Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 7
DOI : 10.15672/hujms.xx

Research Article

On the constant $C_{NJ}^{(p)}(X)$ for the generalized Banaś-Frączek space

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Abstract

In this paper, we study the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for the generalized Banaś-Frączek space and improve related results on the Banaś-Frączek space. The exact value of $C_{NJ}^{(p)}(X)$ will be calculated for X to be the generalized Banaś-Frączek space $\mathbb{R}^2_{a,b,p_1}$ in the case $p \geq 2$ such that $p_1 \geq p \geq 2$ or $p \geq p_1 \geq 1$.

Mathematics Subject Classification (2020). 46B20

Keywords. generalized Banaś-Frączek space, Banaś-Frączek space, generalized von Neumann-Jordan constant, von Neumann-Jordan constant

1. Introduction

The geometric constants have received widespread attention, for the reason that it can essentially reflect the geometric properties of a space. Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson [2] as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ with $\|x\|^2 + \|y\|^2 \neq 0$. Several studies on the constant $C_{NJ}(X)$ have been conducted by many authors (see, for example, [1, 4–13, 18, 19]), and they play an important role in the geometric theory of Banach spaces. Therefore the calculation of geometric constants for some concrete spaces is very important.

Recently, a generalized form of this constant was introduced as follows (see [3])

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}\left(\|x\|^p + \|y\|^p\right)} : \quad x, y \in X, (x, y) \neq (0.0)\right\},$$

where $p \geqslant 1$.

Now let us collect some properties of this constant (see [3, 14–16, 20, 21]).

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Received: 19.10.2023; Accepted: 06.03.2025

(i) $1 \leqslant C_{NJ}^{(p)}(X) \leqslant 2$;
(ii) X is uniformly non-square if and only if $C_{NJ}^{(p)}(x) < 2$.
(iii) Let $r \in (1,2]$ and $\frac{1}{r}+\frac{1}{r'}=1$. Then for $X = L_r[0.1]$, we have

$$C_{NJ}^{(p)}\left(L_r[0,1]\right) = \begin{cases} 2^{2-p}, & 1 \leqslant p \leqslant r; \\ 2^{\frac{p}{r}-p+1}, & r < p \leqslant r'; \\ 1, & r' < p \leqslant \infty; \end{cases}$$

(iv) If $p \geq 2$, then $C_{NJ}^{(p)}(X) \leqslant J(X)$.
(v) For the regular octagon space X, we have
(a) If $p \geq 2$, then $C_{NJ}^{(p)}(X) = 1 + (\sqrt{2}-1)^p$;
(b) If $1 < p \leq 2$, then $C_{NJ}^{(p)}(X) = 2^{2-p}\left[1+(\sqrt{2}-1)^{\frac{p}{p-1}}\right]^{p-1}$.

Recently, C. Yang et al. introduced the Banaś-Frączek space $X_{\lambda,p}$ in [17], i.e., $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{\lambda,p} = \max\{\lambda|x|, \|(x,y)\|_p\},$$

where $\lambda > 1$ and $p \geq 1$. In [17], they showed that if $p \geq 2$ and $(\lambda^p-1)^{p-2}(\lambda^2-1)^p \geq 1$, then

$$C_{NJ}(X_{\lambda,p}) = 1 + (1-\frac{1}{\lambda^p})^{\frac{2}{p}}.$$

Also, Mitani and Stito and Takahashi introduced generalized Banaś-Frączek space $\mathbb{R}^2_{a,b,p}$ which is $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\| = \max\{a|x|, b|y|, \|(x,y)\|_p\}.$$

And for $a > 1$ and $a \geq b \geq 1$ with $a^{-p}+b^{-p} > 1$, they consider the $C_{NJ}(X)$ for this space in [8, 9] as follows:

(i) If $p \geq 2$ and $b \leq a(a^p-1)^{\frac{p-2}{2p}}$, then

$$C_{NJ}(\mathbb{R}^2_{a,b,p}) = 1 + b^2(1-a^{-p})^{\frac{2}{p}}. \tag{1.1}$$

(ii) If $p \geq 2$ and $b \geq a(a^p-1)^{\frac{p-2}{2p}}$, then

$$C_{NJ}(\mathbb{R}^2_{a,b,p}) = b^2\left(1+\left(\frac{b}{a}\right)^{\frac{2p}{2-p}}\right)^{1-\frac{2}{p}}. \tag{1.2}$$

