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Partial Soft Derivative

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Abstract
The concept of soft derivative, introduced by Molodtsov in 1999, is one of the fundamental concepts in soft analysis. The handled paper defines partial soft derivative and studies some of its basic properties, such as the relation between partial soft derivative and boundedness, some basic partial soft derivative rules, e.g., sum rule, constant multiple rule, and difference rule, the relation between soft derivative and partial soft derivative, the relation between classical partial derivative and partial soft derivative, and the geometric interpretation of partial soft derivative. Moreover, it exemplifies the theoretical part of the study and provides figures for the geometric interpretation. Finally, this study discusses the need for further research.

Keywords: Partial soft derivative, Soft analysis, Soft derivative, Soft sets
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1. Introduction

Molodtsov [1] has introduced soft sets and discussed their relationships with several mathematical tools. Moreover, the author has investigated soft sets' applications to stability and regularization, game theory and operations research, and soft analysis. Molodtsov has studied soft limit, soft approximator (soft derivative), and upper and lower Riemann and Perron integrals in soft analysis. Afterward, the author has written a book entitled Soft Set Theory [2] that contains many topics related to soft sets. Then, Molodtsov et al. [3] have widely explored the basic concepts of soft analysis. Further, Molodtsov [4] has suggested higher-order soft derivative and higher-order almost soft derivative. Besides, the author [5, 6] has analyzed the basic concepts of rational analysis. Additionally, Acharjee and Molodtsov [7] have proposed soft rational line integral. However, since most of the aforesaid studies are in Russian, soft analysis studies have not become widespread.

On the other hand, despite the considerable developments in classical analysis, the fact that there are many types of uncertainty in real-life problems and that increasing the need for new mathematical tools makes soft analysis worth studying. Therefore, this paper focuses on the partial soft derivative, one of the essential concepts in soft analysis. Thus, this study aims to increase the widespread impact and make soft analysis studies more accessible. Moreover, the partial soft derivative will shed light on the concepts of higher-order partial soft derivative and soft gradients. Hence, this paper provides ideas concerning further studies to researchers. Section 2 of the present study provides some basic definitions and properties to be required in the next section. Section 3 defines partial soft derivative and studies some of its basic properties. The final section discusses the need for future studies.

2. Preliminaries

This section presents some of the basic definitions and properties to be needed for the following section. Across this paper, the notations $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^{\geq 0}$ represent the set of integer, real, positive real, and non-negative real numbers, respectively. Moreover, $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$ and $P(U)$ denotes the set of all the classical subsets of U.

Definition 2.1. *[1, 2] Let U be a universal set, E be a parameter set, and $f : E \to P(U)$ be a function. Then, f is called a soft set parameterized via E over U (briefly over U).*

Example 2.2. *Let $f : \mathbb{Z} \to P(\mathbb{R})$ be a function defined by $f(x) = [x+2, x+4]$. Then, f is a soft set over $\mathbb{R}$.*

Definition 2.3. *[1, 2] Let M be a set called a model set, U be a universal set, E be a parameter set, and $f : M\times E \to P(U)$ be a function. Then, f is called a soft mapping parameterized via $M\times E$ over U (briefly over U).*

Definition 2.4. *[1, 2, 3] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_f(a) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a (τ,ε)-soft derivative of f at the point a if $x \in \tau_f(a) \Rightarrow |f(x) - f(a) - L(x-a)| \leq \varepsilon(a)$. The set of all the (τ,ε)-soft derivatives of f at the point a is denoted by $D(f,a,\tau,\varepsilon)$. If $D(f,a,\tau,\varepsilon) = \emptyset$, then the (τ,ε)-soft derivative of f at the point a does not exist.*

Here, $\tau : \mathbb{R} \to P(\mathbb{R})$ and $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ are two functions such that $\tau(a)$ is a set of points that are close to the point a but not equal to a. In addition, $\tau_f(a) := \tau(a)\cap \mathrm{Dom}(f)$, for all $a \in \mathbb{R}$, where $\mathrm{Dom}(f)$ stands for the domain set of f.

3. Partial Soft Derivative

This section defines the concept of partial soft derivative and studies some of its basic properties. Throughout this section, let $\tau,\lambda,\kappa : \mathbb{R}^2 \to P\left(\mathbb{R}^2\right)$, $\varepsilon,\alpha,\beta : \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$, and $\delta : \mathbb{R}^2 \to \mathbb{R}^+$ be seven functions such that $\tau(a,b)$, $\lambda(a,b)$, and $\kappa(a,b)$ are sets of points that are close to the point (a,b) but not equal to (a,b). Besides, let $\tau_f(a,b) := \tau(a,b)\cap \mathrm{Dom}(f)$, for all $(a,b) \in \mathbb{R}^2$.

Definition 3.1. *The set of all the points belonging to $\tau(a,b)$ and the plane $y = b$ is defined by $\tau_x(a,b) := \tau(a,b)\cap(\mathbb{R}\times\{b\})$. Similarly, the set of all the points belonging to $\tau(a,b)$ and the plane $x = a$ is defined by $\tau_y(a,b) := \tau(a,b)\cap(\{a\}\times\mathbb{R})$. Therefore, the set of all the points belonging to $\tau(a,b)$ and whose first components are greater than a is defined by $\tau_x^+(a,b) := \tau(a,b)\cap((a,\infty)\times\mathbb{R})$ and the set of all the points belonging to $\tau(a,b)$ and whose first components are less than a is defined by $\tau_x^-(a,b) := \tau(a,b)\cap((-\infty,a)\times\mathbb{R})$. Similarly, the set of all the points belonging to $\tau(a,b)$ and whose second components are greater than b is defined by $\tau_y^+(a,b) := \tau(a,b)\cap(\mathbb{R}\times(b,\infty))$ and the set of all the points belonging to $\tau(a,b)$ and whose second components are less than b is defined by $\tau_y^-(a,b) := \tau(a,b)\cap(\mathbb{R}\times(-\infty,b))$.*

Moreover, if $\tau_x^-(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_x-right mapping, and if $\tau_x^+(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_x-left mapping. Similarly, if $\tau_y^-(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_y-right mapping and if $\tau_y^+(a,b) = \emptyset$, for all $(a,b) \in \mathbb{R}^2$, then this mapping is called by τ_y-left mapping.

Furthermore, $\tau_\delta(a,b)$ is defined by

$$\tau_\delta(a,b) := \left\{(x,y) \in \mathbb{R}^2 \ : 0 < \sqrt{(x-a)^2+(y-b)^2} \leq \delta(a,b)\right\}$$

Thus, $\tau_{x\delta}^+(a,b) := \tau_\delta(a,b)\cap((a,\infty)\times\mathbb{R})$, $\tau_{x\delta}^-(a,b) := \tau_\delta(a,b)\cap((-\infty,a)\times\mathbb{R})$, $\tau_{y\delta}^+(a,b) := \tau_\delta(a,b)\cap(\mathbb{R}\times(b,\infty))$, and $\tau_{y\delta}^-(a,b) := \tau_\delta(a,b)\cap(\mathbb{R}\times(-\infty,b))$.

