http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 486–495 (2024)
DOI:10.31801/cfsuasmas.1377811
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:October 18, 2023; Accepted: March 18, 2024

COMPLEX DEFORMABLE CALCULUS

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Abstract. In this paper, we give a new complex deformable derivative and integral of order λ which coincides with the classical derivative and integral for the special values of the parameters. We examine the basic properties of this derivative and integral. We also investigate the basic concepts of complex analysis for the λ-complex deformable derivative. Finally, we give some applications.

1. Introduction

The derivative of a complex-valued function is defined as a certain limit, similar to the derivative of real-valued functions. The official definition is

$$f'(\zeta_0) = \lim_{\varepsilon \to 0} \frac{f(\zeta_0 + \varepsilon) - f(\zeta_0)}{\varepsilon}. \tag{1}$$

For this limit to exist, it must be equal to the same complex number from any direction. Therefore, the differentiability of a complex-valued function at a point is more complex than the differentiability of a real-valued function at a point. See [1, 7].

The fractional derivative emerged in 1965 when Leibniz asked L'Hospital what does it mean derivative of order 1/2 [3]. Since then, the fractional derivative has attracted the attention of many researchers. Many fractional derivatives have been defined so far. An integral form was generally used in these definitions. Fractional derivatives, introduced by mathematicians such as Caputo, Riemann-Liouville, Hadamard, Riesz, and Grunwald-Letnikov, are the most popular of the fractional derivatives. See [4–6] for more information on the fractional derivative.

2020 *Mathematics Subject Classification.* 30D05, 30D20, 39B32.
Keywords. Complex deformable derivative, complex deformable integral, complex deformable partial derivative, λ-deformable Cauchy-Riemann equations, λ-deformable analytic.

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In 2014, Khlalil et al. [2] presented a limit-based definition of the fractional derivative for real-valued functions, similar to the standard derivative definition. Recently, Zulfeqarr et al. [8] introduced a new derivative called the deformable derivative, similar to Khalil's definition of the derivative. The conformable derivative is defined for functions whose domain is zero and positive numbers. Therefore, this derivative definition lacks to include negative numbers. Deformable derivative overcomes this deficiency in the conformable derivative.

The aim of this study is to introduce the λ-complex deformable derivative. In the second section, we give the relationship between λ-complex differentiability and complex differentiability. In the third section, we investigate the fundamental properties of the λ-complex deformable derivative. On the other hand, we examine the fundamental concepts of complex analysis according to this derivative. In the fourth section, we introduce the deformable integral operator for complex functions and give some of its properties. In the last section, we give some applications.

2. Complex Deformable Derivative

We first give the λ-complex deformable derivative definition.

Definition 1. *Let f be a complex-valued function and $0 \leq \lambda \leq 1$. Then the λ-complex deformable derivative is defined by*

$$D^{\lambda} f(z) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(z+\varepsilon\lambda) - f(z)}{\varepsilon} \tag{2}$$

where $\lambda + \delta = 1$.

Note that this definition is compatible with $\lambda = 0$ and $\lambda = 1$. Because, if $\lambda = 0$, we get $D^0 f(z) = f(z)$, and if $\lambda = 1$, we get $Df(z) = f'(z)$. In this study, we assume that $0 < \lambda \leq 1$ unless otherwise stated.

The first result implies a relationship between the complex differentiability and the λ-complex deformable differentiability.

Theorem 1. *A complex differentiable f at $\zeta_0 \in \mathbb{C}$ is always λ-complex deformable differentiable at that point for any λ. Moreover, we have*

$$D^{\lambda} f(\zeta_0) = \delta f(\zeta_0) + \lambda D f(\zeta_0), \ \text{where } \lambda + \delta = 1. \tag{3}$$

Proof. By definition, we have

$$\begin{aligned} D^{\lambda} f(\zeta_0) &= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(\zeta_0 + \lambda\varepsilon) - f(\zeta_0)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \left(\frac{f(\zeta_0 + \lambda\varepsilon) - f(\zeta_0)}{\varepsilon} + \delta f(\zeta_0 + \lambda\varepsilon) \right) \\ &= \lambda D f(\zeta_0) + \delta \lim_{\varepsilon \to 0} f(\zeta_0 + \lambda\varepsilon). \end{aligned}$$

Since f is differentiable at ζ_0, it is continuous at ζ_0. Hence, $\lim_{\varepsilon \to 0} f(\zeta_0 + \lambda\varepsilon)$ exist. Thus, the proof is complete. □

We know that a differentiable function f is continuous. The following result gives a similar result for λ-complex deformable differentiable functions.

