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On a New Generalization of Bivariate Fibonacci Polynomials and Their Matrix Representations

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Abstract

In this study, we introduce the concept of d-bivariate Fibonacci polynomials, which is a generalization of the classical bivariate Fibonacci polynomials. We obtain several fundamental properties for these new polynomials including the generating function, the Binet's formula, some combinatorial identities and summation formulas. Then, we define the infinite d-bivariate Fibonacci polynomials matrix, which is a Riordan matrix. By Riordan method, we give two new factorizations of the infinite Pascal matrix whose etries are the d-bivariate Fibonacci polynomials.

Keywords: *Bivariate Fibonacci polynomials; d-bivariate Fibonacci polynomials; Pascal matrix; Riordan matrix.*
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1. Introduction

Many researchers have been interested in number sequences and their polynomials for long years since they have many applications in nature and various fields. The Fibonacci numbers are one of the most widely recognized number sequences. Fibonacci numbers are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \ \ n \geq 1$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. In 1883, Fibonacci polynomials, studied by Catalan, were defined by the recurrence relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \ \ n \geq 1$$

with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. In [16], Nalli and Haukkanen introduced $h(x)$-Fibonacci polynomials by

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \ \ n \geq 1$$

with initial conditions $F_{h,0}(x) = 0$ and $F_{h,1}(x) = 1$. Lee and Asci [15] defined (p,q)-Fibonacci polynomials by

$$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x), \ \ n \geq 1$$

with initial conditions $F_{p,q,0}(x) = 0$ and $F_{p,q,1}(x) = 1$. Let $d \in \mathbb{Z}^+ = \{1,2,\dots\}$ and $p_i(x)$ be a real polynomial for each $i = 1,2,\dots,d+1$. Sadaoui and Krelifa [18] generalized (p,q)-Fibonacci polynomials to d- Fibonacci polynomials, which are defined such that

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x), \ n \geq 1$$

with initial conditions $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$.
Catalani [8] defined the bivariate Fibonacci polynomials as

$$F_{n+1}(x,y) = xF_n(x,y) + yF_{n-1}(x,y), \ \ n \geq 1$$

with initial conditions $F_0(x,y) = 0$ and $F_1(x,y) = 1$, where $x \neq 0, y \neq 0$ and $x^2 + 4y \neq 0$. In [2], Bao and Yang introduced homogeneous q-Laguerre polynomials and homogeneous little q-Jacobi polynomials. These polynomials can be viewed separately as solutions to two q-partial differential equations. Özimamoğlu and Kaya [17] defined the Pell-Lucas and the symmetric Pell-Lucas matrices, and studied the factorizations and eigenvalues of these matrices. In [7], Catalani introduced the generalized bivariate Fibonacci polynomials, and presented

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the summation and inversion formulas for these polynomials. Moreover, Catalani obtained some identities of bivariate Fibonacci and Lucas polynomials in [8, 9].
In [13], Lawden defined the $n\times n$ lower triangular Pascal matrix $P=\left[p_{i,j}\right]$ by

$$p_{i,j}=\begin{cases}0, & \text{if } i<j\\ \binom{i-1}{j-1}, & \text{if } i\geq j\end{cases}$$

for $i,j=1,2,\ldots,n$ (see, for example, [4] and [6]). In addition, the infinite Pascal matrix P is given by

$$P=\begin{bmatrix}1&0&0&0&\cdots&0\\1&1&0&0&\cdots&0\\1&2&1&0&\cdots&0\\1&3&3&1&\cdots&0\\1&4&6&4&1&\cdots\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}. \tag{1.1}$$

The Pascal matrices have several applications in probability, numerical analysis, surface reconstruction and combinatorics. In [3], [5] and [22], the authors investigated the linear algebras of the generalized Pascal functional matrix, the Pascal matrix and the generalized Pascal matrix, respectively. In [14] and [21], the authors obtained two factorizations of the Pascal matrix involving the Fibonacci matrix.
In [19], Shapiro et al. defined the Riordan group as follows.
Let $i,j\in\mathbb{N}=\{0,1,2,\ldots\}$ and $A=\left[a_{i,j}\right]$ be an infinite matrix with entries in $\mathbb{C}$. Let $k\in\mathbb{N}$ and $c_k(v)=\sum_{t=0}^{\infty}a_{t,k}v^t$ be the generating function of the kth column of A. The matrix $A=(g(v),f(v))$ is called a *Riordan matrix*, if $c_k(v)=g(v)\left[f(v)\right]^k$, where $g(v)=\sum_{t=0}^{\infty}g_tv^t$ and $f(v)=\sum_{t=1}^{\infty}f_tv^t$ with $g_0=f_1=1$.
We denote by $\mathscr{R}$ the set of Riordan matrices. It is well-known that $\mathscr{R}$ is a group under matrix multiplication $*$, and is called *Riordan group*. We present the following features related to Riordan group.