Note 3.2. *It must be noted that $\tau(a,b) = \tau_x^+(a,b)\cup\tau_x^-(a,b)$, $\tau(a,b) = \tau_y^+(a,b)\cup\tau_y^-(a,b)$, $\tau_\delta(a,b) = \tau_{x\delta}^+(a,b)\cup\tau_{x\delta}^-(a,b)$, and $\tau_\delta(a,b) = \tau_{y\delta}^+(a,b)\cup\tau_{y\delta}^-(a,b)$.*

Definition 3.3. *Let $A\times B \subseteq \mathbb{R}^2$, $f : A\times B \to \mathbb{R}$ be a function, $(a,b) \in A\times B$, $\tau_f(a,b) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a partial (τ,ε)-soft derivative of f with respect to x at the point (a,b) if $(x,b) \in \tau_f(a,b) \Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \varepsilon(a,b)$. The set of all the partial (τ,ε)-soft derivatives of f with respect to x at the point (a,b) is denoted by $D_x(f,(a,b),\tau,\varepsilon)$. If $D_x(f,(a,b),\tau,\varepsilon) = \emptyset$, then the partial (τ,ε)-soft derivative of f with respect to x at the point (a,b) does not exist.*

Definition 3.4. *Let $A\times B \subseteq \mathbb{R}^2$, $f : A\times B \to \mathbb{R}$ be a function, $(a,b) \in A\times B$, $\tau_f(a,b) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a partial (τ,ε)-soft derivative of f with respect to y at the point (a,b) if $(a,y) \in \tau_f(a,b) \Rightarrow |f(a,y) - f(a,b) - L(y-b)| \leq \varepsilon(a,b)$. The set of all the partial (τ,ε)-soft derivatives of f with respect to y at the point (a,b) is denoted by $D_y(f,(a,b),\tau,\varepsilon)$. If $D_y(f,(a,b),\tau,\varepsilon) = \emptyset$, then the partial (τ,ε)-soft derivative of f with respect to y at the point (a,b) does not exist.*

Note 3.5. *Each of the concepts of partial (τ,ε)-soft derivative with respect to x and y is a soft mapping parameterized via $\Phi(A\times B,\mathbb{R})\times(A\times B)\times\Phi\left(\mathbb{R}^2,P\left(\mathbb{R}^2\right)\right)\times\Phi\left(\mathbb{R}^2,\mathbb{R}^{\geq 0}\right)$ over $\mathbb{R}$ such that $\emptyset \neq A\times B \subseteq \mathbb{R}^2$.*

Example 3.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x,y) = x^3 + 2y^2$ and $\varepsilon(-1,3) = 2$. Since*

$$\begin{aligned}(x,3) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2 &\Leftrightarrow 0 < \sqrt{(x+1)^2 + (3-3)^2} \leq \tfrac{1}{2} \\ &\Leftrightarrow 0 < |x+1| \leq \tfrac{1}{2}\end{aligned}$$

then, for all $(x,3) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2$,

$$\begin{aligned}|f(x,3) - f(-1,3) - L(x+1)| \leq 2 &\Leftrightarrow |x^3 + 1 - L(x+1)| \leq 2 \\ &\Leftrightarrow -x^3 - 1 - 2 \leq -L(x+1) \leq -x^3 - 1 + 2 \\ &\Leftrightarrow \tfrac{x^3+1}{x+1} - \tfrac{2}{|x+1|} \leq L \leq \tfrac{x^3+1}{x+1} + \tfrac{2}{|x+1|} \\ &\Leftrightarrow x^2 - x + 1 - \tfrac{2}{|x+1|} \leq L \leq x^2 - x + 1 + \tfrac{2}{|x+1|} \\ &\Leftrightarrow L \in \left[\tfrac{3}{4}, \tfrac{23}{4}\right]\end{aligned}$$

Therefore, $D_x\left(f,(-1,3),\tau_{\frac{1}{2}},\varepsilon\right) = \left[\frac{3}{4}, \frac{23}{4}\right]$. Similarly, as

$$\begin{aligned}(-1,y) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2 &\Leftrightarrow 0 < \sqrt{(-1+1)^2 + (y-3)^2} \leq \tfrac{1}{2} \\ &\Leftrightarrow 0 < |y-3| \leq \tfrac{1}{2}\end{aligned}$$

then, for all $(-1,y) \in \tau_{\frac{1}{2}}(-1,3) \cap \mathbb{R}^2$,

$$\begin{aligned}|f(-1,y) - f(-1,3) - L(y-3)| \leq 2 &\Leftrightarrow |2y^2 - 18 - L(y-3)| \leq 2 \\ &\Leftrightarrow -2y^2 + 18 - 2 \leq -L(y-3) \leq -2y^2 + 18 + 2 \\ &\Leftrightarrow \tfrac{2y^2-18}{y-3} - \tfrac{2}{|y-3|} \leq L \leq \tfrac{2y^2-18}{y-3} + \tfrac{2}{|y-3|} \\ &\Leftrightarrow 2y + 6 - \tfrac{2}{|y-3|} \leq L \leq 2y + 6 + \tfrac{2}{|y-3|} \\ &\Leftrightarrow L \in [9,15]\end{aligned}$$

Thus, $D_y\left(f,(-1,3),\tau_{\frac{1}{2}},\varepsilon\right) = [9,15]$.

Theorem 3.7. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\tau_f(a,b)$ be bounded. If $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $z = f(x,b)$ is bounded on $\tau_f(a,b)$.*

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, $\tau_f(a,b)$ be bounded, and $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$. Then, $\tau_f(a,b) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$\begin{aligned}(x,b) \in \tau_f(a,b) &\Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \varepsilon(a,b) \\ &\Rightarrow f(a,b) + L(x-a) - \varepsilon(a,b) \leq f(x,b) \leq f(a,b) + L(x-a) + \varepsilon(a,b) \\ &\Rightarrow f(a,b) + \inf_{x \in \tau_f(a,b)} \{L(x-a)\} - \varepsilon(a,b) \leq f(x,b) \leq f(a,b) + \sup_{x \in \tau_f(a,b)} \{L(x-a)\} + \varepsilon(a,b)\end{aligned}$$

Since

$$f(a,b) + \inf_{x \in \tau_f(a,b)} \{L(x-a)\} - \varepsilon(a,b) \in \mathbb{R} \quad \text{and} \quad f(a,b) + \sup_{x \in \tau_f(a,b)} \{L(x-a)\} + \varepsilon(a,b) \in \mathbb{R}$$

then $z = f(x,b)$ is bounded on $\tau_f(a,b)$. □

Theorem 3.8. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\tau_f(a,b) \neq \emptyset$. If $z = f(x,b)$ is bounded on $\tau_f(a,b)$, then there exists a function $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$.*

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, $\tau_f(a,b)\neq\emptyset$, and $z=f(x,b)$ be bounded on $\tau_f(a,b)$. Then, there exists an $M\in\mathbb{R}$ such that $|f(x,b)|\leq M$, for all $(x,y)\in\tau_f(a,b)$. Then,

$$\begin{aligned}(x,b)\in\tau_f(a,b) &\Rightarrow |f(x,b)|\leq M\\ &\Rightarrow -M-f(a,b)\leq f(x,b)-f(a,b)\leq M-f(a,b)\\ &\Rightarrow |f(x,b)-f(a,b)-0(x-a)|\leq \max\{|M+f(a,b)|,|M-f(a,b)|\}\end{aligned}$$

Hence, for any function $\varepsilon:\mathbb{R}^2\to\mathbb{R}$ such that $\varepsilon(a,b)=\max\{|M+f(a,b)|,|M-f(a,b)|\}$, $0\in D_x(f,(a,b),\tau,\varepsilon)$. Consequently, $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. □

Theorem 3.9. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $\tau_f(a,b)$ be bounded. If $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then $z=f(a,y)$ is bounded on $\tau_f(a,b)$.*

Theorem 3.10. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $\tau_f(a,b)\neq\emptyset$. If $z=f(a,y)$ is bounded on $\tau_f(a,b)$, then there exists a function $\varepsilon:\mathbb{R}^2\to\mathbb{R}$ such that $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$.*

The proofs are as in Theorems 3.7 and 3.8, respectively.