(iii) If $1 \leq p < 2$ and $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$, then

$$C_{NJ}(\mathbb{R}^2_{a,b,p}) = 1 + b^2(1-a^{-p})^{\frac{2}{p}}. \tag{1.3}$$

In this paper, we consider the constant $C_{NJ}^{(p)}(X)$ for the generalized Banaś-Frączek space and obtain the following results:

(1) For $a > 1, a \geq b \geq 1$ and $p_1 \geq p \geq 2$ with $a^{-p_1}+b^{-p_1} > 1$.
(i) If $b \leq a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, then

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = 1 + b^p(1-a^{-p_1})^{\frac{p}{p_1}}.$$

(ii) If $b \geq a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, then

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = b^p\left(1+\left(\frac{b}{a}\right)^{\frac{pp_1}{p-p_1}}\right)^{1-\frac{p}{p_1}}.$$

(2) For $a > 1, a \geq b \geq 1, p \geq 2$ and $p \geq p_1 \geq 1$ with $a^{-p_1} + b^{-p_1} > 1$. If $a^{\frac{pp_1}{p_1-p}} + b^{\frac{pp_1}{p_1-p}} \leq 1$, then

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = 1 + b^p(1 - a^{-p_1})^{\frac{p}{p_1}}.$$

2. Main results

Recall that the norm $||\cdot||$ on $\mathbb{R}^2$ is said to be absolute if $||(x,y)|| = ||(|x|,|y|)||$ for each $(x,y) \in \mathbb{R}^2$. Before describing the main results, we give some lemmas.

Lemma 2.1. *Let $p \geq 2$ and $\|\cdot\|$, $\|\cdot\|_\sim$ be two absolute norms on $\mathbb{R}^2$ satisfying the following conditions:*
(i) $\|u+v\|_\sim^p + \|u-v\|_\sim^p \leq 2^{p-1}(\|u\|_\sim^p + \|v\|_\sim^p)$ for any $u, v \in \mathbb{R}^2$.
(ii) $\|(x,y)\|_\sim^p = \|(x,0)\|_\sim^p + \|(0,y)\|_\sim^p$ for any $x, y \in \mathbb{R}$.
(iii) $\|(x,y)\| \leq \|(x,y)\|_\sim$ for any $x, y \in \mathbb{R}$.
(iv) $\|(1,0)\| = \|(1,0)\|_\sim$ and $\|(0,1)\| = \|(0,1)\|_\sim$.
Then

$$C_{NJ}^{(p)}((\mathbb{R}^2, \|\cdot\|)) = \beta^p, \text{ where } \beta = \max\left\{\frac{\|(x,y)\|_\sim}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)\right\}.$$

Proof. By $\|\cdot\| \leq \|\cdot\|_\sim \leq \beta\|\cdot\|$ and (i), we have

$$\begin{aligned} C_{NJ}^{(p)}((\mathbb{R}^2, \|\cdot\|)) &= \sup\left\{\frac{\|u+v\|^p + \|u-v\|^p}{2^{p-1}(\|u\|^p + \|v\|^p)} : u, v \in \mathbb{R}^2, (u,v) \neq (0,0)\right\} \\ &\leq \beta^p \sup\left\{\frac{\|u+v\|_\sim^p + \|u-v\|_\sim^p}{2^{p-1}(\|u\|_\sim^p + \|v\|_\sim^p)} : u, v \in \mathbb{R}^2, (u,v) \neq (0,0)\right\} \\ &\leq \beta^p. \end{aligned}$$

On the other hand, for any positive number ε, we can take $(x_0, y_0) \in \mathbb{R}^2$ such that $\frac{\|(x_0,y_0)\|_\sim}{\|(x_0,y_0)\|} > \beta - \varepsilon$. Thus by (ii),

$$\begin{aligned} C_{NJ}^{(p)}((\mathbb{R}^2, \|\cdot\|)) &\geq \frac{\|(2x_0,0)\|^p + \|(0,2y_0)\|^p}{2^{p-1}(\|(x_0,y_0)\|^p + \|(x_0,-y_0)\|^p)} \\ &= \frac{\|(x_0,0)\|_\sim^p + \|(0,y_0)\|_\sim^p}{\|(x_0,y_0)\|_\sim^p} \frac{\|(x_0,y_0)\|_\sim^p}{\|(x_0,y_0)\|^p} \\ &\geq (\beta - \varepsilon)^p. \end{aligned}$$

Hence, $C_{NJ}^{(p)}((\mathbb{R}^2, \|\cdot\|)) = \beta^p$.

