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Research Article

Total coloring of circulant graphs $C_n(1, 4)$

Senyuan Su[1], Chunling Tong*[1,2], Yuansheng Yang[3]

[1] *School of Information Science and Electricity Engineering, Shandong Jiaotong University, Jinan 250357, China*
[2] *Department of Mathematics, Simon Fraser University, Burnaby V5A 1S6, Canada*
[3] *College of Computer Science, Dalian University of Technology, Dalian, 116024, China*

Abstract

Total coloring of circulant graphs has attracted much attention in recent years. Studies on the total chromatic numbers of them, in particular 4-regular circulant graphs, have thrown up a number of interesting results. However, as a challenging issue, the total chromatic numbers of 4-regular circulant graphs $C_n(1, 4)$ remain an open question even after many efforts. In this paper, we solve this question by completely determining total chromatic numbers of $C_n(1, 4)$ for all $n \geq 9$.

Mathematics Subject Classification (2020). 05C15

Keywords. total chromatic number, total coloring, circulant graph

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A *k-total coloring* of a graph G is a map σ: $V(G) \cup E(G) \to \{1, 2, \cdots, k\}$, such that no two adjacent or incident elements of $V(G) \cup E(G)$ receive the same color. The smallest number of colors needed for a total coloring of G is known as the *total chromatic number*, denoted as $\chi''(G)$. Determining total chromatic number is NP-complete [13], and NP-hard even for k-regular bipartite graphs with $k \geq 3$ [9].

There is a long-standing total coloring conjecture formulated by Behzad [1] and Vizing [16] independently. It says $\chi''(G) \leq \Delta(G) + 2$ for a simple graph G, where $\Delta(G)$ is the maximum degree of G. The conjecture implies that for every graph G, $\chi''(G)$ attains one of the two values $\Delta(G) + 1$ or $\Delta(G) + 2$. Usually, a graph with $\chi''(G) = \Delta(G) + 1$ is known as Type I while a graph with $\chi''(G) = \Delta(G) + 2$ is known as Type II. The conjecture has been verified by many graphs, and exact values of total chromatic number for some graphs were determined [5, 6, 14, 15]. However, the total chromatic numbers for most circulant graphs including $C_n(1, 4)$ remain open even after many efforts [2, 3, 6, 7, 10–12].

A circulant graph $C_n(d_1, d_2, \cdots, d_l)$ is the graph that has a vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and an edge set $E = \bigcup_{i=1}^{l} E_i$ with $E_i = \{e_0^i, e_1^i, \cdots, e_{n-1}^i\}$ and $e_m^i = v_m v_{m+d_i}$, where

*Corresponding Author.
Email addresses: susy@stu.sdjtu.edu.cn (S.Y. Su), tongcl@sdjtu.edu.cn (C.L. Tong), yangys@dlut.edu.cn (Y.S. Yang)
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$1 \le d_1 < d_2 < \cdots < d_l \le \lfloor \frac{n}{2} \rfloor$ and indices of the vertices are considered modulo n. When l is taken as 2, it reduces to a 4-regular graph $C_n(d_1, d_2)$, of which a special class is $C_n(1, 4)$.

It is difficult to determine the total chromatic number of a general circulant graph. So far, great efforts have been directed towards studying the total coloring of 4-regular circulant graphs and a number of interesting results have been thrown up. Campos and de Mello proved that $C_n(1, 2)$ is Type I except for graph $C_7(1, 2)$, which is Type II [3]. Khennoufa and Togni proved that $C_{5p}(1, k)$ for any positive integer p, $k < \frac{5p}{2}$ and k (mod 5) = 2,3, and $C_{6p}(1, k)$ for $p \ge 3$, $k < 3p$ and k (mod 3) $\ne 0$ are Type I [7]. Nigro et al. demonstrated that $C_n(3, 2k)$ for $n = (8p + 6q)k$ ($k \ge 1$) with non-negative integers p and q, $C_{3p}(1, 3)$ for $p > 1$ except for $C_{12}(1, 3)$, and $C_{3tp}(1, p)$ for $t \ge 1$ and p multiple of 3 are Type I [11]. Navaneeth et al. proved that $C_{5p}(1, k)$ for any positive integer p, $k < \frac{5p}{2}$ and k (mod 5) = 1,4, $C_{3p}(a, b)$ for odd p, $1 \le a < b < \frac{3p}{2}$, $\gcd(a, b) = 1$ and $\frac{3p}{\gcd(3p,b)} = 3s$ $(s \in N)$, $C_{9p}(1, k)$ for $2 \le k < \frac{9p}{2}$ and $\frac{9p}{\gcd(9p,k)} = 3s$ $(s \in N)$, and $C_{6p}(a, b)$ for even p and a, b (mod 3) $\ne 0$ or odd p, $\gcd(a, b) = 1$ and a, b (mod 3) $\ne 0$ are Type I [10].