(i) $(g(v),f(v))*C(v)=g(v)C(f(v))$, where $C(v)$ is a column vector (matrix multiplication $*$ with $C(v)$),
(ii) $(g(v),f(v))*(h(v),l(v))=(g(v)h(f(v)),l(f(v)))$ (matrix multiplication $*$),
(iii) $i_{\mathscr{R}}=(1,v)$, where $i_{\mathscr{R}}$ is the identity element of $\mathscr{R}$,
(iv) $(g(v),f(v))^{-1}=\left(\frac{1}{g(\bar{f}(v))},\bar{f}(v)\right)$, where $\bar{f}(v)$ is compositional inverse of $f(v)$ (inverse element).

Riordan group has many applications. In [19], the three applications of Riordan group are presented by Euler's problem of the King walks, binomial and inverse identities and a Bessel-Neumann expansion. Also, Cheon et al. [10] gave a generalization of Lucas polynomial sequence by using the Riordan array which is derived from weighted Delannoy numbers.
This paper is structured as follows:
In Section 2, we describe the d-bivariate Fibonacci polynomials. These polynomials are a new generalization of the known bivariate Fibonacci polynomials. We provide a variety of conclusions for the d-bivariate Fibonacci polynomials including the generating function, the Binet's formula, some combinatorial identities and summation formulas. We define the matrix $Q_d(x,y)$, and show that the power of $Q_d(x,y)$ generates the d-bivariate Fibonacci polynomials. In Section 3, we introduce the infinite d-bivariate Fibonacci polynomials matrix, which is a Riordan matrix. Then, we derive two factorizations of the infinite Pascal matrix including d-bivariate Fibonacci polynomials.

2. d-Bivariate Fibonacci Polynomials

In this part, we introduce a new generalization of bivariate Fibonacci polynomials.

Definition 2.1. *Let $d\in\mathbb{Z}^+$ and $r_j(x,y)$ be a real polynomial for $j=1,2,\ldots,d+1$. Then, d-bivariate Fibonacci polynomials $F_n^{(d)}(x,y)$ are defined by the recurrence relation*

$$F_{n+1}^{(d)}(x,y)=r_1(x,y)F_n^{(d)}(x,y)+r_2(x,y)F_{n-1}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{n-d}^{(d)}(x,y),\ n\geq 1$$

with initial conditions $F_n^{(d)}(x,y)=0$ for $n\leq 0$ and $F_1^{(d)}(x,y)=1$.

We provide a few terms of d-bivariate Fibonacci polynomials in Table 1.

Table 1: Some values of d-bivariate Fibonacci polynomials.

n	$F_n^{(d)}(x,y)$
2	$r_1(x,y)$
3	$r_1^2(x,y)+r_2(x,y)$
4	$r_1^3(x,y)+2r_1(x,y)r_2(x,y)+r_3(x,y)$
5	$r_1^4(x,y)+3r_1^2(x,y)r_2(x,y)+2r_1(x,y)r_3(x,y)+r_2^2(x,y)+r_4(x,y)$
6	$r_1^5(x,y)+4r_1^3(x,y)r_2(x,y)+3r_1^2(x,y)r_3(x,y)+3r_1(x,y)r_2^2(x,y)+2r_1(x,y)r_4(x,y)+2r_2(x,y)r_3(x,y)+r_5(x,y)$

In Definition 2.1, if we take $r_1(x,y)=x, r_2(x,y)=y$ and $r_k(x,y)=0$ for $k=3,4,\ldots,d+1$, so we obtain $F_n^{(d)}(x,y)=F_n(x,y)$. Then d-bivariate Fibonacci polynomials are a generalization of the known bivariate Fibonacci polynomials. Also, for the special cases of d-bivariate Fibonacci polynomials $F_n^{(d)}(x,y)$, we obtain the polynomials given in Table 2.
From Definition 2.1, for d-bivariate Fibonacci polynomials, the characteristic equation is given by

$$u^{d+1}-r_1(x,y)u^d-\cdots-r_{d+1}(x,y)=0. \tag{2.1}$$

Table 2: Special cases of $F_n^{(d)}(x,y)$ such that $r_k(x,y)=0$ for $k=3,4,\ldots,d+1$.