Theorem 3.11. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in A\times B$. If $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then*

$$D_x(f,(a,b),\tau,\varepsilon)=\left[\sup_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\right),\inf_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}\right)\right]$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. Then, $\tau_f(a,b)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that, for all $(x,b)\in\tau_f(a,b)$,

$$\begin{aligned}|f(x,b)-f(a,b)-L(x-a)|\leq\varepsilon(a,b) &\Rightarrow -(f(x,b)-f(a,b))-\varepsilon(a,b)\leq -L(x-a)\leq -(f(x,b)-f(a,b))+\varepsilon(a,b)\\ &\Rightarrow \begin{cases}\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{x-a}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{x-a}, & (x,b)\in\tau_x^+(a,b)\cap A\times B\\ \frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{x-a}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{x-a}, & (x,b)\in\tau_x^-(a,b)\cap A\times B\end{cases}\\ &\Rightarrow \begin{cases}\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}, & (x,b)\in\tau_x^+(a,b)\cap A\times B\\ \frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}, & (x,b)\in\tau_x^-(a,b)\cap A\times B\end{cases}\\ &\Rightarrow \frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\leq L\leq\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|},\quad (x,b)\in\tau_f(a,b)\end{aligned}$$

Hence,

$$\sup_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\right)\leq L\quad\text{and}\quad L\leq\inf_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}\right)$$

Consequently,

$$D_x(f,(a,b),\tau,\varepsilon)=\left[\sup_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}-\frac{\varepsilon(a,b)}{|x-a|}\right),\inf_{(x,b)\in\tau_f(a,b)}\left(\frac{f(x,b)-f(a,b)}{x-a}+\frac{\varepsilon(a,b)}{|x-a|}\right)\right]$$

□

Theorem 3.12. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in A\times B$. If $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then*

$$D_y(f,(a,b),\tau,\varepsilon)=\left[\sup_{(a,y)\in\tau_f(a,b)}\left(\frac{f(a,y)-f(a,b)}{y-b}-\frac{\varepsilon(a,b)}{|y-b|}\right),\inf_{(a,y)\in\tau_f(a,b)}\left(\frac{f(a,y)-f(a,b)}{y-b}+\frac{\varepsilon(a,b)}{|y-b|}\right)\right]$$

The proof is as in Theorem 3.11.

Theorem 3.13. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $\beta(a,b)\leq\alpha(a,b)$. If $D_x(f,(a,b),\tau,\beta)\neq\emptyset$, then $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$. Moreover, $D_x(f,(a,b),\tau,\beta)\subseteq D_x(f,(a,b),\tau,\alpha)$.*

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, $\beta(a,b) \leq \alpha(a,b)$, and $D_x(f,(a,b),\tau,\beta) \neq \emptyset$. Then, $\tau_f(a,b) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$(x,b) \in \tau_f(a,b) \Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \beta(a,b) \leq \alpha(a,b)$$

Therefore, $D_x(f,(a,b),\tau,\alpha) \neq \emptyset$. Moreover, since $\beta(a,b) \leq \alpha(a,b)$, then

$$\sup_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} - \frac{\alpha(a,b)}{|x-a|} \right) \leq \sup_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} - \frac{\beta(a,b)}{|x-a|} \right)$$

and

$$\inf_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} + \frac{\beta(a,b)}{|x-a|} \right) \leq \inf_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} + \frac{\alpha(a,b)}{|x-a|} \right)$$

Thus,

$$\begin{aligned} D_x(f,(a,b),\tau,\beta) &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\beta(a,b)}{|x-a|} \right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\beta(a,b)}{|x-a|} \right) \right] \\ &\subseteq \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\alpha(a,b)}{|x-a|} \right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\alpha(a,b)}{|x-a|} \right) \right] \\ &= D_x(f,(a,b),\tau,\alpha) \end{aligned}$$

□

Theorem 3.14. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\beta(a,b) \leq \alpha(a,b)$. If $D_y(f,(a,b),\tau,\beta) \neq \emptyset$, then $D_y(f,(a,b),\tau,\alpha) \neq \emptyset$. Moreover, $D_y(f,(a,b),\tau,\beta) \subseteq D_y(f,(a,b),\tau,\alpha)$.*

The proof is as in Theorem 3.13.

Theorem 3.15. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$. If $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_x(f,(a,b),\lambda,\varepsilon) \neq \emptyset$. Moreover, $D_x(f,(a,b),\tau,\varepsilon) \subseteq D_x(f,(a,b),\lambda,\varepsilon)$.*

Proof. Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$, and $D_x(f,(a,b),\tau,\varepsilon) \neq \emptyset$. Then, there exists an $L \in \mathbb{R}$ such that

$$\begin{aligned} (x,b) \in \lambda_f(a,b) &\Rightarrow (x,b) \in \tau_f(a,b) \\ &\Rightarrow |f(x,b) - f(a,b) - L(x-a)| \leq \varepsilon(a,b) \end{aligned}$$

Therefore, $D_x(f,(a,b),\lambda,\varepsilon) \neq \emptyset$. Moreover, since $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$, then

$$\sup_{(x,b)\in\lambda_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} - \frac{\varepsilon(a,b)}{|x-a|} \right) \leq \sup_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} - \frac{\varepsilon(a,b)}{|x-a|} \right)$$

and

$$\inf_{(x,b)\in\tau_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} + \frac{\varepsilon(a,b)}{|x-a|} \right) \leq \inf_{(x,b)\in\lambda_f(a,b)} \left(\frac{f(x,b) - f(a,b)}{x-a} + \frac{\varepsilon(a,b)}{|x-a|} \right)$$

Thus,

$$\begin{aligned} D_x(f,(a,b),\tau,\varepsilon) &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|} \right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|} \right) \right] \\ &\subseteq \left[\sup_{(x,b)\in\lambda_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|} \right), \inf_{(x,b)\in\lambda_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|} \right) \right] \\ &= D_x(f,(a,b),\lambda,\varepsilon) \end{aligned}$$

□

Theorem 3.16. *Let $A \times B \subseteq \mathbb{R}^2$, $f : A \times B \to \mathbb{R}$ be a function, $(a,b) \in A \times B$, and $\emptyset \neq \lambda_f(a,b) \subseteq \tau_f(a,b)$. If $D_y(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_y(f,(a,b),\lambda,\varepsilon) \neq \emptyset$. Moreover, $D_y(f,(a,b),\tau,\varepsilon) \subseteq D_y(f,(a,b),\lambda,\varepsilon)$.*

The proof is as in Theorem 3.15.