In this paper, we study the total coloring of circulant graphs $C_n(1, 4)$. We aim to find their total chromatic numbers for all $n \ge 9$. The paper is organized as follows. We will first determine the total chromatic numbers of $C_n(1, 4)$ for $n = 5p+11q$ with p and q being arbitrary nonnegative integers in Section 2, and then we determine the total chromatic numbers of $C_n(1, 4)$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$ in Section 3, and for $n = 13$ in Section 4. Section 5 is our conclusion.

2. Total coloring of $C_n(1, 4)$ for $n = 5p + 11q$

Any positive integer n can be always written as $n = 5k, 5k + 1, 5k + 2, 5k + 3$, or $5k + 4$ with k being a nonnegative integer, which can be equivalently recast as $n = 5k + 11 \times 0,\ 5(k - 2) + 11 \times 1,\ 5(k - 4) + 11 \times 2,\ 5(k - 6) + 11 \times 3,\ 5(k - 8) + 11 \times 4$. Therefore, n in $C_{n\ge 9}(1, 4)$ can be expressed as $n = 5p+11q$ with p and q being nonnegative integers, except for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$, and $n = 13$. In this section, we first study the total coloring of $C_n(1, 4)$ for $n = 5p + 11q$.

Lemma 2.1. *$\chi''(C_n(1, 4)) = 5$ for $n = 5p + 11q$ with nonnegative integers p, q.*

Proof. For simplicity, we use $(i_1 i_2 \dots i_t)^p$ to represent $\underbrace{i_1 i_2 \dots i_t \cdots i_1 i_2 \dots i_t}_{p}$, where $i_1, i_2, \cdots, i_t \in \{1, 2, 3, 4, 5\}$. For example, $(24351)^2 = 2435124351$. Let $V = \{v_i : 0 \le i \le n - 1\}$, $E_1 = \{v_i v_{i+1} : 0 \le i \le n - 1\}$ and $E_2 = \{v_i v_{i+4} : 0 \le i \le n - 1\}$. The total coloring of $C_n(1, 4)$ is denoted as:
$\sigma(C_n(1, 4)) = (\sigma(v_0)\sigma(v_1) \cdots \sigma(v_{n-1}), \sigma(v_0v_1)\sigma(v_1v_2) \cdots \sigma(v_{n-1}v_0), \sigma(v_0v_4)\sigma(v_1v_5) \cdots \sigma(v_{n-1}v_3))$.

We construct a 5-total coloring of $C_n(1, 4)$ for $n = 5p + 11q$ as follows.
$\sigma(C_{5p+11q}(1, 4)) = ((24153)^p(23143523121)^q, (12314)^p(12312312353)^q, (53542)^p(54554144542)^q)$.

It is straightforward to verify that the above construction $\sigma(C_n(1, 4))$ is indeed a total coloring of $C_{5p+11q}(1, 4)$. First, the vertices adjacent to v_i are v_{i-1}, v_{i+1}, v_{i-4} and v_{i+4}. The construction indicates $\sigma(v_i) \ne \sigma(v_{i-1}), \sigma(v_{i+1}), \sigma(v_{i-4}), \sigma(v_{i+4})$, which means that two adjacent vertices receive different colors. Second, the edges adjacent to v_iv_{i+1} are $v_{i-1}v_i$, $v_{i+1}v_{i+2}$, $v_{i-4}v_i$, v_iv_{i+4}, $v_{i-3}v_{i+1}$ and $v_{i+1}v_{i+5}$, and the edges adjacent to v_iv_{i+4} are $v_{i-1}v_i$, v_iv_{i+1}, $v_{i+3}v_{i+4}$, $v_{i+4}v_{i+5}$, $v_{i-4}v_i$ and $v_{i+4}v_{i+8}$. The construction indicates $\sigma(v_iv_{i+1}) \ne \sigma(v_{i-1}v_i), \sigma(v_{i+1}v_{i+2}), \sigma(v_{i-4}v_i), \sigma(v_iv_{i+4}), \sigma(v_{i-3}v_{i+1}), \sigma(v_{i+1}v_{i+5})$; and $\sigma(v_iv_{i+4}) \ne \sigma(v_{i-1}v_i), \sigma(v_iv_{i+1}), \sigma(v_{i+3}v_{i+4}), \sigma(v_{i+4}v_{i+5}), \sigma(v_{i-4}v_i), \sigma(v_{i+4}v_{i+8})$, which means that two adjacent edges receive different colors. Third, the edges incident to the vertex v_i are $v_{i-1}v_i$, v_iv_{i+1}, $v_{i-4}v_i$ and v_iv_{i+4}. The construction indicates $\sigma(v_i) \ne$

$\sigma(v_{i-1}v_i), \sigma(v_iv_{i+1}), \sigma(v_{i-4}v_i)$, $\sigma(v_iv_{i+4})$, which means that a vertex receive a different color from its incident edges.

The above construction implies $\chi''(C_n(1,4)) \leq 5$ for $n = 5p + 11q$. On the other hand, there is $\chi''(C_n(1,4)) \geq 5$. Hence, $\chi''(C_n(1,4)) = 5$ for $n = 5p + 11q$. $\square$

Figure 1 shows $\sigma(C_{10}(1,4))$, $\sigma(C_{11}(1,4))$ and $\sigma(C_{16}(1,4))$.