$r_1(x,y)$	$r_2(x,y)$	d-bivariate Fibonacci polynomials $F_n^{(d)}(x,y)$
x	y	Bivariate Fibonacci polynomials $F_n(x,y)$ [7, 8, 9]
x	$-y$	Bivariate Vieta-Fibonacci polynomials $V_n(x,y)$ [12]
$2xy$	y	Bivariate Pell polynomials $P_n(x,y)$ [11]
xy	$2y$	Bivariate Jacobsthal polynomials $J_n(x,y)$ [20]
$3y$	$-2x$	Bivariate Mersenne polynomials $M_n(x,y)$ [1]

Theorem 2.2. *Let $n \geq d$. Then we have*

$$\begin{aligned} u^n &= F_{n-d+1}^{(d)}(x,y)u^d + \left(r_2(x,y)F_{n-d}^{(d)}(x,y) + \cdots + r_{d+1}(x,y)F_{n-2d+1}^{(d)}(x,y)\right)u^{d-1} \\ &\quad + \left(r_3(x,y)F_{n-d}^{(d)}(x,y) + \cdots + r_{d+1}(x,y)F_{n-2d+2}^{(d)}(x,y)\right)u^{d-2} + \cdots + r_{d+1}(x,y)F_{n-d}^{(d)}(x,y). \end{aligned} \tag{2.2}$$

Proof. To prove the theorem, we use mathematical induction on n. For $n=1$, it is clear that the equation (2.2) is true. Suppose that the equation (2.2) satisfies for $n=k$. We will show that the equation (2.2) is true for $n=k+1$. Using Definition 2.1 and the characteristic equation (2.1), we derive

$$\begin{aligned} u^{k+1} &= u^k u \\ &= F_{k-d+1}^{(d)}(x,y)\left(r_1(x,y)u^d + \cdots + r_{d+1}(x,y)\right) + \left(r_2(x,y)F_{k-d}^{(d)}(x,y) + \cdots + r_{d+1}(x,y)F_{k-2d+1}^{(d)}(x,y)\right)u^d \\ &\quad + \left(r_3(x,y)F_{k-d}^{(d)}(x,y) + \cdots + r_{d+1}(x,y)F_{k-2d+2}^{(d)}(x,y)\right)u^{d-1} + \cdots + r_{d+1}(x,y)F_{k-d}^{(d)}(x,y)u \\ &= \left(r_1(x,y)F_{k-d+1}^{(d)}(x,y) + \cdots + r_{d+1}(x,y)F_{k-2d+1}^{(d)}(x,y)\right)u^d + \left(r_2(x,y)F_{k-d+1}^{(d)}(x,y) + \cdots + r_{d+1}(x,y)F_{k-2d+2}^{(d)}(x,y)\right)u^{d-1} \\ &\quad + \cdots + \left(r_d(x,y)F_{k-d+1}^{(d)}(x,y) + r_{d+1}(x,y)F_{k-d}^{(d)}(x,y)\right)u + r_{d+1}(x,y)F_{k-d+1}^{(d)}(x,y), \end{aligned}$$

which completes the proof. □

Theorem 2.3. *The generating function of d-bivariate Fibonacci polynomials is given by*

$$g^{(d)}(u) = \frac{u}{1 - r_1(x,y)u - r_2(x,y)u^2 - \cdots - r_{d+1}(x,y)u^{d+1}}.$$

Proof. We get

$$\begin{aligned} g^{(d)}(u) &= \sum_{i=0}^{\infty} F_i^{(d)}(x,y)u^i \\ &= F_0^{(d)}(x,y) + F_1^{(d)}(x,y)u + F_2^{(d)}(x,y)u^2 + \cdots + F_n^{(d)}(x,y)u^n + \cdots. \end{aligned} \tag{2.3}$$

If we multiply (2.3) by $r_1(x,y)u, r_2(x,y)u^2, \ldots, r_{d+1}(x,y)u^{d+1}$, respectively, then we obtain the following equations.