Theorem 3.17. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, and $(a,b)\in A\times B$. If $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$ and $D_x(g,(a,b),\lambda,\beta)\neq\emptyset$, then $D_x(f+g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f+g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,*

$$D_x(f,(a,b),\tau,\alpha)+D_x(g,(a,b),\lambda,\beta)\subseteq D_x(f+g,(a,b),\kappa,\varepsilon)$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, $(a,b)\in A\times B$, $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$, and $D_x(g,(a,b),\lambda,\beta)\neq\emptyset$. Then, there exist $L_1,L_2\in\mathbb{R}$ such that

$$(x,b)\in\tau_f(a,b)\Rightarrow|f(x,b)-f(a,b)-L_1(x-a)|\leq\alpha(a,b)$$

and

$$(x,b)\in\lambda_g(a,b)\Rightarrow|g(x,b)-g(a,b)-L_2(x-a)|\leq\beta(a,b)$$

Therefore,

$$\begin{aligned}(x,b)\in\kappa_{f+g}(a,b)&\Rightarrow(x,b)\in\tau_f(a,b)\wedge(x,b)\in\lambda_g(a,b)\\&\Rightarrow|f(x,b)-f(a,b)-L_1(x-a)|\leq\alpha(a,b)\wedge|g(x,b)-g(a,b)-L_2(x-a)|\leq\beta(a,b)\\&\Rightarrow-\alpha(a,b)-\beta(a,b)\leq f(x,b)-f(a,b)-L_1(x-a)+g(x,b)-g(a,b)-L_2(x-a)\leq\alpha(a,b)+\beta(a,b)\\&\Rightarrow|(f+g)(x,b)-(f+g)(a,b)-(L_1+L_2)(x-a)|\leq\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)\\&\Rightarrow L_1+L_2\in D_x(f+g,(a,b),\kappa,\varepsilon)\\&\Rightarrow D_x(f+g,(a,b),\kappa,\varepsilon)\neq\emptyset\end{aligned}$$

Moreover, for all $L\in D_x(f,(a,b),\tau,\alpha)+D_x(g,(a,b),\lambda,\beta)$, there exist $L_1\in D_x(f,(a,b),\tau,\alpha)$ and $L_2\in D_x(g,(a,b),\lambda,\beta)$ such that $L=L_1+L_2$. Then,

$$(x,b)\in\tau_f(a,b)\Rightarrow|f(x,b)-f(a,b)-L_1(x-a)|\leq\alpha(a,b)$$

and

$$(x,b)\in\lambda_g(a,b)\Rightarrow|g(x,b)-g(a,b)-L_2(x-a)|\leq\beta(a,b)$$

Hence,

$$(x,b)\in\kappa_{f+g}(a,b)\Rightarrow|(f+g)(x,b)-(f+g)(a,b)-(L_1+L_2)(x-a)|\leq\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$$

Therefore, $L=L_1+L_2\in D_x(f+g,(a,b),\kappa,\varepsilon)$. Thus, $D_x(f,(a,b),\tau,\alpha)+D_x(g,(a,b),\lambda,\beta)\subseteq D_x(f+g,(a,b),\kappa,\varepsilon)$. □

Theorem 3.18. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, and $(a,b)\in A\times B$. If $D_y(f,(a,b),\tau,\alpha)\neq\emptyset$ and $D_y(g,(a,b),\lambda,\beta)\neq\emptyset$, then $D_y(f+g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f+g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,*

$$D_y(f,(a,b),\tau,\alpha)+D_y(g,(a,b),\lambda,\beta)\subseteq D_y(f+g,(a,b),\kappa,\varepsilon)$$

The proof is as in Theorem 3.17.

Theorem 3.19. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $t\neq 0$. Then, $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$ if and only if $D_x(tf,(a,b),\tau,|t|\varepsilon)\neq\emptyset$. Moreover,*

$$tD_x(f,(a,b),\tau,\varepsilon)=D_x(tf,(a,b),\tau,|t|\varepsilon)$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $t\neq 0$.
($\Rightarrow$): Let $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. Then, $\tau_f(a,b)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that

$$\begin{aligned}(x,b)\in\tau_{tf}(a,b)&\Rightarrow(x,b)\in\tau_f(a,b)\\&\Rightarrow|f(x,b)-f(a,b)-L(x-a)|\leq\varepsilon(a,b)\\&\Rightarrow|t||f(x,b)-f(a,b)-L(x-a)|\leq|t|\varepsilon(a,b)\\&\Rightarrow|tf(x,b)-tf(a,b)-tL(x-a)|\leq|t|\varepsilon(a,b)\\&\Rightarrow|(tf)(x,b)-(tf)(a,b)-tL(x-a)|\leq|t|\varepsilon(a,b)\end{aligned}$$

Thus, $tL\in D_x(tf,(a,b),\tau,|t|\varepsilon)$. That is, $D_x(tf,(a,b),\tau,|t|\varepsilon)\neq\emptyset$.

($\Leftarrow$): Let $D_x(tf,(a,b),\tau,|t|\varepsilon)\neq\emptyset$. Then, $\tau_{tf}(a,b)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that

$$\begin{aligned}(x,b)\in\tau_f(a,b) &\Rightarrow (x,b)\in\tau_{tf}(a,b)\\ &\Rightarrow |(tf)(x,b)-(tf)(a,b)-L(x-a)|\leq|t|\varepsilon(a,b)\\ &\Rightarrow |tf(x,b)-tf(a,b)-L(x-a)|\leq|t|\varepsilon(a,b)\\ &\Rightarrow |t|\left|f(x,b)-f(a,b)-\tfrac{L}{t}(x-a)\right|\leq|t|\varepsilon(a,b)\\ &\Rightarrow \left|f(x,b)-f(a,b)-\tfrac{L}{t}(x-a)\right|\leq\varepsilon(a,b)\end{aligned}$$

Thus, $\frac{L}{t}\in D_x(f,(a,b),\tau,\varepsilon)$. That is, $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. Moreover, for all $L\in tD_x(f,(a,b),\tau,\varepsilon)$, there exists an $L^*\in D_x(f,(a,b),\tau,\varepsilon)$ such that $L=tL^*$. Since $L^*\in D_x(f,(a,b),\tau,\varepsilon)$, then $tL^*\in D_x(tf,(a,b),\tau,|t|\varepsilon)$ from the proof of the existence. That is, $L\in D_x(tf,(a,b),\tau,|t|\varepsilon)$. Hence,

$$tD_x(f,(a,b),\tau,\varepsilon)\subseteq D_x(tf,(a,b),\tau,|t|\varepsilon)$$

In addition, for all $L\in D_x(tf,(a,b),\tau,|t|\varepsilon)$, $\frac{L}{t}\in D_x(f,(a,b),\tau,\varepsilon)$ from the proof of the existence. Hence, $L=t\frac{L}{t}\in tD_x(f,(a,b),\tau,\varepsilon)$. Thus,

$$D_x(tf,(a,b),\tau,|t|\varepsilon)\subseteq tD_x(f,(a,b),\tau,\varepsilon)$$

Consequently, $tD_x(f,(a,b),\tau,\varepsilon)=D_x(tf,(a,b),\tau,|t|\varepsilon)$. □

Theorem 3.20. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in A\times B$, and $t\neq0$. Then, $D_y(f,(a,b),\tau,\varepsilon)\neq\emptyset$ if and only if $D_y(tf,(a,b),\tau,|t|\varepsilon)\neq\emptyset$. Moreover,*

$$tD_y(f,(a,b),\tau,\varepsilon)=D_y(tf,(a,b),\tau,|t|\varepsilon)$$

The proof is as in Theorem 3.19.

