4\gamma_4)\beta_2\beta_4\gamma_3}{\alpha_1^2\beta_4^3\,\gamma_3^2\gamma_6}}$ in $\mathcal{B}_3$, we obtain the algebra $L8$.
- If $\theta_2 \neq 0$, then taking $a = \frac{\beta_2\theta_2^2}{\beta_4\gamma_3\theta_1^2}, b = \frac{\beta_2\theta_2}{\beta_4\gamma_3\theta_1}$ in $\mathcal{B}_3$, we get the algebra $L9$.

Case 2. If the bilinear form matrix is $C_1 \oplus C_1$, then the nonzero products in L can be given as:

$[w_1, w_1] = w_7, [w_1, w_2] = \alpha_1 w_3 + \alpha_2 w_4 + \alpha_3 w_5 + \alpha_4 w_6 = -[w_2, w_1], [w_2, w_2] = w_7, [w_1, w_3] = \beta_1 w_4 + \beta_2 w_5 + \beta_3 w_6 = -[w_3, w_1], [w_2, w_3] = \beta_4 w_4 + \beta_5 w_5 + \beta_6 w_6 = -[w_3, w_2], [w_1, w_4] = \gamma_1 w_5 + \gamma_2 w_6 = -[w_4, w_1], [w_2, w_4] = \gamma_3 w_5 + \gamma_4 w_6 = -[w_4, w_2], [w_3, w_4] = \gamma_5 w_5 + \gamma_6 w_6 = -[w_4, w_3], [w_1, w_5] = \theta_1 w_6 = -[w_5, w_1], [w_2, w_5] = \theta_2 w_6 = -[w_5, w_2], [w_3, w_5] = \theta_3 w_6 = -[w_5, w_3], [w_3, w_5] = \theta_4 w_6 = -[w_5, w_3].$

Then, again from the Leibniz identity, we arrive Equations that obtained in case 1. Take a transition matrix switching the basis $W = \{w_1, w_2, \ldots, w_7\}$ to the basis $\zeta = \{\zeta_1, \zeta_2, \ldots, \zeta_7\}$

$$\mathcal{B}_4 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2\alpha_2 & a^3\alpha_1\beta_1 & 0 & 0 & 0 \\ 0 & 0 & a^2\alpha_3 & a^3(\alpha_1\beta_2 + \alpha_2\gamma_1) & a^4\alpha_1\beta_1\gamma_1 & 0 & 0 \\ 0 & 0 & a^2\alpha_4 & a^3(\alpha_1\beta_3 + \alpha_3\theta_1) & a^4(\alpha_1\beta_1\gamma_2 + \alpha_1\beta_2\theta_1 + \alpha_2\gamma_1\theta_1) & a^5\alpha_1\beta_1\gamma_1\theta_1(or\ a^5\alpha_1\beta_1\gamma_1\theta_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2 \end{pmatrix}$$

First, let $\gamma_3 = 0$. Then, Equation 3.4 imply $\beta_4 = 0, \beta_1, \gamma_1 \neq 0$. Suppose $\theta_2 = 0$. Then, from the equations 3.4 and 3.5, we get $\gamma_6 = 0$ and $\beta_5\theta_1 - \beta_1\gamma_4 = 0$.

- If $\gamma_4 = \beta_6 = 0$, then plugging $a = 1$ in $\mathcal{B}_4$ results with the algebra $L10$.
- If $\gamma_4 = 0$ and $\beta_6 \neq 0$, then plugging $a = \sqrt{\frac{\beta_6}{\beta_1\gamma_1\theta_1}}$ in $\mathcal{B}_4$, we get the algebra $L11$.
- If $\gamma_4 \neq 0$, then by taking $a = \frac{\gamma_4}{\gamma_1\theta_1}$ in $\mathcal{B}_4$, we get the algebra $L12$.