Theorem 2. *If f is λ-complex deformable differentiable at a point ζ_0, then f is continuous at ζ_0.*

Proof. By hypothesis, the limits $\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}$ and $\lim_{\varepsilon\to 0}\varepsilon$ exist and equal $D^{\lambda}f(\zeta_0)$ and 0, respectively. Hence, we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}((1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)) &= \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}\varepsilon\\ &= \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}\lim_{\varepsilon\to 0}\varepsilon\\ &= D^{\lambda}f(\zeta_0).0=0.\end{aligned}$$

Since $\lim_{\varepsilon\to 0}(1+\varepsilon\delta)=1$, we get $\lim_{\varepsilon\to 0}f(\zeta_0+\varepsilon\lambda)=f(\zeta_0)$. Thus, f is continuous at ζ_0.

□

Theorem 3. *Let f be λ-complex deformable differentiable function at ζ_0. Then f is differentiable at ζ_0.*

Proof. By the description of complex differentiability

$$\begin{aligned}Df(\zeta_0) &= \frac{1}{\lambda}\lim_{\varepsilon\to 0}\frac{f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}\\ &= \frac{1}{\lambda}\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)-\varepsilon\delta f(\zeta_0+\varepsilon\lambda)}{\varepsilon}\\ &= \frac{1}{\lambda}\left(\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda)-f(\zeta_0)}{\varepsilon}-\delta\lim_{\varepsilon\to 0}f(\zeta_0+\varepsilon\lambda)\right).\end{aligned}$$

Since the function f is λ-complex deformable differentiable at point ζ_0, it is continuous at the same point. Thus, the proof is complete. □

Theorem 4. *A function f is λ-complex deformable differentiable at ζ_0 if and only if it is differentiable at ζ_0.*

Definition 2. *Suppose that f is an m-times differentiable at ζ_0. For $\lambda\in(m,m+1]$, λ-complex deformable differentiable at ζ_0 is defined as*

$$D^{\lambda}f(\zeta_0)=\lim_{\varepsilon\to 0}\frac{(1+\varepsilon\{\delta\})D^m f(\zeta_0+\varepsilon\{\lambda\})-D^m f(\zeta_0)}{\varepsilon}$$

where $\{\lambda\}$ is the fractional part of λ and $\{\lambda\}$+$\{\delta\}$=1.

By the above definition, if f is (m+1)-times differentiable, we get

$$D^{\lambda}f(\zeta_0)=\{\delta\}D^m f(\zeta_0)+\{\lambda\}D^{m+1}f(\zeta_0).$$

3. Basic Properties of Complex Deformable Derivative

In this part, we investigate certain properties of λ-complex deformable derivative.

Theorem 5. *The operator D^λ provides the following properties:*

(a) $D^\lambda(\alpha f(z) + \beta g(z)) = \alpha D^\lambda f(z) + \beta D^\lambda g(z)$ *(Linearity),*
(b) $D^{\lambda_1}.D^{\lambda_2} = D^{\lambda_2}.D^{\lambda_1}$. *(Commutativity),*
(c) *For any constant c, $D^\lambda(c) = \delta c$,*
(d) $D^\lambda(fg)(z) = (D^\lambda f(z))g(z) + \lambda f(z)Dg(z)$.