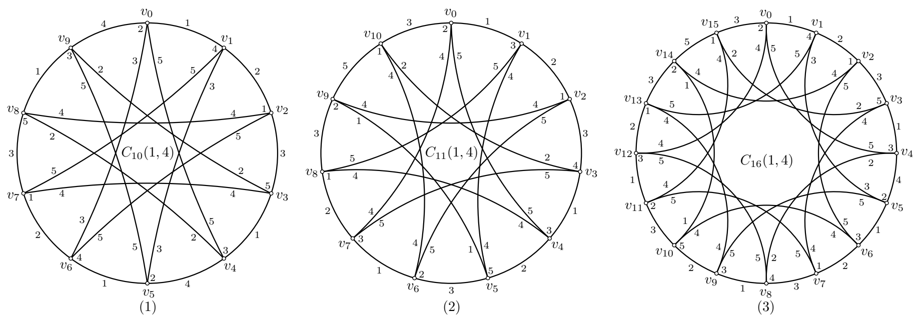

Figure 1. $\sigma(C_{10}(1,4))$ in subfigure (1), $\sigma(C_{11}(1,4))$ in subfigure (2) and $\sigma(C_{16}(1,4))$ in subfigure (3)

In the following sections, we will study the total coloring of $C_n(1,4)$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$ and $n = 13$, which are not included in the expression $n = 5p + 11q$.

3. Total coloring of $C_n(1,4)$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$

Lemma 3.1. $\chi''(C_n(1,4)) = 5$ *for* $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$.

Proof. We construct a 5-total coloring of $C_n(1,4))$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$ as follows, respectively.

$\sigma(C_9(1,4)) = (241251231, 123124125, 455533344)$, as illustrated in Figure 2(1),

$\sigma(C_{12}(1,4)) = (535345453434, 121212121212, 343453534545)$, as illustrated in Figure 2(2),

$\sigma(C_{14}(1,4)) = (25434352143431, 12121213212124, 34345545435355)$,

$\sigma(C_{17}(1,4)) = (24535345453434531, 12121212121212123, 45343453534545345)$,

$\sigma(C_{18}(1,4)) = (453453453453453453, 121212121212121212, 345345345345345345)$,

$\sigma(C_{19}(1,4)) = (2545345343121535321, 1212121212342142145, 4334554534515323453)$,

$\sigma(C_{23}(1,4)) = (25454354343215253241213, 12121212121324124525425, 34335545435451331314354)$,

$\sigma(C_{24}(1,4)) = (535345453434535345453434, 121212121212121212121212, 343453534545343453534545)$,

$\sigma(C_{28}(1,4)) = (2543435214343125434352143431, 1212121321212412121213212124, 3434554543535534345545435355)$,

$\sigma(C_{29}(1,4)) = (24535345453434535345453434531, 12121212121212121212121212123, 45343453534545343453534545345)$,

$\sigma(C_{34}(1,4)) = (2453534545343453124535345453434531, 1212121212121212312121212121212123, 4534345353454534545343453534545345)$,

$\sigma(C_{39}(1,4)) = (241532415324153231435231215353121425321, 123141231412314123123123531214352132153, 535425354253542545541445422435514344542)$.

By examining the colors of all adjacent or incident elements of $C_n(1,4)$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$, it is straightforward to verify that each construction is indeed a total coloring of $C_n(1,4)$. The existence of the above constructions indicates that the total chromatic number of $C_n(1,4)$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$ is not larger than 5, but on the other hand, the total chromatic number cannot be less than 5 too. Hence, there must be $\chi''(C_n(1,4)) = 5$ for $n = 9, 12, 14, 17, 18, 19, 23, 24, 28, 29, 34, 39$. $\square$

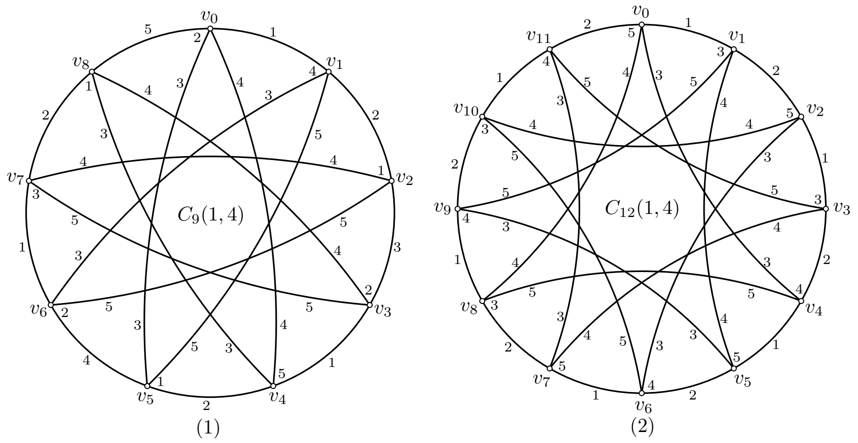

Figure 2. $\sigma(C_9(1,4))$ in subfigure (1) and $\sigma(C_{12}(1,4))$ in subfigure (2)

4. Total coloring of $C_{13}(1,4)$

We recall the result obtained by Chetwynd and Hilton [4] presented in Lemma 4.1, as it will be used in the proof of Lemma 4.2.