□

If $a > 1, a \geq b \geq 1$ and $a^{-p_1} + b^{-p_1} \leq 1$, by taking $u = \left(\frac{1}{a}, \frac{1}{b}\right)$ and $v = \left(\frac{1}{a}, -\frac{1}{b}\right)$, then

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) \geq \frac{\|u+v\|^p + \|u-v\|^p}{2^{p-1}(\|u\|^2 + \|v\|^2)} = 2.$$

Thus, $C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = 2$. Therefore, we only to consider the case $a^{-p_1} + b^{-p_1} > 1$.

Lemma 2.2. *Let $a > 1, a \geq b \geq 1$ and $p_1 \geq p \geq 1$ with $a^{-p_1} + b^{-p_1} > 1$. If $f(t) = \frac{(a^pt^p+b^p)^{\frac{1}{p}}}{(1+t^{p_1})^{\frac{1}{p_1}}}$, $t_1 = (a^{p_1}-1)^{-\frac{1}{p_1}}, t_2 = (\frac{a}{b})^{\frac{p}{p_1-p}}$ and $t_3 = (b^{p_1}-1)^{\frac{1}{p_1}}$, then*

(i) f is non-decreasing on $(0, t_2)$ and is non-increasing on (t_2, ∞). Hence f has the maximum at t_2.

(ii) $f(t_3) \le f(t_1)$.

(iii) If $b \le a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, *then* $t_3 \le t_1 \le t_2$ *and*

$$\max\{f(t): t_3 \le t \le t_1\} = f(t_1) = \left(1 + b^p(1-a^{-p_1})^{\frac{p}{p_1}}\right)^{\frac{1}{p}}.$$

(iv) If $b \ge a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, *then* $t_3 \le t_2 \le t_1$ *and*

$$\max\{f(t): t_3 \le t \le t_1\} = f(t_2) = b\left(1 + \left(\frac{b}{a}\right)^{\frac{pp_1}{p-p_1}}\right)^{\frac{1}{p}-\frac{1}{p_1}}.$$

Proof. (i) Since $f'(t) = (a^pt^p + b^p)^{\frac{1}{p}-1}(1+t^{p_1})^{-\frac{1}{p_1}-1}t^{p-1}[a^p - b^pt^{p_1-p}]$, we have (i).

(ii) It is easy to see that

$$f(t_1) = \left(1 + \left(\frac{b}{a}\right)^p (a^{p_1}-1)^{\frac{p}{p_1}}\right)^{\frac{1}{p}}$$

and

$$f(t_3) = \left(1 + \left(\frac{a}{b}\right)^p (b^{p_1}-1)^{\frac{p}{p_1}}\right)^{\frac{1}{p}}.$$

Hence $f(t_3) \le f(t_1)$ if and only if $\frac{b}{a}(a^{p_1}-1)^{\frac{1}{p_1}} \ge \frac{a}{b}(b^{p_1}-1)^{\frac{1}{p_1}}$, that is $(b^{p_1}-a^{p_1})(a^{p_1}+b^{p_1}-a^{p_1}b^{p_1}) \le 0$. Thus it follow from $a \ge b$ and $a^{-p_1}+b^{-p_1} > 1$.

(iii) Since $b \le a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, we obtain $t_1 \le t_2$. By $a^{-p_1}+b^{-p_1} > 1$, we can get $(a^{p_1}-1)(b^{p_1}-1) < 1$, and this implies $t_3 < t_1$. Thus it follows from (i) that (iii) is valid.

(iv) From $b \ge a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, we see that $t_1 \ge t_2$ and $a \ge b \ge a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$. Thus, $b^{p_1}-1 \le a^{p_1}-1 \le 1$ and $t_3 \le 1 \le t_2 \le t_1$. Hence (iv) can also be obtained from (i). □

Theorem 2.3. *$a > 1, a \ge b \ge 1$ and $p_1 \ge p \ge 2$ with $a^{-p_1}+b^{-p_1} > 1$.*

(i) If $b \le a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, *then*

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = 1 + b^p(1-a^{-p_1})^{\frac{p}{p_1}}. \tag{2.1}$$

(ii) If $b \ge a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, *then*

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = b^p\left(1 + \left(\frac{b}{a}\right)^{\frac{pp_1}{p-p_1}}\right)^{1-\frac{p}{p_1}}. \tag{2.2}$$

Proof. We define the norm $\|\cdot\|_\sim$ on $\mathbb{R}^2$ by $\|(x,y)\|_\sim = \|(ax,by)\|_p$.

