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A GENERALIZATION OF THE REGULAR TRIBONACCI-LUCAS MATRIX

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ABSTRACT

We define a generalization of a regular Tribonacci-Lucas matrix and give some factorizations by some special matrices. We find the inverse and the k-th power of the matrix. We also present several identities and a relation between an exponential of a matrix and the defined matrix.

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1 INTRODUCTION

There have been several studies about Fibonacci and Lucas numbers and their generalizations as they have many applications on several fields, see [8, 9, 12—14, 16, 17]. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by the recurrence

$$F_{n+2} = F_{n+1} + F_n$$

with initial conditions $F_0 = 0, F_1 = 1$. The Lucas sequence $\{L_n\}_{n\geq 0}$ is defined by $L_0 = 2, L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n.$$

A third order generalization of these sequences are called as Tribonacci sequence $\{t_n\}_{n\geq 0}$ and Tribonacci-Lucas sequence $\{v_n\}_{n\geq 0}$. These sequences are defined by the recurrences

$$t_{n+3} = t_{n+2} + t_{n+1} + t_n$$

with initial conditions $t_0 = 0, t_1 = 1, t_2 = 1$ and

$$v_{n+3} = v_{n+2} + v_{n+1} + v_n$$

with initial conditions $v_0 = 3, v_1 = 1, v_2 = 3$, respectively. The first few terms of $\{t_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ are given in Table 1.

***Table 1**. The first few terms of the Tribonacci and Tribonacci-Lucas sequences.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12
t_n	0	1	1	2	4	7	13	24	44	81	149	274	504
v_n	3	1	3	7	11	21	39	71	131	241	443	815	1499

There are many studies on Tribonacci and Tribonacci-Lucas numbers and their various properties in the literature. Several sums formulas of these sequences such as

$$\sum_{k=1}^{n} t_k = \frac{t_{n+2} + t_n - 1}{2}$$
$$\sum_{k=1}^{n} v_k = \frac{v_{n+2} + v_n - 6}{2}$$

are also obtained, see [4–6, 10, 11, 20, 24–28, 30].

Matrices whose entries are chosen from special numbers are also found interesting and some factorizations of these matrices have been considered by many researchers, see [1, 2, 7, 19, 21, 32]. In [31], a matrix of order $n+1$ with entries $[t_{i,j}]$

$$t_{i,j} = \begin{cases} \dfrac{2t_j}{t_{i+2} + t_i - 1}, & \text{if } 0 \leq j \leq i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

is defined and the Tribonacci space sequences $\ell_p(T)$ are introduced. In [22], a two variables generalization of the matrix given in (1) is defined and some factorizations of the defined matrix are obtained.

Recently, a new regular Tribonacci-Lucas matrix $V = [v_{i,j}]$ is defined by

$$v_{i,j} = \begin{cases} \dfrac{2v_j}{v_{i+2} + v_i - 6}, & \text{if } 0 \leq j \leq i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

see [18]. They give some relations and inclusion results between the defined matrix and some well-known summability matrices. In this paper, we define a generalization of the matrix given in (2) and present several properties. We obtain some factorizations of the defined matrix and give a relation with an exponential of a special matrix.

2 A GENERALIZATION OF THE REGULAR TRİBONACCİ-LUCAS MATRİX

We define a generalization of the matrix (2) for two variables. Let $V_n(x,y) = [v_{i,j}(x,y)]$ be the matrix of order $n+1$ with entries

$$v_{i,j}(x,y)=\begin{cases}\dfrac{2v_j}{v_{i+2}+v_i-6}x^{i-j}y^j, & \text{if } 0\le j\le i,\\ 0, & \text{otherwise.}\end{cases}$$

Here $v_{i,j}(x,y)$ will be zero for x or y is zero and so we assume that x and y are non-zero real numbers. It is clear that for $x=y=1$ we have

$$v_{i,j}(1,1)=v_{i,j}$$

and so, in this case we obtain the regular Tribonacci-Lucas matrix (2).