Lemma 4.1. *If regular graph G has a $(\Delta(G)+1)$-total coloring, then it has a vertex-coloring with colors $1, 2, \ldots, \Delta(G)+1$ such that $|V_j| \equiv |V(G)|$ $(mod\, 2)$ $(1 \leq j \leq \Delta(G)+1)$.*

Here, V_j represents the set of vertices assigned with the color j and will be used throughout the paper. It can be explicitly denoted as $V_j = \{v_{j_0}, v_{j_1}, \cdots v_{j_{(t-1)}}\}$, $1 \leq j \leq \Delta(G)+1$. We further define $d_{js} = (n + j_{s+1} - j_s) \pmod n$ (indices of j are modulo t), $0 \leq s \leq t-1$. Then, there is $d_{js} \in \{2, 3, 5, 6, 7, 8, 10, 11\}$ for $C_{13}(1,4)$.

Lemma 4.2 presents a necessary condition for the graph $C_{13}(1,4)$ to have a 5-total coloring.

Lemma 4.2. *If the circulant graph $C_{13}(1,4)$ has a 5-total coloring, then $|V_1| = |V_2| = |V_3| = |V_4| = 3$ and $|V_5| = 1$.*

Proof. Without loss of generality, we assume $|V_1| \geq |V_2| \geq |V_3| \geq |V_4| \geq |V_5|$. Since the maximum independent set in $C_{13}(1,4)$ has size 5, then $|V_j| \in \{5, 3, 1\} (1 \leq j \leq 5)$ by Lemma 4.1. From $\Sigma_{1 \leq j \leq 5} |V_j| = 13$, we have $|V_1| \in \{5, 3\}$.

We first consider the case of $|V_1| = 5$. Since $d_{1s} \in \{2, 3, 5, 6, 7, 8, 10, 11\}$ and $\sum_{0 \leq s \leq 4} d_{1s} = 13$, then $|\{d_{1s} | d_{1s} = 2\}| \in \{4, 2\}$. If $|\{d_{1s} | d_{1s} = 2\}| = 4$, there must be an integer i such that $\sigma(v_i) = \sigma(v_{i+4}) = 1$, a contradiction to the requirements of total coloring. If $|\{d_{1s} | d_{1s} = 2\}| = 2$, without loss of generality, we may let $V_1 = \{v_0, v_2, v_5, v_7, v_{10}\}$. We then have $\sigma(v_1), \sigma(v_1v_0), \sigma(v_1v_2), \sigma(v_1v_{10}), \sigma(v_1v_5) \neq 1$, a contradiction to the precondition that $C_{13}(1,4)$ has a 5-total coloring (see Figure 3(1)). Hence, $|V_1| = 3$. Since $\Sigma_{1 \leq j \leq 5} |V_j| = 13$, we have $|V_1| = |V_2| = |V_3| = |V_4| = 3$ and $|V_5| = 1$. □

Lemma 4.3. *If the circulant graph $C_{13}(1,4)$ has a 5-total coloring, then $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,3,8\},4), (\{2,5,6\},3), (\{2,5,6\},4), (\{3,3,7\},3)\}$ and there are at least three colors j such that $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,5,6\},3), (\{3,3,7\},3)\}$ for $1 \leq j \leq 4$, where e_j is the number of edges assigned with the color j.*

Proof. If the circulant graph $C_{13}(1,4)$ has a 5-total coloring, then by Lemma 4.2, $|V_1| = |V_2| = |V_3| = |V_4| = 3$. Since $d_{js} \in \{2, 3, 5, 6, 7, 8, 10, 11\}$ and $d_{j0} + d_{j1} + d_{j2} = 13$ for $1 \leq j \leq 4$, we have $\{d_{j0}, d_{j1}, d_{j2}\} \in \{\{2,3,8\}, \{2,5,6\}, \{3,3,7\}, \{3,5,5\}\}$.

Case 1. $\{d_{j0}, d_{j1}, d_{j2}\} = \{2, 3, 8\}$. Without loss of generality, we may let $V_j = \{v_0, v_2, v_5\}$. Then $\sigma(v_1), \sigma(v_1v_0), \sigma(v_1v_2), \sigma(v_1v_5) \neq j$, $\sigma(v_1v_{10}) = j$. It follows $\sigma(v_6v_7) = j$, $\sigma(v_{11}v_{12}) = j$, $\sigma(v_3v_4) = j$, $\sigma(v_8v_9) = j$ and $e_j = 4$. (see Figure 3(2)).