Corollary 3.21. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, and $(a,b)\in A\times B$. If $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$ and $D_x(g,(a,b),\lambda,\beta)\neq\emptyset$, then $D_x(f-g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f-g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,*

$$D_x(f,(a,b),\tau,\alpha)-D_x(g,(a,b),\lambda,\beta)\subseteq D_x(f-g,(a,b),\kappa,\varepsilon)$$

Proof. Let $A,B\subseteq\mathbb{R}$, $f,g:A\times B\to\mathbb{R}$ be two functions, $(a,b)\in A\times B$, $D_x(f,(a,b),\tau,\alpha)\neq\emptyset$, and $D_x(g,(a,b),\lambda,\beta)\neq\emptyset$. From Theorem 3.19, for $t=-1$, $-D_x(g,(a,b),\lambda,\beta)=D_x(-g,(a,b),\lambda,\beta)$. Therefore, from Theorem 3.17, $D_x(f-g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f-g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$ and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,

$$D_x(f,(a,b),\tau,\alpha)-D_x(g,(a,b),\lambda,\beta)=D_x(f,(a,b),\tau,\alpha)+D_x(-g,(a,b),\lambda,\beta)\subseteq D_x(f+(-g),(a,b),\kappa,\varepsilon)=D_x(f-g,(a,b),\kappa,\varepsilon)$$

□

Corollary 3.22. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, and $(a,b)\in A\times B$. If $D_y(f,(a,b),\tau,\alpha)\neq\emptyset$ and $D_y(g,(a,b),\lambda,\beta)\neq\emptyset$, then $D_y(f-g,(a,b),\kappa,\varepsilon)\neq\emptyset$ such that $\emptyset\neq\kappa_{f-g}(a,b)\subseteq\tau_f(a,b)\cap\lambda_g(a,b)$, and $\alpha(a,b)+\beta(a,b)\leq\varepsilon(a,b)$. Moreover,*

$$D_y(f,(a,b),\tau,\alpha)-D_y(g,(a,b),\lambda,\beta)\subseteq D_y(f-g,(a,b),\kappa,\varepsilon)$$

The proof is as in Corollary 3.21.

Theorem 3.23. *Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, $(a,b)\in A\times B$, and $k,l\in\mathbb{R}$. If $g(x,y)=f(x,y)+kx+ly$, for all $(x,y)\in\tau_f(a,b)=\tau_g(a,b)$, and $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$, then $D_x(g,(a,b),\tau,\varepsilon)\neq\emptyset$. Moreover,*

$$D_x(g,(a,b),\tau,\varepsilon)=D_x(f,(a,b),\tau,\varepsilon)+k$$

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f,g:A\times B\to\mathbb{R}$ be two functions, $(a,b)\in A\times B$, $k,l\in\mathbb{R}$, $g(x,y)=f(x,y)+kx+ly$, for all $(x,y)\in\tau_f(a,b)=\tau_g(a,b)$, and $D_x(f,(a,b),\tau,\varepsilon)\neq\emptyset$. Then, $\tau_f(a,b)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that

$$(x,b)\in\tau_f(a,b)\Rightarrow|f(x,b)-f(a,b)-L(x-a)|\leq\varepsilon(a,b)$$

Therefore, for $L^* = k + L$ and for all $(x,b) \in \tau_g(a,b) = \tau_f(a,b)$,

$$\begin{aligned} |g(x,b) - g(a,b) - L^*(x-a)| &= |f(x,b) + kx + lb - f(a,b) - ka - lb - (k+L)(x-a)| \\ &= |f(x,y) - f(a,b) - L(x-a)| \\ &\leq \varepsilon(a) \end{aligned}$$

Thus, $L^* \in D_x(g,(a,b),\tau,\varepsilon)$. That is, $D_x(g,(a,b),\tau,\varepsilon) \neq \emptyset$. Moreover,

$$\begin{aligned} D_x(g,(a,b),\tau,\varepsilon) &= \left[\sup_{(x,b)\in\tau_g(a,b)} \left(\tfrac{g(x,b)-g(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right), \inf_{(x,b)\in\tau_g(a,b)} \left(\tfrac{g(x,b)-g(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right)\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)+kx+lb-f(a,b)-ka-lb}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)+kx+lb-f(a,b)-ka-lb}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right)\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|} + k\right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|} + k\right)\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right) + k, \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right) + k\right] \\ &= \left[\sup_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} - \tfrac{\varepsilon(a,b)}{|x-a|}\right), \inf_{(x,b)\in\tau_f(a,b)} \left(\tfrac{f(x,b)-f(a,b)}{x-a} + \tfrac{\varepsilon(a,b)}{|x-a|}\right)\right] + k \\ &= D_x(f,(a,b),\tau,\varepsilon) + k \end{aligned}$$

□

Theorem 3.24. *Let $A \times B \subseteq \mathbb{R}^2$, $f, g : A \times B \to \mathbb{R}$ be two functions, $(a,b) \in A \times B$, and $k, l \in \mathbb{R}$. If $g(x,y) = f(x,y) + kx + ly$, for all $(x,y) \in \tau_f(a,b) = \tau_g(a,b)$, and $D_y(f,(a,b),\tau,\varepsilon) \neq \emptyset$, then $D_y(g,(a,b),\tau,\varepsilon) \neq \emptyset$. Moreover,*

$$D_y(g,(a,b),\tau,\varepsilon) = D_y(f,(a,b),\tau,\varepsilon) + l$$

The proof is as in Theorem 3.23

Example 3.25. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$, $\tau, \lambda, \kappa : \mathbb{R}^2 \to P\left(\mathbb{R}^2\right)$, and $\alpha, \beta, \varepsilon : \mathbb{R}^2 \to \mathbb{R}^{\geq 0}$ be seven functions defined by $f(x,y) = x^2 + 2y^2$, $g(x,y) = 2x + y$, $\tau(x,y) = \tau_1(x,y)$,*

$$\lambda(x,y) = \left\{(x_0,y_0) \in \mathbb{R}^2 \ : 0 < \sqrt{\frac{(x-x_0)^2}{9} + \frac{(y-y_0)^2}{16}} \leq 1\right\}$$