Example 1. For $n=5$, the matrix $V_5(x,y)$ will be of the form

$$V_5(x,y)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 & 0\\ \frac{1}{11}x^2 & \frac{3}{11}xy & \frac{7}{11}y^2 & 0 & 0 & 0\\ \frac{1}{22}x^3 & \frac{3}{22}x^2y & \frac{7}{22}xy^2 & \frac{11}{22}y^3 & 0 & 0\\ \frac{1}{43}x^4 & \frac{3}{43}x^3y & \frac{7}{43}x^2y^2 & \frac{11}{43}xy^3 & \frac{21}{43}y^4 & 0\\ \frac{1}{49}x^5 & \frac{3}{49}x^4y & \frac{7}{49}x^3y^2 & \frac{11}{49}x^2y^3 & \frac{21}{49}xy^4 & \frac{39}{49}y^5 \end{bmatrix}$$

2.1 Properties of the Tribonacci-Lucas Matrices $V_n(x,y)$

We give some interesting properties and applications of the matrix $V_n(x,y)$. Throughout the paper, we will denote the (i,j) entry of a matrix A as $(A)_{i,j}$. For $n,j\in\mathbb{N}$, we define

$$(x\oplus y)_j^n := \sum_{k=0}^{n} v_{k+j,k+j}x^{n-k}y^k.$$

Theorem 2.1. For any positive integer n and any real numbers x,y,z and w, we have

$$(V_n(x,y)V_n(w,z))_{i,j}=\left(V_n\left((x\oplus yw)_j, yz\right)\right)_{i,j}. \quad \textbf{(3)}$$

Proof. It is clear from the definition that $v_{i,j+1}v_{j+1,j} = v_{j+1,j+1}v_{i,j}$. Then we have

$$\begin{aligned}(V_n(x,y)V_n(w,z))_{i,j} &= \sum_{k=j}^{i} v_{i,k}(x,y)v_{k,j}(w,z)\\ &= v_{i,j}v_{j,j}x^{i-j}y^jz^j + v_{i,j+1}v_{j+1,j}x^{i-j-1}y^{j+1}wz^j + \cdots + v_{i,i}v_{i,j}y^iw^{i-j}z^j\\ &= v_{i,j}y^jz^j\left(v_{j,j}x^{i-j} + v_{j+1,j+1}x^{i-j-1}yw + \cdots + v_{i,i}y^{i-j}w^{i-j}\right)\\ &= v_{i,j}y^jz^j(x \oplus yw)_j^{i-j}\\ &= \left(V_n\left((x \oplus yw)_j, yz\right)\right)_{i,j}.\end{aligned}$$

We can obtain the $k-$ th power of the matrix $V_n(x,y)$ by using Theorem 2.1. For $w = x$ and $z = y$ in (3), we get

$$(V_n^2(x,y))_{i,j} = \left(V\left(x(1 \oplus y)_j, y^2\right)\right)_{i,j}.$$

Using formula (3) again, multiplying $V_n^2(x,y)$ and $V_n(x,y)$, we get

$$(V_n^3(x,y))_{i,j} = \left(V\left(x\left((1 \oplus y)_j \oplus y^2\right)_j, y^3\right)\right)_{i,j}.$$

Then using the mathematical induction method, we have

$$(V_n^k(x,y))_{i,j} = \left(V\left(x\left(\left(\ldots\left((1 \oplus y)_j \oplus y^2\right)_j \oplus y^3\right)_j \ldots \oplus y^{k-1}\right)_j, y^k\right)\right)_{i,j}.$$

The inverse of the Tribonacci-Lucas matrix $V_n(x,y)$ which is denoted by $V_n^{-1}(x,y) = [v_{i,j}^{-1}(\mathrm{x,y})]$ is given by the following theorem.