Theorem 6. *The operator D^λ possesses the following property*

$$D^\lambda\left(\frac{f}{g}\right)(z) = \frac{g(z)D^\lambda(f(z)) - \lambda f D(g(z))}{g^2(z)}.$$

Proof. We have

$$\begin{aligned}
D^\lambda\left(\frac{f}{g}\right)(z) &= \delta\left(\frac{f(z)}{g(z)}\right) + \lambda D\left(\frac{f(z)}{g(z)}\right) \\
&= \delta\left(\frac{f(z)}{g(z)}\right) + \lambda\left(\frac{(Df(z))g(z) - f(z)(Dg(z))}{g^2(z)}\right) \\
&= \frac{g(z)\left(\lambda Df(z) + \delta f(z)\right) - \lambda f(z)(Dg(z))}{g^2(z)} \\
&= \frac{g(z)D^\lambda f(z) - \lambda f(z)(Dg(z))}{g^2(z)}.
\end{aligned}$$

□

The following result gives the chain rule for the λ-complex deformable derivative.

Theorem 7. *Suppose f and g are λ-complex deformable differentiable at ζ_0. Then,*

$$D^\lambda(fog)(\zeta_0) = \delta(fog)(\zeta_0) + \lambda D(fog)(\zeta_0).$$

Proof. Since

$$D^\lambda f(\zeta_0) = \lim_{\varepsilon\to 0} \frac{(1+\varepsilon\delta)f(\zeta_0+\varepsilon\lambda) - f(\zeta_0)}{\varepsilon} = \lim_{\varepsilon\to 0}\left[\frac{f(\zeta_0+\varepsilon\lambda) - f(\zeta_0)}{\varepsilon} + \delta f(\zeta_0+\varepsilon\lambda)\right],$$

we have

$$\begin{aligned}
D^\lambda f(g(\zeta_0)) &= \lim_{\varepsilon\to 0}\left[\frac{f(g(\zeta_0+\varepsilon\lambda)) - f(g(\zeta_0))}{\varepsilon} + \delta f(g(\zeta_0+\varepsilon\lambda))\right] \\
&= \lim_{\varepsilon\to 0}\left[\frac{f(g(\zeta_0+\varepsilon\lambda)) - f(g(\zeta_0))}{g(\zeta_0+\varepsilon\lambda) - g(\zeta_0)}\,\frac{g(\zeta_0+\varepsilon\lambda) - g(\zeta_0)}{\varepsilon} + \delta f(g(\zeta_0+\varepsilon\lambda))\right] \\
&= \lim_{\varepsilon\to 0}\left[\frac{f(g(\zeta_0)+\varepsilon_0) - f(g(\zeta_0))}{\varepsilon_0}\,\frac{g(\zeta_0+\varepsilon\lambda) - g(\zeta_0)}{\varepsilon} + \delta f(g(\zeta_0+\varepsilon\lambda)\right],
\end{aligned}$$

where $\varepsilon_0 \to 0$ as $\varepsilon \to 0$. We obtain

$$\begin{aligned} D^\lambda f(g(\zeta_0)) &= \lim_{\varepsilon_0 \to 0} \frac{f(g(\zeta_0)+\varepsilon_0) - f(g(\zeta_0))}{\varepsilon_0} \lim_{\varepsilon \to 0} \frac{g(\zeta_0+\varepsilon\lambda) - g(\zeta_0)}{\varepsilon} + \lim_{\varepsilon \to 0} \delta f(g(\zeta_0+\varepsilon\lambda)) \\ &= Df(g(\zeta_0))\lambda Dg(\zeta_0) + \delta f(g(\zeta_0)) \\ &= \lambda D[f(g(\zeta_0))] + \delta f(g(\zeta_0)). \end{aligned}$$

The proof is completed. □

Proposition 1.

(a) $D^\lambda(z^n) = \delta z^n + n\lambda z^{n-1}$, $n \in \mathbb{R}$.
(b) $D^\lambda(e^z) = e^z$.
(c) $D^\lambda(sinz) = \delta sinz + \lambda cosz$.
(d) $D^\lambda(logz) = \delta logz + \frac{\lambda}{z}$.

We now give the notion of real deformable partial derivatives.