By Clarkson's inequality, for any $u, v \in \mathbb{R}^2$, we have

$$\|u+v\|_\sim^p + \|u-v\|_\sim^p \le 2^{p-1}(\|u\|_\sim^p + \|v\|_\sim^p).$$

Also, $\|(x,y)\|_\sim^p = a^p|x|^p + b^p|y|^p = \|(x,0)\|_\sim^p + \|(0,y)\|_\sim^p$ for any $x, y \in \mathbb{R}$. It is clear that $\|(x,y)\| \le \|(x,y)\|_\sim$ for any $(x,y) \in \mathbb{R}^2$ by $a \ge b \ge 1$ and $p_1 \ge p \ge 1$.

Put $\beta = \max\left\{\frac{\|(x,y)\|_\sim}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2, (x,y) \ne (0,0)\right\}$.

(i) If $b \le a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, then we can show that $\beta = f(t_1)$.

In the case where $x = 0$ or $y = 0$, since $f(t_1) \ge 1$, we can assume that $x \ne 0$ and $y \ne 0$. Put $t = \frac{|x|}{|y|}$. We first consider the case $\|(x,y)\| = a|x|$. Since $a|x| \ge \|(x,y)\|_{p_1}$, we have

$t \geq (a^{p_1}-1)^{-\frac{1}{p_1}} = t_1$. Hence

$$\frac{\|(x,y)\|_\sim}{\|(x,y)\|} = \frac{\|(ax,by)\|_p}{a|x|} = \|(1,\frac{b}{at})\|_p \leq \|(1,\frac{b}{at_1})\|_p = f(t_1).$$

Next, we consider the case $\|(x,y)\| = b|y|$. Since $b|y| \geq \|(x,y)\|_{p_1}$, we have $t \leq (b^{p_1}-1)^{\frac{1}{p_1}} = t_3$. Hence by Lemma 2.2

$$\frac{\|(x,y)\|_\sim}{\|(x,y)\|} = \frac{\|(ax,by)\|_p}{b|y|} = \|(\frac{at}{b},1)\|_p \leq \|(\frac{at_3}{b},1)\|_p = f(t_3) \leq f(t_1).$$

Finally we consider the case $\|(x,y)\| = \|(x,y)\|_{p_1}$. Since $\|(x,y)\|_{p_1} \geq a|x|$ and $\|(x,y)\|_{p_1} \geq b|y|$, it follows that $t_3 \leq t \leq t_1$. Then by Lemma 2.2, we also have

$$\frac{\|(x,y)\|_\sim}{\|(x,y)\|} = \frac{\|(ax,by)\|_p}{\|(x,y)\|_{p_1}} = f(t) \leq f(t_1).$$

Thus, $\beta \leq f(t_1)$. Moreover, we have $\frac{\|(x,y)\|_\sim}{\|(x,y)\|} = f(t_1)$ for $(x,y) = (t_1,1)$. Hence $\beta = f(t_1)$ and (i) is valid by Lemma 2.1.

(ii) If $b \geq a(a^{p_1}-1)^{\frac{p_1-p}{pp_1}}$, similar to (i), then we can get $\beta = f(t_2)$. Hence (ii) is also valid by Lemma 2.1.

□

Remark 2.4. By taking $p=2$ and $p_1=p$ in Theorem 2.3, we can see that (2.1) and (2.2) can imply (1.1) and (1.2) respectively.

Next, we consider the case $p \geq p_1$.

Lemma 2.5. *Let $a > 1, a \geq b \geq 1$ and $p \geq p_1 \geq 1$ with $a^{-p_1}+b^{-p_1} > 1$. If $f(t) = \frac{(a^pt^p+b^p)^{\frac{1}{p}}}{(1+t^{p_1})^{\frac{1}{p_1}}}$, $t_1 = (a^{p_1}-1)^{-\frac{1}{p_1}}, t_2 = (\frac{a}{b})^{\frac{p}{p_1-p}}$ and $t_3 = (b^{p_1}-1)^{\frac{1}{p_1}}$, then*

(i) f is non-increasing on $(0,t_2)$ and is non-decreasing on (t_2,∞). Hence f has the minimum at t_2.

(ii) $f(t_3) \leq f(t_1)$.

Proof. (i) Since $f'(t) = (a^pt^p+b^p)^{\frac{1}{p}-1}(1+t^{p_1})^{-\frac{1}{p_1}-1}t^{p_1-1}[a^pt^{p-p_1}-b^p]$, we have (i).