Theorem 2.2. The $(i,j)-$ entry of the inverse of the matrix $V_n(x,y)$ is

$$v_{i,j}^{-1}(x,y) = \begin{cases} \dfrac{v_{i+2} + v_i - 6}{2v_jy^i}, & \text{if } i = j,\\ \dfrac{-(v_{i+2} + v_i - 6)x}{2v_{j+2}y^i}, & \text{if } i = j+1,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. It is clear that $(V_n(x,y)V_n^{-1}(x,y))_{i,j} = 0$ in the case of $i \neq j$ and $i \neq j+1$. For $i = j$, we obtain that

$$(V_n(x,y)V_n^{-1}(x,y))_{i,i} = \sum_{k=i}^{i} v_{i,k}(x,y)v_{k,i}^{-1}(x,y) = v_{ii}(x,y)v_{ii}^{-1}(x,y)$$
$$= \frac{2v_i y^i}{v_{i+2}+v_i-6}\frac{v_{i+2}+v_i-6}{2v_i y^i} = 1$$

and for $i = j+1$ we get

$$\begin{aligned}(V_n(x,y)V_n^{-1}(x,y))_{i,j} &= \sum_{k=j}^{i} v_{i,k}(x,y)v_{k,j}^{-1}(x,y) \\ &= v_{ij}(x,y)v_{jj}^{-1}(x,y) + v_{i,j+1}(x,y)v_{j+1,j}^{-1}(x,y) \\ &= \frac{2v_j x^{i-j}y^j}{v_{i+2}+v_i-6}\frac{v_{j+2}+v_j-6}{2v_j y^j} + \frac{2v_{j+1}x^{i-j-1}y^{j+1}}{v_{i+2}+v_i-6}\frac{(v_{j+2}+v_j-6)(-x)}{2v_{j+1}y^{j+1}} \\ &= \frac{(v_{j+2}+v_j-6)x^{i-j}}{v_{i+2}+v_i-6} - \frac{(v_{j+2}+v_j-6)x^{i-j}}{v_{i+2}+v_i-6} \\ &= 0.\end{aligned}$$

Thus, the result follows.

2.2 Factorizations of the Tribonacci-Lucas Matrices $V_n(x,y)$

We give some factorizations of the matrix $V_n(x,y)$. For this purpose, we need to define the following matrices of order $n+1$

$$(S_n(x,y))_{i,j} = \begin{cases} v_{i,j+1}(x,y)v_{j,j-1}^{-1}(x,y) + v_{i,j}(x,y)v_{j-1,j-1}^{-1}(x,y), & \text{if } 0 \le j \le i, \\ 0, & \text{otherwise} \end{cases}$$
$$\bar{V}_{n-1}(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & V_{n-1} \end{bmatrix},$$
$$G_k = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k \end{bmatrix} \text{ for } 1 \le k \le n-1, \text{ and } G_n(x,y) = S_n(x,y).$$

Lemma 2.1. For any positive integer n and any real numbers x and y, we have

$$V_n(x,y) = S_n(x,y)\bar{V}_{n-1}(x,y).$$

Proof. We denote the inverse of the matrix $\bar{V}_n(x,y)$ as $\bar{V}_n^{-1}(x,y) := [\bar{v}_{i,j}^{-1}(x,y)]$. Then

$$(V_n(x,y)\bar{V}_{n-1}^{-1}(x,y))_{i,j} = \sum_{k=j}^{i} v_{i,k}(x,y)\bar{v}_{k,j}^{-1}(x,y) = \sum_{k=j}^{i} v_{i,k}(x,y)v_{k-1,j-1}^{-1}(x,y).$$

Here the sum is nonzero only for $k-1 = j-1$ and $k-1 = j$. So we get

$$\sum_{k=j}^{i} v_{i,k}(x,y)v_{k-1,j-1}^{-1}(x,y) = v_{i,j+1}(x,y)v_{j,j-1}^{-1}(x,y) + v_{i,j}(x,y)v_{j-1,j-1}^{-1}(x,y) = S_n(x,y).$$

Example 2.