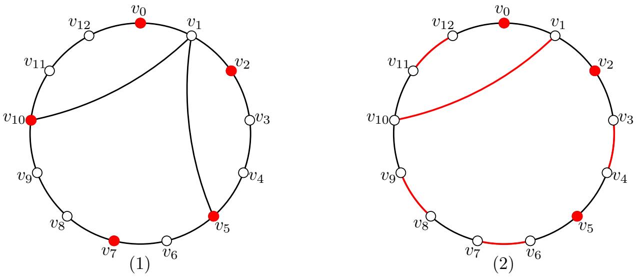

Figure 3. $\sigma(C_{13}(1,4))$ for $|V_1| = 5$ in subfigure (1), and for $|V_1| = 3$ and $\{d_{j0}, d_{j1}, d_{j2}\} = \{2, 3, 8\}$ in subfigure (2)

Case 2. $\{d_{j0}, d_{j1}, d_{j2}\} = \{2, 5, 6\}$. Let $V_j = \{v_0, v_2, v_7\}$. Then $\sigma(v_3), \sigma(v_3v_2), \sigma(v_3v_7) \neq j$, $\sigma(v_3v_{12}) = j$ or $\sigma(v_3v_4) = j$. If $\sigma(v_3v_{12}) = j$, then $\sigma(v_{11}v_{10}) = j$, $\sigma(v_1v_5) = j$. It follows $\sigma(v_6), \sigma(v_6v_5), \sigma(v_6v_7), \sigma(v_6v_2), \sigma(v_6v_{10}) \neq j$, a contradiction to the precondition (see Figure 4(1)). Hence, $\sigma(v_3v_4) = j$. Since $\sigma(v_1)$, $\sigma(v_1v_0)$, $\sigma(v_1v_2) \neq j$, then $\sigma(v_1v_{10}) = j$ or $\sigma(v_1v_5) = j$. If $\sigma(v_1v_{10}) = j$, then $\sigma(v_{11}v_{12}) = j$, $\sigma(v_8v_9) = j$, $\sigma(v_5v_6) = j$ and $e_j = 4$ (see Figure 4(2)). If $\sigma(v_1v_5) = j$, then $\sigma(v_6v_{10}) = j$, $\sigma(v_{11}v_{12}) = j$, $\sigma(v_8v_9) = j$, and $e_j = 3$ (see Figure 4(3)).

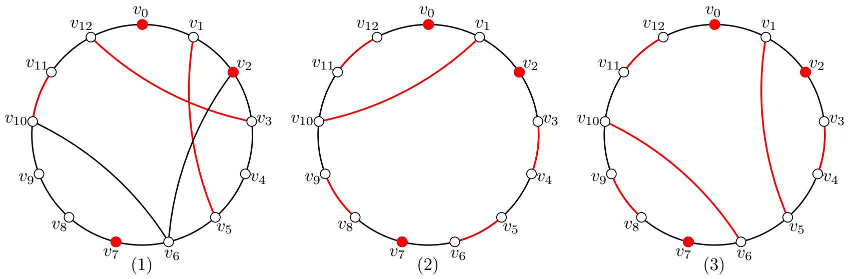

Figure 4. $C_{13}(1,4)$ for $|V_1| = 3$ and $\{d_{j0}, d_{j1}, d_{j2}\} = \{2, 5, 6\}$. $\sigma(v_3v_{12}) = j$ in subfigure (1), $\sigma(v_3v_4) = j$ and $\sigma(v_1v_{10}) = j$ in subfigure (2) and $\sigma(v_3v_4) = j$ and $\sigma(v_1v_5) = j$ in subfigure (3)

Case 3. $\{d_{j0}, d_{j1}, d_{j2}\} = \{3, 3, 7\}$. Let $V_j = \{v_0, v_3, v_6\}$, then $\sigma(v_2)$, $\sigma(v_2v_3)$, $\sigma(v_2v_6) \neq j$, $\sigma(v_2v_{11}) = j$ or $\sigma(v_2v_1) = j$. If $\sigma(v_2v_{11}) = j$, then $\sigma(v_7v_8) = j$. It follows $\sigma(v_{12}), \sigma(v_{12}v_0)$, $\sigma(v_{12}v_{11}), \sigma(v_{12}v_3), \sigma(v_{12}v_8) \neq j$, a contradiction to the precondition (see Figure 5(1)). Hence, $\sigma(v_2v_1) = j$. Since $\sigma(v_{10}), \sigma(v_{10}v_1), \sigma(v_{10}v_6) \neq j$, then $\sigma(v_{10}v_{11}) = j$ or $\sigma(v_{10}v_9) = j$. If $\sigma(v_{10}v_{11}) = j$, then $\sigma(v_7v_8) = j$. It follows $\sigma(v_{12}), \sigma(v_{12}v_0), \sigma(v_{12}v_{11}), \sigma(v_{12}v_3), \sigma(v_{12}v_8)$ $\neq j$, a contradiction to the precondition (see Figure 5(2)). If $\sigma(v_{10}v_9) = j$, then $\sigma(v_5v_4) = j$. It follows $\sigma(v_8v_7) = j$, $\sigma(v_{12}v_{11}) = j$ and $e_j = 5$ (see Figure 5(3)) or $\sigma(v_8v_{12}) = j$, $\sigma(v_7v_{11}) = j$ and $e_j = 3$ (see Figure 5(4)).