$$\begin{aligned} r_1(x,y)ug^{(d)}(u) &= r_1(x,y)F_0^{(d)}(x,y)u + r_1(x,y)F_1^{(d)}(x,y)u^2 + r_1(x,y)F_2^{(d)}(x,y)u^3 + \cdots \\ r_2(x,y)u^2g^{(d)}(u) &= r_2(x,y)F_0^{(d)}(x,y)u^2 + r_2(x,y)F_1^{(d)}(x,y)u^3 + r_2(x,y)F_2^{(d)}(x,y)u^4 + \cdots \\ &\vdots \\ r_{d+1}(x,y)u^{d+1}g^{(d)}(u) &= r_{d+1}(x,y)F_0^{(d)}(x,y)u^{d+1} + r_{d+1}(x,y)F_1^{(d)}(x,y)u^{d+2} + \cdots. \end{aligned}$$

If the necessary calculations are made, by Definition 2.1 we have

$$g^{(d)}(u)\left(1 - r_1(x,y)u - r_2(x,y)u^2 - \cdots - r_{d+1}(x,y)u^{d+1}\right) = F_0^{(d)}(x,y) + \left(F_1^{(d)}(x,y) - r_1(x,y)F_0^{(d)}(x,y)\right)u$$

and so

$$g^{(d)}(u) = \frac{u}{1 - r_1(x,y)u - r_2(x,y)u^2 - \cdots - r_{d+1}(x,y)u^{d+1}}.$$

□

Let the set of the roots of (2.1) be $\{\alpha_1(x,y), \alpha_2(x,y), \ldots, \alpha_{d+1}(x,y)\}$. Namely, we get

$$\frac{u}{1 - r_1(x,y)u - r_2(x,y)u^2 - \cdots - r_{d+1}(x,y)u^{d+1}} = \sum_{i=1}^{d+1} \frac{C_i(x,y)}{1 - \alpha_i(x,y)u}.$$

By Theorem 2.3, we find that

$$\sum_{i=0}^{\infty} F_i^{(d)}(x,y)u^i = \sum_{i=1}^{d+1} C_i(x,y) \sum_{j=0}^{\infty} \alpha_i^j(x,y)u^j.$$

Then, we can obtain the Binet's formula for $F_n^{(d)}(x,y)$ in the following corollary.

Corollary 2.4. *The Binet's formula for d-bivariate Fibonacci polynomials is given by*

$$F_n^{(d)}(x,y)=\sum_{j=1}^{d+1}C_j(x,y)\alpha_j^n(x,y).$$

The multinomial coefficients, in particular, allow us to offer the explicit form of the d-bivariate Fibonacci polynomials.

Theorem 2.5. *For $n\geq 0$, then we have*

$$F_{n+1}^{(d)}(x,y)=\sum_{\substack{i_1,i_2,\dots,i_{d+1}\\ i_1+2i_2+\cdots+(d+1)i_{d+1}=n}}\binom{i_1+i_2+\cdots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}}r_1^{i_1}(x,y)r_2^{i_2}(x,y)\cdots r_{d+1}^{i_{d+1}}(x,y).$$

Proof. From Theorem 2.3, we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}F_{i+1}^{(d)}(x,y)u^i &= \frac{1}{1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1}}\\ &= \sum_{i=0}^{\infty}\left(r_1(x,y)u+r_2(x,y)u^2+\cdots+r_{d+1}(x,y)u^{d+1}\right)^i\\ &= \sum_{i=0}^{\infty}\left[\sum_{i_1+i_2+\cdots+i_{d+1}=i}\binom{i}{i_1,i_2,\dots,i_{d+1}}r_1^{i_1}(x,y)\cdots r_{d+1}^{i_{d+1}}(x,y)\right]\\ &\quad u^{i_1+2i_2+\cdots+(d+1)i_{d+1}}\\ &= \sum_{i=0}^{\infty}\left[\sum_{\substack{i_1,i_2,\dots,i_{d+1}\\ i_1+2i_2+\cdots+(d+1)i_{d+1}=i}}\binom{i_1+i_2+\cdots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}}r_1^{i_1}(x,y)\cdots r_{d+1}^{i_{d+1}}(x,y)\right]u^i.\end{aligned}$$

Hence the proof is completed. □

Theorem 2.6. *The sum of d-bivariate Fibonacci polynomials is given by*

$$\sum_{i=0}^{\infty}F_i^{(d)}(x,y)=\frac{1}{1-r_1(x,y)-r_2(x,y)-\cdots-r_{d+1}(x,y)}.$$