$S_5(x,y)\bar{V}_4(x,y) =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 & 0 \\ \frac{1}{11}x^2 & \frac{2}{33}xy & \frac{28}{11}y & 0 & 0 & 0 \\ \frac{1}{22}x^3 & \frac{1}{33}x^2y & \frac{32}{231}xy & \frac{11}{14}y & 0 & 0 \\ \frac{1}{43}x^4 & \frac{2}{129}x^3y & \frac{64}{903}x^2y & -\frac{26}{301}xy & \frac{42}{43}y & 0 \\ \frac{1}{49}x^5 & \frac{2}{147}x^4y & \frac{64}{1029}x^3y & -\frac{26}{343}x^2y & \frac{8}{343}xy & \frac{559}{343}y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 \\ 0 & \frac{1}{11}x^2 & \frac{3}{11}xy & \frac{7}{11}y^2 & 0 & 0 \\ 0 & \frac{1}{22}x^3 & \frac{3}{22}x^2y & \frac{7}{22}xy^2 & \frac{11}{22}y^3 & 0 \\ 0 & \frac{1}{43}x^4 & \frac{3}{43}x^3y & \frac{7}{43}x^2y^2 & \frac{11}{43}xy^3 & \frac{21}{43}y^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 & 0 \\ \frac{1}{11}x^2 & \frac{3}{11}xy & \frac{7}{11}y^2 & 0 & 0 & 0 \\ \frac{1}{22}x^3 & \frac{3}{22}x^2y & \frac{7}{22}xy^2 & \frac{11}{22}y^3 & 0 & 0 \\ \frac{1}{43}x^4 & \frac{3}{43}x^3y & \frac{7}{43}x^2y^2 & \frac{11}{43}xy^3 & \frac{21}{43}y^4 & 0 \\ \frac{1}{49}x^5 & \frac{3}{49}x^4y & \frac{7}{49}x^3y^2 & \frac{11}{49}x^2y^3 & \frac{21}{49}xy^4 & \frac{39}{49}y^5 \end{bmatrix}$$

$= V_5(x,y).$

Theorem 2.3. The matrix $V_n(x,y)$ can be factorized as

$$V_n(x,y) = G_n(x,y)G_{n-1}(x,y) \dots G_1(x,y).$$

In particular,

$$V_n = G_n G_{n-1} \dots G_1$$

where $V_n := V_n(1,1), G_k := G_k(1,1), k = 1,2,\dots,n.$

Proof. By the definition of the matrices $G_k(x,y)$ and Lemma 2.1, we get the desired decomposition of the matrix $V_n(x,y)$.

It is clear that the inverse matrix $V_n^{-1}(x,y)$ can be factorized as

$$V_n^{-1}(x,y) = G_1^{-1}(x,y)G_2^{-1}(x,y) \dots G_n^{-1}(x,y).$$

Example 3. Since

$$V_5(x,y)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 & 0 \\ \frac{1}{11}x^2 & \frac{3}{11}xy & \frac{7}{11}y^2 & 0 & 0 & 0 \\ \frac{1}{22}x^3 & \frac{3}{22}x^2y & \frac{7}{22}xy^2 & \frac{11}{22}y^3 & 0 & 0 \\ \frac{1}{43}x^4 & \frac{3}{43}x^3y & \frac{7}{43}x^2y^2 & \frac{11}{43}xy^3 & \frac{21}{43}y^4 & 0 \\ \frac{1}{49}x^5 & \frac{3}{49}x^4y & \frac{7}{49}x^3y^2 & \frac{11}{49}x^2y^3 & \frac{21}{49}xy^4 & \frac{39}{49}y^5 \end{bmatrix}$$