Case 4. $\{d_{j0}, d_{j1}, d_{j2}\} = \{3, 5, 5\}$. Let $V_j = \{v_0, v_3, v_8\}$. Then $\sigma(v_{12}), \sigma(v_{12}v_0)$, $\sigma(v_{12}v_3), \sigma(v_{12}v_8) \neq j$, $\sigma(v_{12}v_{11}) = j$. It follow $\sigma(v_7v_6) = j$, $\sigma(v_2v_1) = j$, $\sigma(v_{10}v_9) = j$, $\sigma(v_5v_4) = j$ and $e_j = 5$ (see Figure 6).

By Cases 1-4, we have $e_j \in \{3, 4, 5\}$ for $1 \leq j \leq 4$. Since $\sum_{1 \leq j \leq 4} e_j \leq 13$, we have $e_j \leq 4$. It follows $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,3,8\},4), (\{2,5,6\},3), (\{2,5,6\},4), (\{3,3,7\},3)\}$ and there are at least three colors j such that $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,5,6\},3), (\{3,3,7\},3)\}$. □

Further, according to Case 2 and Case 3 in the above, we have the following lemma.

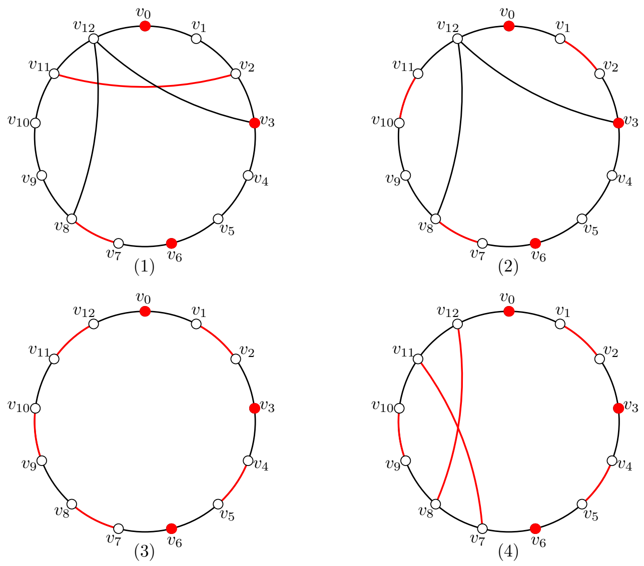

Figure 5. $C_{13}(1,4)$ for $|V_1| = 3$ and $\{d_{j0}, d_{j1}, d_{j2}\} = \{3,3,7\}$. $\sigma(v_2v_{11}) = j$ in subfigure (1), $\sigma(v_2v_1) = j$ and $\sigma(v_{10}v_{11}) = j$ in subfigure (2), $\sigma(v_2v_1) = j$, $\sigma(v_{10}v_9) = j$ and $\sigma(v_8v_7) = j$ in subfigure (3), and $\sigma(v_2v_1) = j$, $\sigma(v_{10}v_9) = j$ and $\sigma(v_8v_{12}) = j$ in subfigure (4)

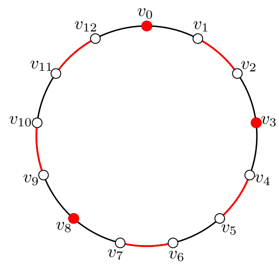

Figure 6. $C_{13}(1,4)$ for $|V_1| = 3$ and $\{d_{j0}, d_{j1}, d_{j2}\} = \{3,5,5\}$

Lemma 4.4. *If the circulant graph $C_{13}(1,4)$ has a 5-total coloring, then $\sigma(v_{j_s+1}v_{j_s+2}) = \sigma(v_{j_s+4}v_{j_s+5}) = \sigma(v_{j_s+9}v_{j_s+10}) = \sigma(v_{j_s})$ for $d_{js} = 6$ or $d_{js} + d_{j(s+1)} = 6$, where $1 \le j \le 4$ and indices of v are modulo n=13.*

For $1 \le j_1, j_2 \le 4$, let $d^2_{j_1,j_2} = i_1 - i_2$ where $\sigma(v_{i_1}) = \sigma(v_{i_1+6}) = j_1$ and $\sigma(v_{i_2}) = \sigma(v_{i_2+6}) = j_2$. Then $d^2_{j_1,j_2} \in \{1,2,3,4,5,8,9,10,11,12\}$.

Lemma 4.5. *If the circulant graph $C_{13}(1,4)$ has a 5-total coloring, then $d^2_{j_1,j_2} \in \{1,2,4,9,11,12\}$.*

Proof. If the circulant graph $C_{13}(1,4)$ has a 5-total coloring, then there are at least three colors j such that $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,5,6\},3), (\{3,3,7\},3)\}$ by Lemma 4.3. Without loss of gennerality, let $(\{d_{j_10}, d_{j_11}, d_{j_12}\}, e_{j_1}), (\{d_{j_20}, d_{j_21}, d_{j_22}\}, e_{j_2}) \in \{(\{2,5,6\},3), (\{3,3,7\},3)\}$, and let $\sigma(v_0) = \sigma(v_6) = j_1$. By Lemma 4.4, we have $\sigma(v_1v_2) = \sigma(v_4v_5) = \sigma(v_9v_{10}) = j_1$. If $d^2_{j_1,j_2} = 3$, then $\sigma(v_4v_5) = j_2$, a contradiction (see Figure 7(1)). If $d^2_{j_1,j_2} = 5$, then $\sigma(v_9v_{10}) = \sigma(v_1v_2) = j_2$, a contradiction (see Figure 7(2)). So, $d^2_{j_1,j_2} \notin \{3,5\}$. By symmetry, $d^2_{j_1,j_2} \notin \{10,8\}$. Hence, $d^2_{j_1,j_2} \in \{1,2,4,9,11,12\}$. □

Lemma 4.6. $\chi''(C_{13}(1,4)) = 6$.