Proof. We get

$$\sum_{i=0}^{\infty}F_i^{(d)}(x,y)=F_0^{(d)}(x,y)+F_1^{(d)}(x,y)+\cdots+F_n^{(d)}(x,y)+\cdots. \tag{2.4}$$

Multiplying (2.4) by $r_1(x,y),r_2(x,y),\dots,r_{d+1}(x,y)$, respectively, then we have

$$\begin{aligned}r_1(x,y)\sum_{i=0}^{\infty}F_i^{(d)}(x,y) &= r_1(x,y)F_0^{(d)}(x,y)+r_1(x,y)F_1^{(d)}(x,y)+\cdots+r_1(x,y)F_n^{(d)}(x,y)+\cdots\\ r_2(x,y)\sum_{i=0}^{\infty}F_i(x,y) &= r_2(x,y)F_0^{(d)}(x,y)+r_2(x,y)F_1^{(d)}(x,y)+\cdots+r_2(x,y)F_n^{(d)}(x,y)+\cdots\\ &\vdots\\ r_{d+1}(x,y)\sum_{i=0}^{\infty}F_i^{(d)}(x,y) &= r_{d+1}(x,y)F_0^{(d)}(x,y)+r_{d+1}(x,y)F_1^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_n^{(d)}(x,y)+\cdots.\end{aligned}$$

If we take the necessary calculations, from Definition 2.1 we can have

$$\sum_{i=0}^{\infty}F_i^{(d)}(x,y)\left(1-r_1(x,y)-r_2(x,y)-\cdots-r_{d+1}(x,y)\right)=F_0^{(d)}(x,y)+\left(F_1^{(d)}(x,y)-r_1(x,y)F_0^{(d)}(x,y)\right)$$

and so

$$\sum_{i=0}^{\infty}F_i^{(d)}(x,y)=\frac{1}{1-r_1(x,y)-r_2(x,y)-\cdots-r_{d+1}(x,y)}.$$

□

Nalli and Haukkanen [16] defined the matrix

$$Q_h(x)=\begin{bmatrix}h(x) & 1\\ 1 & 0\end{bmatrix},$$

Lee and Asci [15] introduced the matrix

$$Q_{p,q}(x)=\begin{bmatrix}p(x) & q(x)\\ 1 & 0\end{bmatrix},$$

that plays the role of the Fibonacci matrix

$$Q=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}.$$

Then in [18], Sadaoui and Krelifa defined the matrix

$$Q_d(x)=\begin{bmatrix} p_1(x) & p_2(x) & \cdots & & p_{d+1}(x)\\ 1 & 0 & & & 0\\ 0 & \ddots & & & \\ & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & 0\end{bmatrix}. \tag{2.5}$$

Now we introduce the matrix $Q_d(x,y)$ which is a generalization of the $Q_d(x)$ in (2.5) as follows:

$$Q_d(x,y)=\begin{bmatrix} r_1(x,y) & r_2(x,y) & \cdots & & r_{d+1}(x,y)\\ 1 & 0 & & & 0\\ 0 & \ddots & & & \\ & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & 0\end{bmatrix}. \tag{2.6}$$

This implies instantly that the determinant of $Q_d(x,y)$ is the polynomial $(-1)^d r_{d+1}(x,y)$. We present matrix representation of $F_n^{(d)}(x,y)$ in the next theorem.

Theorem 2.7. *For $n\geq 1$, then we have*

$$Q_d^n(x,y)=\begin{bmatrix} F_{n+1}^{(d)}(x,y) & r_2(x,y)F_n^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{n-d+1}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_n^{(d)}(x,y)\\ F_n^{(d)}(x,y) & r_2(x,y)F_{n-1}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{n-d}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{n-1}^{(d)}(x,y)\\ \vdots & \vdots & \vdots & \vdots\\ F_{n-d+1}^{(d)}(x,y) & r_2(x,y)F_{n-d}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{n-2d+1}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{n-d}^{(d)}(x,y)\end{bmatrix}.$$