we can factorize this matrix as

$$G_5(x,y)G_4(x,y)G_3(x,y)G_2(x,y)G_1(x,y)=$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 & 0 \\ \frac{1}{11}x^2 & \frac{2}{33}xy & \frac{28}{11}y & 0 & 0 & 0 \\ \frac{1}{22}x^3 & \frac{1}{33}x^2y & \frac{32}{231}xy & \frac{11}{14}y & 0 & 0 \\ \frac{1}{43}x^4 & \frac{2}{129}x^3y & \frac{64}{903}x^2y & -\frac{26}{301}xy & \frac{42}{43}y & 0 \\ \frac{1}{49}x^5 & \frac{2}{147}x^4y & \frac{64}{1029}x^3y & -\frac{26}{343}x^2y & \frac{8}{343}xy & \frac{559}{343}y \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 \\ 0 & \frac{1}{11}x^2 & \frac{2}{33}xy & \frac{28}{11}y & 0 & 0 \\ 0 & \frac{1}{22}x^3 & \frac{1}{33}x^2y & \frac{32}{231}xy & \frac{11}{14}y & 0 \\ 0 & \frac{1}{43}x^4 & \frac{2}{129}x^3y & \frac{64}{903}x^2y & -\frac{26}{301}xy & \frac{42}{43}y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}x & \frac{3}{4}y & 0 & 0 \\ 0 & 0 & \frac{1}{11}x^2 & \frac{2}{33}xy & \frac{28}{11}y & 0 \\ 0 & 0 & \frac{1}{22}x^3 & \frac{1}{33}x^2y & \frac{32}{231}xy & \frac{11}{14}y \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}x & \frac{3}{4}y & 0 \\ 0 & 0 & 0 & \frac{1}{11}x^2 & \frac{2}{33}xy & \frac{28}{11}y \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4}x & \frac{3}{4}y \end{bmatrix}.$$

We can also separate the variables x and y from the matrices $V_n(x,y)$ and $V_n(-x,y)$.

Theorem 2.4. Let $D_n(x):=\operatorname{diag}(1,x,x^2,x^3,\ldots,x^n)$ be a diagonal matrix. For any positive integer k and any non-zero real numbers x and y , we have

$$V_k(x,y)=V_k(x,1)D_k(y),$$
$$V_k(-x,y)=V_k(-x,1)D_k(y).$$

Now, we present a relation between the matrices $V_n(x,ay)$ and $V_n(x,-y)$ for a nonzero real number a.

Theorem 2.5. For a nonzero real number a, the matrices $V_n(x,ay)$ and $V_n(x,-y)$ satisfy the following

$$V_n\left(x,\frac{y}{a}\right)^{-1} = V_n^{-1}(x,-y)V_n(x,ay)V_n^{-1}(x,-y).$$

Proof. The proof can be done easily by definition of the matrices and matrix multiplication.

Theorem 2.6. Let $K_n(x,y) = [k_{i,j}]$ be a matrix with entries $k_{i,j} = v_j x^{i-j} y^j$ and $D'_n = [d'_{i,j}]$ be a diagonal matrix with diagonal entries $d'_{i,i} = \frac{2}{v_{i+2}+v_i-6}$. Then we have

$$V_n(x,y) = D'_n K_n(x,y).$$

Proof. By matrix multiplication, we have

$$\begin{aligned}(D'_n K_n(x,y))_{i,j} &= \sum_{k=0}^{n} d'_{i,k} k_{k,j}(x,y) = d'_{i,i} k_{i,j}(x,y) \\ &= \frac{2}{v_{i+2}+v_i-6} v_j x^{i-j} y^j \\ &= \frac{2v_j}{v_{i+2}+v_i-6} x^{i-j} y^j = (V_n(x,y))_{i,j}.\end{aligned}$$