Now suppose $\theta_2 \neq 0$. Then, with the change of basis $x_1 = w_1, x_2 = w_2, x_3 = w_3, x_4 = \theta_2 w_4 - \gamma_4 w_5, x_5 = w_5, x_6 = w_6, x_7 = w_7$, we can make $\gamma_4 = 0$. Consequently, by taking $a = 1$ in $\mathcal{B}_4$, we get the algebra $L13$.

Finally, consider the case $\gamma_3 \neq 0$. If $\gamma_1 = 0$, then the change of basis $x_1 = w_2, x_2 = w_1, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = w_7$ forces $\gamma_3 = 0$ and therefore L is isomorphic to $L10, L11, L12$ or $L13$. Hence, let $\gamma_1 \neq 0$. Note that when $\gamma_1^2 + \gamma_3^2 \neq 0$ with the following change of basis $x_1 = \gamma_1 w_1 + \gamma_3 w_2, x_2 = \gamma_3 w_1 - \gamma_1 w_2, x_3 = w_3, x_4 = w_4, x_5 = w_5, x_6 = w_6, x_7 = (\gamma_1^2 + \gamma_3^2) w_7$, we can force $\gamma_3 = 0$ and again L is isomorphic to $L10, L11, L12$ or $L13$. Thus, $\gamma_1^2 + \gamma_3^2 = 0$. From Equation 3.4 we obtain $\beta_1^2 + \beta_4^2 = 0$.

- If $\gamma_6 = 0$, then from Equation 3.5, we obtain $\theta_1^2 + \theta_2^2 = 0$. Hence, by taking $a = 1$ in $\mathcal{B}_4$, we get the algebra $L14$.
- If $\gamma_6 \neq 0$, then by taking $a = 1$ in the change of basis matrix $\mathcal{B}_4$, we obtain the algebra $L15$.

This completes the proof. ■

4. Conclusion

We reached the complete classification of nilpotent Leibniz algebras of dimension seven, whose derived algebra is of dimension five, while the Leib ideal is one dimensional with the restriction $dim(L^3) < 4$. The classifications given in this paper and in [14] achieve that result. There exist 19 single algebras, 4 one-parameter continuous families, 2 two-parameter continuous families, 1 three-parameter continuous family, and 1 one-parameter continuous family. It is left to look at the case $dim(L^3) = 4$ in order to complete the classification of nilpotent Leibniz algebra of dimension seven, whose derived algebra is of dimension five while Leib ideal is one dimensional. Notice that $\chi(L) = (7,5,4,3,2,1)$ is filiform Leibniz algebra, and the classification of this subclass with one-dimensional Leib ideal can be obtained by Theorem 2.2 in [4]. TLb_7 in that Theorem will produce desired algebras. In fact, the classification of the subclass TLb_7 is given in [6]. According to that classification, there are 13 single algebras and 9 one-parameter continuous families of filiform non-Lie Leibniz algebras of dimension seven. Furthermore, there is no Leibniz algebra for the case $\chi(L) = (7,5,4)$, because here $L^3 = Z(L)$ with the Lemma 2.4 give $dim(L^2) \leq 4$, which contradicts L^2 being a 5-dimensional ideal. Hence, we have the following six cases to classify:

i. $\chi(L) = (7,5,4,3,2)$
ii. $\chi(L) = (7,5,4,3,1)$
iii. $\chi(L) = (7,5,4,3)$
iv. $\chi(L) = (7,5,4,2,1)$
v. $\chi(L) = (7,5,4,2)$
vi. $\chi(L) = (7,5,4,1)$

Our approach of bilinear forms can also be utilized to classify these cases. It is known that there are 119 single algebras and six one-parameter continuous families of Lie algebra of dimension seven over an algebraically closed field [16]. Even though, we restrict our attention to seven-dimensional nilpotent Leibniz algebras whose Leib ideal is one dimensional and derived algebra is of dimension five, we get 32 single algebras, 13 one-parameter continuous families, two two-parameter continuous families, one three-parameter continuous family, and one one-parameter continuous family so far and there are still some cases to cover. As a future work, classification of higher dimensional nilpotent Leibniz algebras with one dimensional Leib ideal and/or classification of nilpotent Leibniz algebras of higher dimensions with the derived algebra of codimension two can be obtained by the congruence classes of bilinear forms method.