(ii) The same as the proof of (ii) in Lemma 2.2.

□

Theorem 2.6. *$a > 1, a \geq b \geq 1, p \geq 2$ and $p \geq p_1 \geq 1$ with $a^{-p_1}+b^{-p_1} > 1$. If $a^{\frac{pp_1}{p_1-p}} + b^{\frac{pp_1}{p_1-p}} \leq 1$, then*

$$C_{NJ}^{(p)}(\mathbb{R}^2_{a,b,p_1}) = 1 + b^p(1-a^{-p_1})^{\frac{p}{p_1}}. \tag{2.3}$$

Proof. We define the norm $\|\cdot\|_\sim$ on $\mathbb{R}^2$ by $\|(x,y)\|_\sim = \|(ax,by)\|_p$.

By Clarkson's inequality, for any $u,v \in \mathbb{R}^2$, we have

$$\|u+v\|_\sim^p + \|u-v\|_\sim^p \leq 2^{p-1}(\|u\|_\sim^p + \|v\|_\sim^p).$$

Also, $\|(x,y)\|_\sim^p = a^p|x|^p + b^p|y|^p = \|(x,0)\|_\sim^p + \|(0,y)\|_\sim^p$ for any $x,y \in \mathbb{R}$.

By Hölder's inequality, we have

$$\|(x,y)\|_{p_1}^{p_1} = |ax|^{p_1}a^{-p_1} + +|by|^{p_1}b^{-p_1} \leq (|ax|^p+|by|^p)^{\frac{p_1}{p}}(a^{\frac{pp_1}{p_1-p}} + b^{\frac{pp_1}{p_1-p}})^{1-\frac{p_1}{p}},$$

which implies $\|(x,y)\| \leq \|(x,y)\|_\sim$ for any $(x,y) \in \mathbb{R}^2$ by $a^{\frac{pp_1}{p_1-p}} + b^{\frac{pp_1}{p_1-p}} \leq 1$.

Put $\beta = \max\left\{\frac{\|(x,y)\|_\sim}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)\right\}$.

Firstly, we show that $\beta \leq f(t_1)$.

In the case where $x=0$ or $y=0$, since $f(t_1)\geq 1$, we can assume that $x\neq 0$ and $y\neq 0$. Put $t=\frac{|x|}{|y|}$. We first consider the case $\|(x,y)\|=a|x|$. Since $a|x|\geq \|(x,y)\|_{p_1}$, we have $t\geq (a^{p_1}-1)^{-\frac{1}{p_1}}=t_1$. Hence

$$\frac{\|(x,y)\|_{\sim}}{\|(x,y)\|}=\frac{\|(ax,by)\|_p}{a|x|}=\|(1,\frac{b}{at})\|_p\leq \|(1,\frac{b}{at_1})\|_p=f(t_1).$$

Next, we consider the case $\|(x,y)\|=b|y|$. Since $b|y|\geq \|(x,y)\|_{p_1}$, we have $t\leq (b^{p_1}-1)^{\frac{1}{p_1}}=t_3$. Hence by Lemma 2.5

$$\frac{\|(x,y)\|_{\sim}}{\|(x,y)\|}=\frac{\|(ax,by)\|_p}{b|y|}=\|(\frac{at}{b},1)\|_p\leq \|(\frac{at_3}{b},1)\|_p=f(t_3)\leq f(t_1).$$

Finally, we consider the case $\|(x,y)\|=\|(x,y)\|_{p_1}$. Since $\|(x,y)\|_{p_1}\geq a|x|$ and $\|(x,y)\|_{p_1}\geq b|y|$, it follows that $t_3\leq t\leq t_1$. Then by Lemma 2.5, we also have

$$\frac{\|(x,y)\|_{\sim}}{\|(x,y)\|}=\frac{\|(ax,by)\|_p}{\|(x,y)\|_{p_1}}=f(t)\leq f(t_1).$$

Thus, $\beta\leq f(t_1)$. Moreover, we have $\frac{\|(x,y)\|_{\sim}}{\|(x,y)\|}=f(t_1)$ for $(x,y)=(t_1,1)$. Hence $\beta=f(t_1)$ and (2.3) is valid by Lemma 2.1.

□

Remark 2.7. By taking $p=2$ and $p_1=p$ in Theorem 2.6, we can see that (2.3) can imply (1.3).

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This work was supported by the National Natural Science Foundation of China (Nos. 12471130; 12371139; 11771126).

Data availability. No data was used for the research described in the article.

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