Example 4. For $n = 5$, we have

$$V_5(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4}x & \frac{3}{4}y & 0 & 0 & 0 & 0 \\ \frac{1}{11}x^2 & \frac{3}{11}xy & \frac{7}{11}y^2 & 0 & 0 & 0 \\ \frac{1}{22}x^3 & \frac{3}{22}x^2y & \frac{7}{22}xy^2 & \frac{11}{22}y^3 & 0 & 0 \\ \frac{1}{43}x^4 & \frac{3}{43}x^3y & \frac{7}{43}x^2y^2 & \frac{11}{43}xy^3 & \frac{21}{43}y^4 & 0 \\ \frac{1}{49}x^5 & \frac{3}{49}x^4y & \frac{7}{49}x^3y^2 & \frac{11}{49}x^2y^3 & \frac{21}{49}xy^4 & \frac{39}{49}y^5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{43} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{49} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x & 3y & 0 & 0 & 0 & 0 \\ x^2 & 3xy & 7y^2 & 0 & 0 & 0 \\ x^3 & 3x^2y & 7xy^2 & 11y^3 & 0 & 0 \\ x^4 & 3x^3y & 7x^2y^2 & 11xy^3 & 21y^4 & 0 \\ x^5 & 3x^4y & 7x^3y^2 & 11x^2y^3 & 21xy^4 & 39y^5 \end{bmatrix}$$
$$= D_5' K_5(x,y).$$

3 SOME APPLICATIONS OF THE TRIBONACCI-LUCAS MATRIX $V_n(x,y)$

The following result gives the sum of squares of the first n Tribonacci-Lucas numbers.

Lemma 3.1 ([23]). For $n \geq 1$, the Tribonacci-Lucas numbers v_n satisfy

$$\sum_{k=1}^{n} v_k^2 = \frac{-v_{n+1}^2 - v_{n-1}^2 + v_{2n+3} + v_{2n-2} - 4}{2}.$$

Now, we consider a matrix whose Cholesky factorization includes the matrix $V_n(1,1)$.

Theorem 3.1. A matrix $Q_n = [c_{i,j}]$ with entries

$$c_{i,j} = \frac{2(-v_{k+1}^2 - v_{k-1}^2 + v_{2k+3} + v_{2k-2} - 4)}{(v_{i+2} + v_i - 6)(v_{j+2} + v_j - 6)},$$

where $k = \min\{i,j\}$, is a symmetric matrix and its Cholesky factorization is $V_n(1,1)V_n(1,1)^T$.

Proof. Since

$$c_{i,j} = \frac{2(-v_{k+1}^2 - v_{k-1}^2 + v_{2k+3} + v_{2k-2} - 4)}{(v_{i+2} + v_i - 6)(v_{j+2} + v_j - 6)} = c_{j,i}$$

the matrix Q_n is symmetric. We now show that $Q_n = V_n(1,1)V_n(1,1)^T$.

$$
\begin{aligned}
V_n(1,1)V_n(1,1)^T &= \sum_{k=0}^{n} v_{i,k}v_{j,k} = \sum_{k=0}^{n} \frac{2v_k}{v_{i+2}+v_i-6}\frac{2v_k}{v_{j+2}+v_j-6} \\
&= \frac{4}{(v_{i+2}+v_i-6)(v_{j+2}+v_j-6)}\sum_{k=0}^{n} v_k^2 \\
&= \frac{4}{(v_{i+2}+v_i-6)(v_{j+2}+v_j-6)}\frac{-v_{n+1}^2-v_{n-1}^2+v_{2n+3}+v_{2n-2}-4}{2} \\
&= \frac{2(-v_{k+1}^2-v_{k-1}^2+v_{2k+3}+v_{2k-2}-4)}{(v_{i+2}+v_i-6)(v_{j+2}+v_j-6)} \\
&= Q_n.
\end{aligned}
$$

Hence, we obtain the result.

For any square matrix M, the exponential of M is defined to be the matrix

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^k}{k!} + \cdots$$

Thus, we have the following result for a square matrix M.

Theorem 3.2 ([3, 29]). (i) For any numbers r and s, we have $e^{(r+s)M} = e^{rM}e^{sM}$.