Definition 3. *Suppose $f(x_1, x_2, ..., x_j)$ is real function. Then the formula for the partial derivative of f with respect to x_i is given by*

$$\frac{\partial^\lambda}{\partial x_i^\lambda} f(x_1, x_2, ..., x_j) = \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(x_1, ..., x_{i-1}, x_i + \lambda\varepsilon, ..., x_j) - f(x_1, x_2, ..., x_j)}{\varepsilon}. \quad (4)$$

$\frac{\partial^\lambda}{\partial x_i^\lambda} f$ can be also represented $f_{x_i}^{(\lambda)}$.

Theorem 8. *Let $f(z) = \mathfrak{u}(x,y) + i\mathfrak{v}(x,y)$ be an λ-complex deformable differentiable at $\zeta_0 = x_0 + iy_0$. Then the λ-complex deformable derivative of f*

$$D^\lambda f(\zeta_0) = \mathfrak{u}_x^{(\lambda)}(x_0, y_0) + i\mathfrak{v}_x^{(\lambda)}(x_0, y_0) = \mathfrak{v}_y^{(\lambda)}(x_0, y_0) - i\mathfrak{u}_y^{(\lambda)}(x_0, y_0). \quad (5)$$

Proof. Let $\varepsilon = a + ib$. For $b = 0$ and $a \to 0$, we get

$$\begin{aligned} D^\lambda f(\zeta_0) &= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(\zeta_0 + \varepsilon\lambda) - f(\zeta_0)}{\varepsilon} \\ &= \lim_{a \to 0} \left[\frac{(1+a\delta)\mathfrak{u}(x_0 + a\lambda, y_0) - \mathfrak{u}(x_0, y_0)}{a} + i \frac{(1+a\delta)\mathfrak{v}(x_0 + a\lambda, y_0) - \mathfrak{v}(x_0, y_0)}{a} \right] \\ &= \mathfrak{u}_x^{(\lambda)}(x_0, y_0) + i\mathfrak{v}_x^{(\lambda)}(x_0, y_0). \end{aligned}$$

For $a = 0$ and $b \to 0$, we get

$$\begin{aligned} D^\lambda f(\zeta_0) &= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon\delta) f(\zeta_0 + \varepsilon\lambda) - f(\zeta_0)}{\varepsilon} \\ &= \lim_{b \to 0} \left[\frac{(1+ib\delta)\mathfrak{u}(x_0, y_0 + ib\lambda) - \mathfrak{u}(x_0, y_0)}{ib} + i \frac{(1+ib\delta)\mathfrak{v}(x_0, y_0 + ib\lambda) - \mathfrak{v}(x_0, y_0)}{ib} \right] \\ &= \mathfrak{v}_y^{(\lambda)}(x_0, y_0) - i\mathfrak{u}_y^{(\lambda)}(x_0, y_0). \end{aligned}$$

Therefore, we have

$$D^\lambda f(\zeta_0) = \mathfrak{u}_x^{(\lambda)}(x_0, y_0) + i\mathfrak{v}_x^{(\lambda)}(x_0, y_0) = \mathfrak{v}_y^{(\lambda)}(x_0, y_0) - i\mathfrak{u}_y^{(\lambda)}(x_0, y_0).$$

□

Corollary 1. *Let $f(z) = \mathfrak{u}(x,y)+i\mathfrak{v}(x,y)$ be an λ-complex deformable differentiable at ζ_0. Then, $\mathfrak{u}(x,y)$ and $\mathfrak{v}(x,y)$ satisfy the λ-deformable Cauchy-Riemann equations as*

$$\mathfrak{u}_x^{(\lambda)} = \mathfrak{v}_y^{(\lambda)} \text{ and } \mathfrak{u}_y^{(\lambda)} = -\mathfrak{v}_x^{(\lambda)}. \tag{6}$$

The conversely of Corollary 1 is not always true. For example, consider the function

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

For $z = x + iy \neq 0$, we have

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \left(\frac{x - iy}{x + iy}\right)^2.$$

This limit is equal to 1 when approaching the origin along the real axis, and -1 when approaching along the $y = x$ line. Then the function f is not differentiable at $z = 0$. Therefore, f is not λ-complex deformable differentiable at $z = 0$. On the other hand, since

$$\mathfrak{u}_x^{(\lambda)}(0,0) = \lambda = \mathfrak{v}_y^{(\lambda)}(0,0)$$

and

$$\mathfrak{u}_y^{(\lambda)}(0,0) = 0 = -\mathfrak{v}_x^{(\lambda)}(0,0)$$

the λ-deformable Cauchy Riemann equations satisfy at $z = 0$.