Proof. Suppose that $C_{13}(1,4)$ has a 5-total coloring. By Lemma 4.2–4.3, $|V_j| = 3$, $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,3,8\},4), (\{2,5,6\},3), (\{2,5,6\},4), (\{3,3,7\},3)\}$, and there are

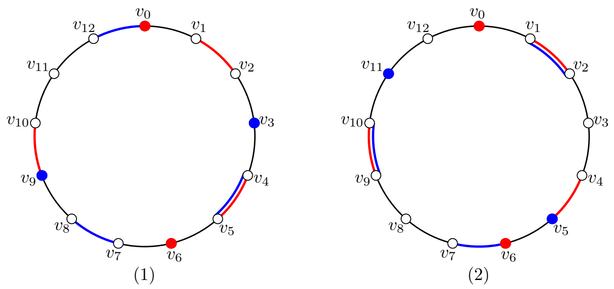

Figure 7. $C_{13}(1,4)$ for $d^2_{j_1,j_2} = 3$ in subfigure (1), and for $d^2_{j_1,j_2} = 5$ in subfigure (2)

at least three colors j such that $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,5,6\},3), (\{3,3,7\},3)\}$ for $1 \le j \le 4$. Without loss of generality, let $(\{d_{j0}, d_{j1}, d_{j2}\}, e_j) \in \{(\{2,5,6\},3), (\{3,3,7\},3)\}$ for $j = 1,2,3$, and let $\sigma(v_0) = \sigma(v_6) = 1$. Then $\sigma(v_1v_2) = \sigma(v_4v_5) = \sigma(v_9v_{10}) = 1$. By symmetry, we need only consider $d^2_{1,2} \in \{1,2,4\}$.

Case 1. $d^2_{1,2} = 1$. Then $\sigma(v_2v_3) = \sigma(v_5v_6) = \sigma(v_{10}v_{11}) = 2$ and $d^2_{1,3} \in \{2,4,9,11,12\}$. If $d^2_{1,3} \in \{4,9,11\}$, then $d^2_{2,3} \notin \{1,2,4,9,11,12\}$. Hence, $d^2_{1,3} \in \{2,12\}$.

Case 1.1 $d^2_{1,3} = 2$. Then $\sigma(v_3v_4) = \sigma(v_6v_7) = \sigma(v_{11}v_{12}) = 3$ and $d^2_{1,4} \in \{4,9,11,12\}$. If $d^2_{1,4} \in \{4,9,11\}$, then $d^2_{2,4} \notin \{1,2,4,9,11,12\}$. If $d^2_{1,4} = 12$, then $d^2_{3,4} \notin \{1,2,4,9,11,12\}$. So, $(\{d_{40}, d_{41}, d_{42}\}, e_4) = (\{2,3,8\},4)$. We have that $\sigma(v_7v_8) = \sigma(v_8v_9) = \sigma(v_{12}v_0) = \sigma(v_0v_1) = 4$ (see Figure 8 (1)), a contradiction to the the requirements of total coloring.

Case 1.2 $d^2_{1,3} = 12$. Then $\sigma(v_0v_1) = \sigma(v_3v_4) = \sigma(v_8v_9) = 3$ and $d^2_{1,4} \in \{2,4,9,11\}$. If $d^2_{1,4} \in \{4,9,11\}$, then $d^2_{2,4} \notin \{1,2,4,9,11,12\}$. If $d^2_{1,4} = 2$, then $d^2_{3,4} \notin \{1,2,4,9,11,12\}$. So, $(\{d_{40}, d_{41}, d_{42}\}, e_4) = (\{2,3,8\},4)$. We have that $\sigma(v_6v_7) = \sigma(v_7v_8) = \sigma(v_{11}v_{12}) = \sigma(v_{12}v_0) = 4$ (see Figure 8 (2)), a contradiction.