(ii) $(e^M)^{-1} = e^{-M}$.

(iii) By taking the derivative with respect to x of each entry of e^{Mx}, we get the matrix $\frac{d}{dx}e^{Mx} = Me^{Mx}$.

In the last part of this section, we will give a relation between the matrix $V_n(x,y)$ and the exponential of a special matrix.

Definition 1. The matrix $M_n = [m_{i,j}]$ is defined by

$$m_{i,j} = \begin{cases} \frac{v_j}{v_i}, & \text{if } i = j+1, \\ 0, & \text{otherwise.} \end{cases} \quad \textbf{(4)}$$

We want to obtain a relation between $V_n(x,y)$ and $e^{M_n x}$, so we prove the following auxiliary result.

Lemma 3.2. For every nonnegative integer k, the entries of the matrix M_n^k are given by

$$(M_n^k)_{i,j} = \begin{cases} \frac{v_j}{v_i}, & \text{if } i = j + k \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.3. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$(V_n^{-1}(0,1)V_n(x,1))_{i,j} = (i-j)!\,(e^{M_n x})_{i,j}.$$

Proof. Suppose that there is a matrix Y_n such that $(V_n^{-1}(0,1)V_n(x,1))_{i,j} = (i-j)!\,(e^{M_n x})_{i,j}$. Then we have

$$\frac{d}{dx}(V_n^{-1}(0,1)V_n(x,1))_{i,j} = Y_n(i-j)(e^{Y_n x})_{i,j} = Y_n(V_n^{-1}(0,1)V_n(x,1))_{i,j}$$

and so

$$\frac{d}{dx}(V_n^{-1}(0,1)V_n(x,1))_{i,j}\Big|_{x=0} = Y_n.$$

Thus, there is at most one matrix Y_n such that $(V_n^{-1}(0,1)V_n(x,1))_{i,j} = (i-j)!\,(e^{Y_n x})_{i,j}$. It can be easily seen that $Y_n = M_n$, where M_n is the matrix given Definition 1, by calculating $\frac{d}{dx}(V_n^{-1}(0,1)V_n(x,1))_{i,j}\Big|_{x=0}$. We conclude that $M_n^k = 0$ for $n+1 \leq k$, thus

$$e^{M_n x} = \sum_{k=0}^{n} M_n^k \frac{x^k}{k!}.$$

For $i < j$, we see that $(e^{M_n x})_{i,j} = 0$ and we also have $(e^{M_n x})_{i,i} = 1$. Now, suppose that $i > j$ and let $i = j + k$

$$(e^{M_n x})_{i,j} = (M_n^k)_{i,j}\frac{x^k}{k!} = \frac{v_j}{v_{j+k}}\frac{x^k}{k!} = \frac{1}{k!}(V_n^{-1}(0,1)V_n(x,1))_{i,j}.$$

Example 5. We obtain the matrix $\frac{d}{dx}(V_5^{-1}(0,1)V_5(x,1))$ by taking the derivative of each entry of the matrix $V_5^{-1}(0,1)V_5(x,1)$ with respect to x. Thus,

$$\frac{d}{dx}(V_5^{-1}(0,1)V_5(x,1)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{7}x & \frac{3}{7} & 0 & 0 & 0 & 0 \\ \frac{3}{11}x^2 & \frac{6}{11}x & \frac{7}{11} & 0 & 0 & 0 \\ \frac{4}{21}x^3 & \frac{9}{21}x^2 & \frac{14}{21}x & \frac{11}{21} & 0 & 0 \\ \frac{5}{39}x^4 & \frac{12}{39}x^3 & \frac{21}{39}x^2 & \frac{22}{39}x & \frac{21}{39} & 0 \end{bmatrix},$$

Hence, we have

$$M_5 = V_5^{-1}(0,1)\frac{d}{dx}V_5(x,1)\Big|_{x=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{21}{39} & 0 \end{bmatrix}$$

and

$$M_5^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{39} & 0 & 0 \end{bmatrix}$$

$$M_5^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{39} & 0 & 0 & 0 \end{bmatrix}$$

$$M_5^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{39} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_5^5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{39} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let M_n be the matrix defined in (4) and $U_n(x) = e^{M_n x}$. At the end of this section, we will find the explicit inverse of the matrix $R_n(x) = [I_n - \lambda U_n(x)]^{-1}$ for a real number λ such that $|\lambda| < 1$. To achieve this, we need the following result.