$\kappa(x,y) = \kappa_{\frac{1}{4}}(x,y)$, $\alpha(x,y) = |x| + |y|$, $\beta(x,y) = \max\{|x|,|y|\}$, and $\varepsilon(x,y) = 2(|x| + |y|)$, respectively. Here, for all $(x,y) \in \mathbb{R}^2$, $\tau(x,y) \subseteq \lambda(x,y)$, $\kappa(x,y) \subseteq \tau(x,y) \cap \lambda(x,y)$, $\beta(x,y) \leq \alpha(x,y)$, and $\alpha(x,y) + \beta(x,y) \leq \varepsilon(x,y)$. From Theorem 3.11, for $(2,-1) \in \mathbb{R}^2$,

$$D_x(f,(2,-1),\tau,\beta) = [3,5]$$

and

$$D_x(g,(2,-1),\lambda,\beta) = \left[\frac{4}{3}, \frac{8}{3}\right]$$

From Theorem 3.13, since $\beta(2,-1) \leq \alpha(2,-1)$, $D_x(f,(2,-1),\tau,\alpha) \neq \emptyset$. Therefore, from Theorem 3.11,

$$D_x(f,(2,-1),\tau,\alpha) = [2,6]$$

and thus,

$$D_x(f,(2,-1),\tau,\beta) = [3,5] \subseteq [2,6] = D_x(f,(2,-1),\tau,\alpha)$$

From Theorem 3.15, as $\tau_f(2,-1) \subseteq \lambda_f(2,-1)$, $D_x(g,(2,-1),\tau,\beta) \neq \emptyset$. Hence, from Theorem 3.11,

$$D_x(g,(2,-1),\tau,\beta) = [0,4]$$

and thus,

$$D_x(g,(2,-1),\lambda,\beta) = \left[\frac{4}{3},\frac{8}{3}\right] \subseteq [0,4] = D_x(g,(2,-1),\tau,\beta)$$

Moreover, from Theorem 3.19, $D_x(2f,(2,-1),\tau,|2|\alpha) \neq \emptyset$ *and* $D_x(-g,(2,-1),\lambda,|-1|\beta) \neq \emptyset$*. Thereby, from Theorem 3.11,*

$$D_x(2f,(2,-1),\tau,|2|\alpha) = [4,12]$$

and

$$D_x(-g,(2,-1),\lambda,|-1|\beta) = \left[-\frac{8}{3},-\frac{4}{3}\right]$$

Therefore,

$$2D_x(f,(2,-1),\tau,\alpha) = 2[2,6] = [4,12] = D_x(2f,(2,-1),\tau,|2|\alpha)$$

and

$$-D_x(g,(2,-1),\lambda,\beta) = -\left[\frac{4}{3},\frac{8}{3}\right] = \left[-\frac{8}{3},-\frac{4}{3}\right] = D_x(-g,(2,-1),\lambda,|-1|\beta)$$

From Theorem 3.17, because $\emptyset \neq \kappa_{f+g}(2,-1) \subseteq \tau_f(2,-1) \cap \lambda_g(2,-1)$ *and* $\alpha(2,-1)+\beta(2,-1) \leq \varepsilon(2,-1)$*, then* $D_x(f+g,(2,-1),\kappa,\varepsilon) \neq \emptyset$*. Hereby, from Theorem 3.11,*

$$D_x(f+g,(2,-1),\kappa,\varepsilon) = \left[-\frac{71}{4},\frac{119}{4}\right]$$

and thus,

$$\begin{aligned} D_x(f,(2,-1),\tau,\alpha) + D_x(g,(2,-1),\lambda,\beta) &= [2,6] + \left[\tfrac{4}{3},\tfrac{8}{3}\right] \\ &= \left[\tfrac{10}{3},\tfrac{26}{3}\right] \\ &\subseteq \left[-\tfrac{71}{4},\tfrac{119}{4}\right] \\ &= D_x(f+g,(2,-1),\kappa,\varepsilon) \end{aligned}$$

From Corollary 3.21, since $\emptyset \neq \kappa_{f-g}(2,-1) \subseteq \tau_f(2,-1) \cap \lambda_g(2,-1)$ *and* $\alpha(2,-1)+\beta(2,-1) \leq \varepsilon(2,-1)$*, then* $D_x(f-g,(2,-1),\kappa,\varepsilon) \neq \emptyset$*. Herewith, from Theorem 3.11,*

$$D_x(f-g,(2,-1),\kappa,\varepsilon) = \left[-\frac{87}{4},\frac{103}{4}\right]$$

and thus,

$$\begin{aligned} D_x(f,(2,-1),\tau,\alpha) - D_x(g,(2,-1),\lambda,\beta) &= [2,6] - \left[\tfrac{4}{3},\tfrac{8}{3}\right] \\ &= \left[-\tfrac{2}{3},\tfrac{14}{3}\right] \\ &\subseteq \left[-\tfrac{87}{4},\tfrac{103}{4}\right] \\ &= D_x(f-g,(2,-1),\kappa,\varepsilon) \end{aligned}$$

Besides, for the function $h:\mathbb{R}^2 \to \mathbb{R}$ *defined by* $h(x,y) = f(x,y) + 3x - 5y$*, from Theorem 3.23,* $D_x(h,(2,-1),\tau,\alpha) \neq \emptyset$*. Hence, from Theorem 3.11,*

$$D_x(h,(2,-1),\tau,\alpha) = [5,9]$$

and thus,

$$D_x(f,(2,-1),\tau,\alpha) + 3 = [2,6] + 3 = [5,9] = D_x(h,(2,-1),\tau,\alpha)$$

Note 3.26. *For the functions* f *and* τ *in Example 3.25,* $(a,b) = (2,-1)$*, and* $\varepsilon^*(2,-1) = \frac{3}{2}$*,* $D_x(f,(2,-1),\tau,\varepsilon^*) = \left[\frac{7}{2},\frac{9}{2}\right]$ *and* $D_y(f,(2,-1),\tau,\varepsilon^*) = \emptyset$*. Similarly, for the function* $h:\mathbb{R}^2 \to \mathbb{R}$ *defined by* $h(x,y) = 2x^2 + y^2$*,* $D_x(h,(2,-1),\tau,\varepsilon^*) = \emptyset$ *and* $D_y(h,(2,-1),\tau,\varepsilon^*) = \left[-\frac{5}{2},-\frac{3}{2}\right]$*. Hence, it is clear that the existence of partial soft derivative with respect to x does not require the existence of partial soft derivative with respect to y and vice versa.*

Note 3.27. *As in classical analysis, for a function with the variables x and y, if taking the partial soft derivative with respect to x, then y is fixed and vice versa. Thus, partial soft derivative turns into soft derivative. In other words, for a function* $f: A\times B \to \mathbb{R}$ *and* $(a,b) \in A \times B$*, if* $L \in D_x(f,(a,b),\tau,\varepsilon)$*, then* $L \in D(g,a,\tau^*,\varepsilon^*)$ *such that* $g: A \to \mathbb{R}$*,* $\tau^*: \mathbb{R} \to P(\mathbb{R})$*, and* $\varepsilon^*: \mathbb{R} \to \mathbb{R}^{\geq 0}$ *are three functions defined by* $g(x) = f(x,b)$*,* $\tau^*(x) = \{x \in \mathbb{R} : (x,b) \in \tau(a,b)\}$*, and* $\varepsilon^*(x) = \varepsilon(x,b)$*, for all* $x \in A$*, respectively. Similarly, for a function* $f: A \times B \to \mathbb{R}$ *and* $(a,b) \in A \times B$*, if* $L \in D_y(f,(a,b),\tau,\varepsilon)$*, then* $L \in D(h,b,\tau^{**},\varepsilon^{**})$ *such that* $h: B \to \mathbb{R}$*,* $\tau^{**}: \mathbb{R} \to P(\mathbb{R})$*, and* $\varepsilon^{**}: \mathbb{R} \to \mathbb{R}^{\geq 0}$ *are three functions defined by* $h(y) = f(a,y)$*,* $\tau^{**}(y) = \{y \in \mathbb{R} : (a,y) \in \tau(a,b)\}$*, and* $\varepsilon^{**}(y) = \varepsilon(a,y)$*, for all* $y \in B$*, respectively.*

In Theorems 3.28 and 3.29, the notations $(A\times B)^\circ$ and $(A\times B)'$ denote the set of all the interior and accumulation points of $A\times B$ according to the usual topology in $\mathbb{R}^2$, respectively.