Author's Contributions

İsmail Demir: Drafted and wrote the manuscript, performed the experiment and result analysis.

Ethics

There are no ethical issues after the publication of this manuscript.

References

[1]. Bloh, A. 1965. On a Generalization of Lie Algebra Notion. *Mathematics in USSR Doklady*; 165(3): 471-473.

[2]. Loday, JL. 1993. Une Version Non-Commutative des Algebres de Lie: Les Algebres de Leibniz. *L'Enseignement Mathematique*; 39(3-4): 269-293.

[3]. Albeverio, S, Omirov, BA, Rakhimov, IS. 2006. Classification of 4-Dimensional Nilpotent Complex Leibniz Algebra. *Extracta Mathematicae*; 21(3): 197-210.

[4]. Rakhimov, IS, Bekbaev, UD. 2010. On Isomorphisms and Invariants of Finite Dimensional Complex Filiform Leibniz algebras. *Communications in Algebra*; 38: 4705-4738.

[5]. Casas, JM, Insua, MA, Ladra, M, Ladra, S. 2012. An Algorithm for the Classification of 3-Dimensional Complex Leibniz Algebras. *Linear Algebra and its Applications*; 9: 3747-3756.

[6]. Abdulkareem, AO, Rakhimov, IS, Husain, SK. On Seven-Dimensional Filiform Leibniz Algebras, In: Kilicman, A., Leong, W., Eshkuvatov, Z. (eds) International Conference on Mathematical Sciences and Statistics, 2014, pp 1-11.

[7]. Gomez, JR, Omirov, BA. 2015. On Classification of Filiform Leibniz Algebras. *Algebra Colloquium*; 22: 757-774.

[8]. Demir, I, Misra, KC, Stitzinger, E. 2017. On Classification of Four-Dimensional Nilpotent Leibniz Algebras. *Communications in Algebra*; 45(3): 1012-1018.

[9]. Rakhimov, IS, Khudoyberdiyev, AK, Omirov, BA. 2017. On Isomorphism Criterion for a Subclass of Complex Filiform Leibniz Algebras. *International Journal of Algebra and Computation*; 27(7): 953-972.

[10]. Demir, I. Classification of 5-Dimensional Complex Nilpotent Leibniz Algebras, In: N. Jing, K. C. Misra (Eds.), Representations of Lie Algebras, Quantum Groups and Related Topics, Contemporary Mathematics, Volume 713, American Mathematical Society, 2018, pp. 95-120.

[11]. Mohamed, NS, Husain, SK, Rakhimov, IS. 2019. Classification of a Subclass of 10-Dimensional Complex Filiform Leibniz Algebras. *Malaysian Journal of Mathematical Sciences*; 13(3): 465-485.

[12]. Demir, I. 2020. Classification of Some Subclasses of 6-Dimensional Nilpotent Leibniz Algebras. *Turkish Journal of Mathematics*; 44: 1925-1940.

[13]. Farris, L. Finite Dimensional Nilpotent Leibniz Algebras with Isomorphic Maximal Algebras, Doctoral Dissertation, North Carolina State University, 2022.

[14]. Demir, I. On Classification of 7-Dimensional Odd-Nilpotent Leibniz Algebras. Hacettepe Journal of Mathematics and Statistics, (in press).

[15]. Teran, F. 2016. Canonical Forms for Congruence of Matrices and T-palindromic Matrix Pencils: a Tribute to H. W. Turnbull and A. C. Aitken. *SeMA Journal: Bulletin of the Spanish Society of Applied Mathematics*; 73: 7-16.

[16]. Gong, MP. Classification of Nilpotent Lie Algebras of Dimension 7 (over Algebraically Closed Field $\mathbb{F}$ and $\mathbb{R}$), Doctoral Dissertation, University of Waterloo, 1998.