Lemma 3.3 ([15], Corollary 5.6.16). A matrix A of order n is nonsingular if there is a matrix norm $\|\cdot\|$ such that $\| I - A \| < 1$. If this condition is satisfied,

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k.$$

Theorem 3.4. The matrix $R_n(x)$ is defined for real number λ such that $|\lambda| < 1$. The entries of the matrix are

$$(R_n(x))_{i,i} = \frac{1}{1-\lambda}$$

and

$$(R_n(x))_{i,i} = (U_n(x))_{i,j} Li_{j-i}(\lambda),$$

for $i > j$, where $Li_n(z)$ is the polylogarithm function

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

Proof. By Lemma 3.3, for $|\lambda| < 1$, we have

$$(R_n(x))_{i,i} = \sum_{k=0}^{\infty} (U_n(x))^k \lambda^k = \sum_{k=0}^{\infty} (U_n(xk))_{i,j} \lambda^k = (U_n(x))_{i,j} \sum_{k=0}^{\infty} \lambda^k k^{i-j}.$$

We get the result by writing the sum for $i = j$ and $i > j$.

Example 6.

$$I_4 - \lambda U_4(x) = I_4 - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{x}{3} & 1 & 0 & 0 & 0 \\ \frac{x^2}{14} & \frac{3x}{7} & 1 & 0 & 0 \\ \frac{x^3}{66} & \frac{3x^2}{22} & \frac{7x}{11} & 1 & 0 \\ \frac{x^4}{528} & \frac{3x^3}{132} & \frac{7x^2}{44} & \frac{11x}{22} & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ \frac{-\lambda x}{3} & 1-\lambda & 0 & 0 & 0 \\ \frac{-\lambda x^2}{14} & \frac{-3\lambda x}{7} & 1-\lambda & 0 & 0 \\ \frac{-\lambda x^3}{66} & \frac{-3\lambda x^2}{22} & \frac{-7\lambda x}{11} & 1-\lambda & 0 \\ \frac{-\lambda x^4}{528} & \frac{-3\lambda x^3}{132} & \frac{-7\lambda x^2}{44} & \frac{-11\lambda x}{22} & 1-\lambda \end{bmatrix}.$$

The inverse of this matrix equals

$$\begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0 & 0 \\ \frac{\lambda x}{3(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 & 0 & 0 \\ \frac{(\lambda+\lambda^2)x^2}{14(1-\lambda)^3} & \frac{3\lambda x}{7(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 & 0 \\ \frac{(\lambda+4\lambda^2+\lambda^3)x^3}{66(1-\lambda)^4} & \frac{(\lambda+\lambda^2)3x^2}{22(1-\lambda)3} & \frac{7\lambda x}{11(1-\lambda)^2} & \frac{1}{1-\lambda} & 0 \\ \frac{(\lambda+11\lambda^2+11\lambda^3+\lambda^4)x^4}{528(1-\lambda)^5} & \frac{(\lambda+4\lambda^2+\lambda^3)3x^3}{132(1-\lambda)^4} & \frac{(\lambda+\lambda^2)7x^2}{44(1-\lambda)^3} & \frac{11\lambda x}{22(1-\lambda)^2} & \frac{1}{1-\lambda} \end{bmatrix}.$$

Conflict of Interest

There is no conflict of interest between the authors.

Authors contributions

All authors contributed equally.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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