Now we give the notion of an λ-deformable analytic function using a complex deformable derivative.

Definition 4. *The function that is λ-complex deformable differentiable at every point of an open set U is called to be λ-deformable analytic in U.*

Definition 5. *A function that is analytic at every point in the complex plane is called a λ-deformable entire function.*

Definition 6. *A mapping is called conformal at the point ζ_0 if it preserves the angles between pairs of regular curves intersecting at ζ_0.*

Theorem 9. *Let f be λ-deformable analytic in D. If $D^\lambda f(\zeta_0) \neq 0$ at $\zeta_0 \in D$, then f is conformal at ζ_0.*

4. Complex Deformable Integral

In this section, we introduce λ-complex deformable integral and examine some of its basic properties.

Definition 7. *Let C be a smooth curve given by the equation $z(t) = x(t) + iy(t)$, $a \leq t \leq b$. If the function $f(z)$ is defined and continuous on C, then $f(z(t))$ is also continuous and we can set*

$$I_C^\lambda f = \frac{1}{\lambda} e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z(t)} f(z(t)) z'(t) dt.$$

Definition 8. *Let $f : [a, b] \to \mathbb{C}$ be a continuous function. We define λ-complex deformable integral of f,*

$$I_a^\lambda f = \frac{1}{\lambda} e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z} f(z) dz. \tag{7}$$

Proposition 2. *The operator $I_a^\lambda f$ possesses the following properties:*

(a) $I_a^\lambda(bf + cg) = bI_a^\lambda f + cI_a^\lambda g$,

(b) $I_a^{\lambda_1} I_a^{\lambda_2} = I_a^{\lambda_2} I_a^{\lambda_1}$, *where* $\lambda_i + \delta_i = 1, i = 1, 2$.

Definition 9. *Let f be continuous on D. If there exists a function F such that $D^\lambda(F)(z) = f(z)$ for every z in D, then F is called an anti-λ-complex deformable derivative of f.*

We now give another version of the fundamental theorem of calculus.

Theorem 10. *Let a function f be continuous on a domain D. Then $I_a^\lambda f$ is λ-complex deformable differentiable in D.*

Proof. If we set $F = I_a^\lambda f$ then we have

$$D^\lambda(I_a^\lambda f(z)) = D^\lambda(F(z)) = \lambda DF(z) + \delta F(z).$$

Moreover, a particular solution of the differential equation $\lambda DF + \delta F = f$ is obtained as

$$F(z) = \frac{1}{\lambda} e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z} f(z) dz.$$

The proof is completed. □

Theorem 11. *Let a function f be continuous on D and F is a continuous anti-λ-complex deformable derivative of f in D. Then we have*

$$I_a^\lambda \left(D^\lambda f(z)\right) = I_a^\lambda \left(g(z)\right) = f(t) - e^{\frac{\delta}{\lambda}(a-z)} f(a).$$

Proof. Since

$$F(z) \;\; = \;\; \left(D^\lambda f(z)\right) = \lambda Df + \delta f$$

we get

$$\begin{aligned} I_a^\lambda F(z) &= \lambda I_a^\lambda Df(z) + \delta I_a^\lambda f(z) \\ &= e^{\frac{-\delta}{\lambda} z} \int_a^b e^{\frac{\delta}{\lambda} z} Df(z) dz + \delta I_a^\lambda f(z) \end{aligned}$$

$$= e^{\frac{-\delta}{\lambda}z}\left(\left[e^{\frac{\delta}{\lambda}z}Df(z)\right]_a^b - \frac{\delta}{\lambda}\int_a^b e^{\frac{\delta}{\lambda}z}f(z)dz\right) + \delta I_a^\lambda f(z)$$
$$= f(z) - e^{\frac{\delta}{\lambda}(a-z)}f(a).$$