Theorem 3.28. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in(A\times B)^\circ\cap(A\times B)'$. If $f_x(a,b)\in\mathbb{R}$, then there exist τ and ε^* such that $D_x(f,(a,b),\tau,\varepsilon^*)\neq\emptyset$.*

Proof. Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, $(a,b)\in(A\times B)^\circ\cap(A\times B)'$, and $f_x(a,b)\in\mathbb{R}$. Then, there exists an $L\in\mathbb{R}$ such that

$$f_x(a,b)=\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}=L$$

Therefore,

$$\forall\varepsilon>0,\exists\delta_\varepsilon>0\ni\left((x,b)\in B_0((a,b),\delta_\varepsilon)\cap A\times B\Rightarrow\left|\frac{f(x,b)-f(a,b)}{x-a}-L\right|\leq\varepsilon\right)$$

Thus,

$$\forall\varepsilon>0,\exists\delta_\varepsilon>0\ni\left((x,b)\in B_0((a,b),\delta_\varepsilon)\cap A\times B\Rightarrow\left|f(x,b)-f(a,b)-L(x-a)\right|\leq\varepsilon|x-a|\leq\varepsilon\delta_\varepsilon\right)$$

Hence, for an $\varepsilon>0$,

$$(x,b)\in\tau_f(a,b)\Rightarrow\left|f(x,b)-f(a,b)-L(x-a)\right|\leq\varepsilon^*(a,b)$$

such that $\tau(a,b)=B_0((a,b),\delta_\varepsilon)$ and $\varepsilon^*(a,b)=\varepsilon\delta_\varepsilon$. Thereby, $L\in D_x(f,(a,b),\tau,\varepsilon^*)$. Consequently, $D_x(f,(a,b),\tau,\varepsilon^*)\neq\emptyset$. □

Theorem 3.29. *Let $A\times B\subseteq\mathbb{R}^2$, $f:A\times B\to\mathbb{R}$ be a function, and $(a,b)\in(A\times B)^\circ\cap(A\times B)'$. If $f_y(a,b)\in\mathbb{R}$, then there exist τ and ε^* such that $D_y(f,(a,b),\tau,\varepsilon^*)\neq\emptyset$.*

The proof is as in Theorem 3.28.

Remark 3.30. *The geometric interpretation of the partial soft derivative of a function f with respect to x at a point (a,b) is the tangent of the slope angle of the bandwidth 2ε bounded by two linear functions $f(a,b)+L(x-a)+\varepsilon(a,b)$ and $f(a,b)+L(x-a)-\varepsilon(a,b)$ which contain the entire graph of $z=f(x,b)$ on the set $\tau_x(a,b)\cap\mathrm{Dom}(f)$. Similarly, the geometric interpretation of the partial soft derivative of a function f with respect to y at a point (a,b) is the tangent of the slope angle of the bandwidth 2ε bounded by two linear functions $f(a,b)+L(y-b)+\varepsilon(a,b)$ and $f(a,b)+L(y-b)-\varepsilon(a,b)$ which contain the entire graph of $z=f(a,y)$ on the set $\tau_y(a,b)\cap\mathrm{Dom}(f)$. For example, for the functions f, τ, and α and the point $(2,-1)\in\mathbb{R}^2$ in Example 3.25, $D_x(f,(2,-1),\tau,\alpha)=[2,6]$. Moreover, consider the following linear functions and ordered pairs:*

for $L=2\in[2,6]$,	$g_1(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=2x+5$	$A_1=(x,g_1(x))$
	$h_1(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=2x-1$	$B_1=(x,h_1(x))$
for $L=3\in[2,6]$,	$g_2(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=3x+3$	$A_2=(x,g_2(x))$
	$h_2(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=3x-3$	$B_2=(x,h_2(x))$
for $L=4\in[2,6]$,	$g_3(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=4x+1$	$A_3=(x,g_3(x))$
	$h_3(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=4x-5$	$B_3=(x,h_3(x))$
for $L=5\in[2,6]$,	$g_4(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=5x-1$	$A_4=(x,g_4(x))$
	$h_4(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=5x-7$	$B_4=(x,h_4(x))$
for $L=6\in[2,6]$,	$g_5(x)=f(2,-1)+L(x-2)+\alpha(2,-1)=6x-3$	$A_5=(x,g_5(x))$
	$h_5(x)=f(2,-1)+L(x-2)-\alpha(2,-1)=6x-9$	$B_5=(x,h_5(x))$

Then, it is clear that for all $i\in I_5=\{1,2,3,4,5\}$ and for all $(x,-1)\in\tau_x(2,-1)\cap\mathbb{R}^2$, $h_i(x)\leq f(x)\leq g_i(x)$ and the Euclidean distance of the ordered pairs $A_i=(x,g_i(x))$ and $B_i=(x,h_i(x))$ is 2α such that $|A_iB_i|=\sqrt{(x-x)^2+(g_i(x)-h_i(x))^2}=6=2\alpha$. Figures 3.1 and 3.2 show the graphs of the functions h_i, f, and g_i, for all $i\in I_5$, on the set $\tau_x(2,-1)\cap\mathbb{R}^2$ from different perspectives.