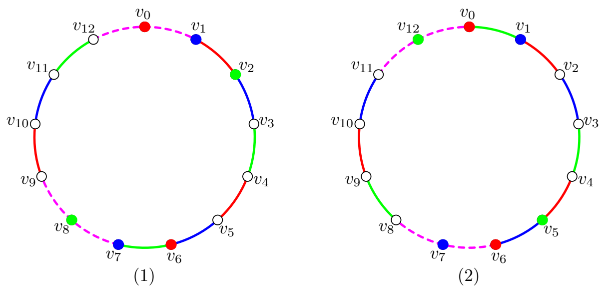

Figure 8. $C_{13}(1,4)$ for $d^2_{1,2} = 1$. $d^2_{1,3} = 2$ in subfigure (1) and $d^2_{1,3} = 12$ in subfigure (2)

Case 2. $d^2_{1,2} = 2$. Then $\sigma(v_3v_4) = \sigma(v_6v_7) = \sigma(v_{11}v_{12}) = 2$ and $d^2_{1,3} \in \{1,4,9,11,12\}$. If $d^2_{1,3} \in \{9,12\}$, $d^2_{2,3} \notin \{1,2,4,9,11,12\}$. Hence, $d^2_{1,3} \in \{1,4,11\}$.

Case 2.1 $d^2_{1,3} = 1$. This case is analogous to Case 1.1.

Case 2.2 $d^2_{1,3} = 4$. Then $\sigma(v_5v_6) = \sigma(v_8v_9) = \sigma(v_0v_1) = 3$ and $d^2_{1,4} \in \{1,9,11,12\}$. If $d^2_{1,4} \in \{9,12\}$, then $d^2_{2,4} \notin \{1,2,4,9,11,12\}$. If $d^2_{1,4} \in \{1,11\}$, then $d^2_{3,4} \notin \{1,2,4,9,11,12\}$. So, $(\{d_{40}, d_{41}, d_{42}\}, e_4) = (\{2,3,8\},4)$. We have that $\sigma(v_2v_3) = \sigma(v_7v_8) = \sigma(v_{10}v_{11}) = \sigma(v_{12}v_0) = 4$. We then have $\sigma(v_3), \sigma(v_7), \sigma(v_{11}), \sigma(v_{12}) \neq 4$. It follows $\{d_{40}, d_{41}, d_{42}\} = \{4,4,5\} \neq \{2,3,8\}$(see Figure 9 (1)), a contradiction.

Case 2.3 $d^2_{1,3} = 11$. Then $\sigma(v_{12}v_0) = \sigma(v_2v_3) = \sigma(v_7v_8) = 3$ and $d^2_{1,4} \in \{1,4,9,12\}$. If $d^2_{1,4} \in \{9,12\}$, then $d^2_{2,4} \notin \{1,2,4,9,11,12\}$. If $d^2_{1,4} \in \{1,4\}$, then $d^2_{3,4} \notin \{1,2,4,9,11,12\}$. So, $(\{d_{40}, d_{41}, d_{42}\}, e_4) = (\{2,3,8\},4)$. We have that $\sigma(v_0v_1) = \sigma(v_5v_6) = \sigma(v_8v_9) =$

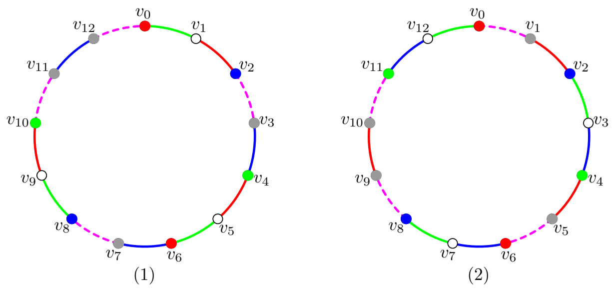

Figure 9. $C_{13}(1,4)$ for $d^2_{1,2}=2$. $d^2_{1,3}=4$ in subfigure (1) and $d^2_{1,3}=11$ in subfigure (2)

$\sigma(v_{10}v_{11})=4$. We then have $\sigma(v_1),\sigma(v_5),\sigma(v_9),\sigma(v_{10})\neq 4$. It follows $\{d_{40},d_{41},d_{42}\}=\{4,4,5\}\neq\{2,3,8\}$ (see Figure 9 (2)), a contradiction.

Case 3. $d^2_{1,2}=4$. Then $d^2_{1,3}\in\{1,2,9,11,12\}$. If $d^2_{1,3}\in\{1,9,11,12\}$, $d^2_{2,3}\notin\{1,2,4,9,11,12\}$. Hence, $d^2_{1,3}=2$, which is analogous to Case 2.2.

From Cases 1-3, the assumption does not hold. Thus, $C_{13}(1,4)$ does not have a 5-total coloring, i.e. $\chi''(C_{13}(1,4))\geq 6$. However, according to reference [8], $\chi''(C_{13}(1,4))\leq 6$. So, $\chi''(C_{13}(1,4))=6$. □

5. Conclusion

In conclusion, we have completely determined the total chromatic numbers of $C_n(1,4)$ for all $n\geq 9$. By combining Lemmas 2.1, 3.1 and 4.6, we obtain the following theorem.

Theorem 5.1. *$\chi''(C_n(1,4))=6$ for $n=13$, and $\chi''(C_n(1,4))=5$ for all others.*

In other words, circulant graphs $C_n(1,4)$ are Type I for all $n\geq 9$ except for 13, which is Type II. These results contribute to the conjecture that 4-regular circulant graphs are all Type 1 graphs except for a finite number of Type 2 graphs, proposed by Khennoufa and Togni [7].

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