The proof is completed. □

5. Applications to Differential Equations

The linear first-order λ-complex differential equation can be expressed in the form

$$D^\lambda w + p(z)w = q(z)$$

where $\omega = f(z)$ be a complex valued function and p(z) is continuous complex valued function. Using expression (3), we get

$$D\omega + \frac{\delta + p(z)}{\lambda}\omega = q(z).$$

Then commonly written as

$$\omega = \frac{1}{\mu(z)}\int \mu(z)q(z)dz + \frac{C}{\mu(z)},$$

with

$$\mu(z) = e^{\int \frac{\delta + p(z)}{\lambda}dz}$$

the integrating factor. Thus, we obtain the general solution of the linear first-order λ-complex deformable differential equation is given by

$$w = e^{\frac{-(\delta + \int P(z)dz)}{\lambda}}\int e^{\frac{(\delta + \int P(z)dz)}{\lambda}}q(z)dz + Ce^{\frac{-(\delta + \int P(z)dz)}{\lambda}} \tag{8}$$

where C is arbitrary complex constant.

Example 1. *Suppose $\omega = f(z)$ is λ-complex deformable differentiable function. Solve*

$$D^\lambda\omega + \omega = 0.$$

Using expression (3), we can write

$$D\omega + \left(\frac{\delta + 1}{\lambda}\right)\omega + \omega = 0.$$

Hence, by using an integrating factor

$$\mu(z) = e^{\int\left(\frac{\delta+1}{\lambda}\right)dz} = e^{\left(\frac{\delta+1}{\lambda}\right)z},$$

we have

$$\omega = Ce^{-\left(\frac{\delta+1}{\lambda}\right)z}$$

where C is an arbitrary complex constant.

Example 2. *Suppose $\omega = f(z)$ is λ-complex deformable differentiable function. Solve*

$$D^{\frac{1}{3}}\omega + \omega = e^{2z}.$$

Using expression (3), we can write

$$\frac{1}{3}D\omega + \frac{2}{3}\omega + \omega = e^{2z},$$

or equally

$$D\omega + 5\omega + \omega = 3e^{2z}.$$

Hence, by using an integrating factor

$$\mu(z) = e^{\int 5dz} = e^{5z},$$

we have

$$\begin{aligned} \omega &= 3e^{-5z}\int e^{7z}dz + Ce^{-5z} \\ &= \frac{3}{7}e^{2z} + Ce^{-5z} \end{aligned}$$

where C is an arbitrary complex constant.

Example 3. *Suppose $\omega = f(z)$ is λ-complex deformable differentiable function. Solve*

$$D^{\frac{1}{2}}\omega + \frac{1}{z}\omega = z^2\omega^2.$$

To solve the $\frac{1}{2}$-complex deformable differential equation, the expansion (3) is first used, and then both sides of the obtained equation are multiplied by ω^{-2}. Then we have

$$\omega^{-2}D\omega + \left(1 + \frac{2}{z}\right)\omega^{-1} = 2z^2.$$

By substituting $\omega^{-1} = \eta$ in the above equation, we get

$$D\eta - \left(1 + \frac{2}{z}\right)\eta = -2z^2.$$

Since the integrating factor of the last differential equation is

$$\mu(z) = e^{-z}z^{-2},$$

we find the general solutions

$$\eta = e^z z^2 \int e^{-z}z^{-2}(-2z^2)dz + Ce^z z^2 = z^2(2 + Ce^z)$$

where C is an arbitrary complex constant. Thus, its general solution may be expressed as

$$\omega = \frac{1}{z^2(2 + Ce^z)}.$$

Author Contribution Statements This manuscript is single-authored and the author has read and approved the final version of the article.

Declaration of Competing Interests The author declares that he has no known competing financial interests or personal relationships that could influence the work reported in this article.

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