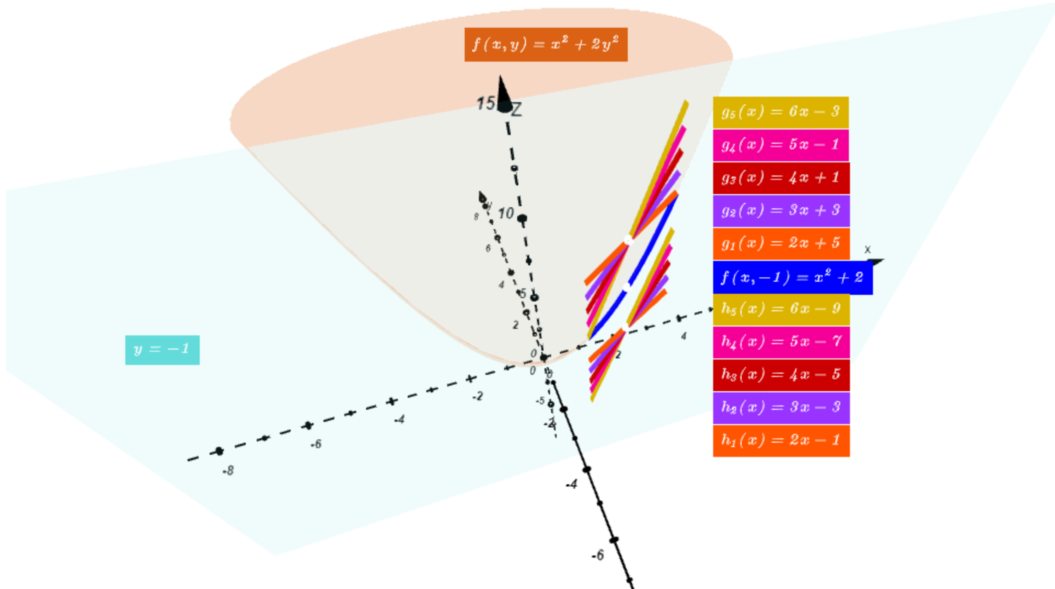

Figure 3.1. Graphs of the functions h_i, f, and g_i, for all $i \in I_5$, on the set $\tau_x(2,-1) \cap \mathbb{R}^2$

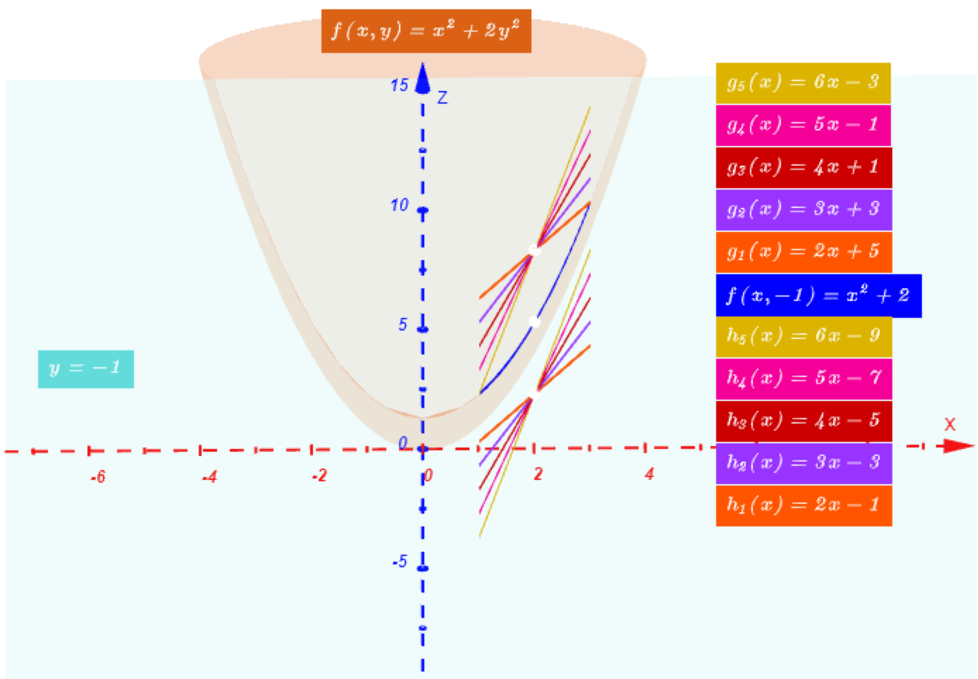

Figure 3.2. Graphs of the functions h_i, f, and g_i, for all $i \in I_5$, on the set $\tau_x(2,-1) \cap \mathbb{R}^2$ (another perspective)

Besides, for all $L \in D_x(f,(2,-1),\tau,\alpha) = [2,6]$, the pairs of all the linear functions h and g form two bundles of lines (see Figure 3.3).

4. Conclusion

This study defined partial soft derivative and investigated some of its basic properties. This paper demonstrated that

- Every function with a partial soft derivative is bounded,
- Every bounded function has a partial soft derivative under certain conditions,
- A partial soft derivative of a function can be considered a soft derivative of the function (see Note 3.27), and

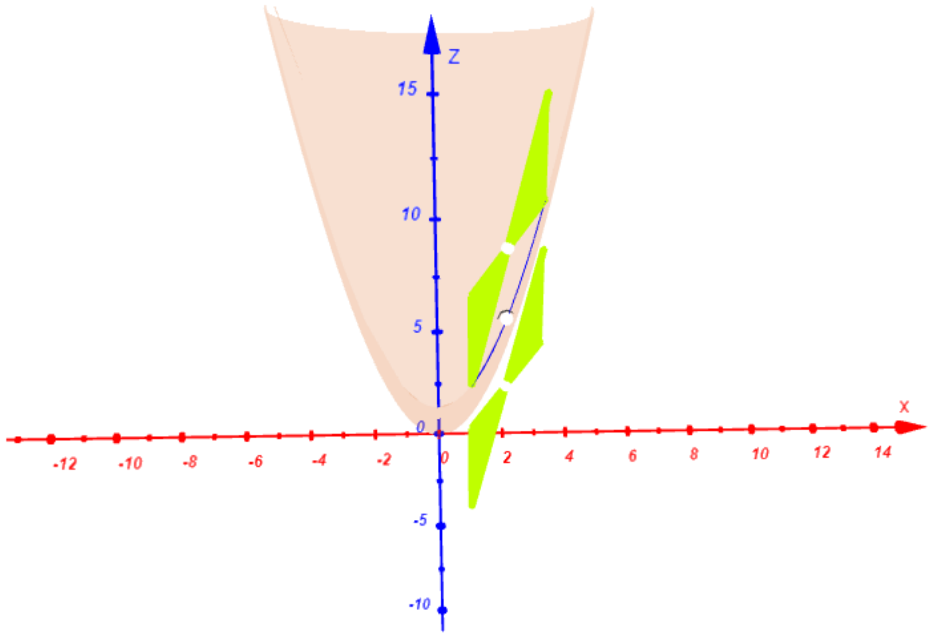

Figure 3.3. Bundles of lines formed by the pairs of all the linear functions h and g, for all $L \in [2,6]$

- Every function with a classical partial derivative has a partial soft derivative under certain conditions

and investigated algebraic properties and the geometric interpretation of partial soft derivative. Moreover, it clarified the theoretical section by examples and provided figures for the geometric interpretation. When the results herein are compared with those of in the classical analysis, the following comments can be briefly made:

- While the classical partial derivative of a function (if any) is equal to a real number, the partial soft derivative of a function (if any) is equal to a closed interval.
- While a bounded function does not always have a classical partial derivative, it has a partial soft derivative (see Theorems 3.8 and 3.10).
- While equality is valid for the sum rule in the partial derivative, inclusion is valid for the partial soft derivative. Similarly, while equality holds for the difference rule in the partial derivative, inclusion holds for the partial soft derivative.
- Geometrically, while a tangent line is obtained in the partial derivative, two bundles of lines are obtained in the partial soft derivative.

Partial soft derivative is a fundamental concept of soft analysis. Therefore, researchers can study this concept and its applications. Moreover, the concepts of higher-order partial soft derivative and soft gradient, associated with partial soft derivative, and the concept of directional soft derivative are also worth studying.

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