Proof. We will use induction method on n to demonstrate the theorem. Let $n=1$. By using Definition 2.1 and (2.6), we have

$$Q_d(x,y)=\begin{bmatrix} F_2^{(d)}(x,y) & r_2(x,y)F_1^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{2-d}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_1^{(d)}(x,y)\\ F_1^{(d)}(x,y) & r_2(x,y)F_0^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{1-d}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_0^{(d)}(x,y)\\ \vdots & \vdots & \vdots & \vdots\\ F_{2-d}^{(d)}(x,y) & r_2(x,y)F_{1-d}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{2-2d}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{1-d}^{(d)}(x,y)\end{bmatrix}.$$

Assume that the hypothesis is true for $n=k$. That is,

$$Q_d^k(x,y)=\begin{bmatrix} F_{k+1}^{(d)}(x,y) & r_2(x,y)F_k^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{k-d+1}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_k^{(d)}(x,y)\\ F_k^{(d)}(x,y) & r_2(x,y)F_{k-1}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{k-d}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{k-1}^{(d)}(x,y)\\ \vdots & \vdots & \vdots & \vdots\\ F_{k-d+1}^{(d)}(x,y) & r_2(x,y)F_{k-d}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{k-2d+1}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{k-d}^{(d)}(x,y)\end{bmatrix}.$$

We will show that it is true for $n=k+1$. So

$$\begin{aligned} Q_d^{k+1}(x,y) &= Q_d^k(x,y)Q_d(x,y)\\ &= \begin{bmatrix} F_{k+2}^{(d)}(x,y) & r_2(x,y)F_{k+1}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{k-d+2}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{k+1}^{(d)}(x,y)\\ F_{k+1}^{(d)}(x,y) & r_2(x,y)F_k^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{k-d+1}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_k^{(d)}(x,y)\\ \vdots & \vdots & \vdots & \vdots\\ F_{k-d+2}^{(d)}(x,y) & r_2(x,y)F_{k-d+1}^{(d)}(x,y)+\cdots+r_{d+1}(x,y)F_{k-2d+2}^{(d)}(x,y) & \cdots & r_{d+1}(x,y)F_{k-d+1}^{(d)}(x,y)\end{bmatrix}, \end{aligned}$$

which completes the proof. □

For $n,t\geq 0$, we have $Q_d^n(x,y)Q_d^t(x,y)=Q_d^{n+t}(x,y)$. Also, we know that the first entry of matrix $Q_d^{n+t}(x,y)$ is the production of the first row in $Q_d^n(x,y)$ by the first column of $Q_d^t(x,y)$. So by Theorem 2.7, we can obtain the following corollary.

Corollary 2.8. *Let $n,k\geq 0$. Then we get*

$$\begin{aligned} F_{n+k+1}^{(d)}(x,y) &= F_{n+1}^{(d)}(x,y)F_{k+1}^{(d)}(x,y)+r_2(x,y)F_n^{(d)}(x,y)F_k^{(d)}(x,y)+r_3(x,y)\left(F_{n-1}^{(d)}(x,y)F_k^{(d)}(x,y)+F_n^{(d)}(x,y)F_{k-1}^{(d)}(x,y)\right)\\ &\quad +r_4(x,y)\left(F_{n-2}^{(d)}(x,y)F_k^{(d)}(x,y)+F_{n-1}^{(d)}(x,y)F_{k-1}^{(d)}(x,y)+F_n^{(d)}(x,y)F_{k-2}^{(d)}(x,y)\right)\\ &\quad +\cdots+r_{d+1}(x,y)\left(F_{n-d+1}^{(d)}(x,y)F_k^{(d)}(x,y)+\cdots+F_n^{(d)}(x,y)F_{k-d+1}^{(d)}(x,y)\right). \end{aligned}$$

Theorem 2.9. *For $n \geq 0$, then we have*

$$F^{(d)}_{(d+1)n}(x,y) = \sum_{\substack{i_1,i_2,\dots,i_{d+1} \\ (d+1)i_1+di_2+\dots+i_{d+1}=n}} \binom{i_1+i_2+\dots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}} r_1^{i_1}(x,y) r_2^{i_2}(x,y) \cdots r_{d+1}^{i_{d+1}}(x,y) F^{(d)}_{n-(i_1+i_2+\dots+i_{d+1})}(x,y). \tag{2.7}$$

Proof. We denote the right hand side of (2.7) by $S(x,y)$. From Binet's formula in Corollary 2.4 and the characteristic equation in (2.1), for $n \geq 2$, we obtain

$$\begin{aligned}
S(x,y) &= \sum_{\substack{i_1,i_2,\dots,i_{d+1} \\ (d+1)i_1+di_2+\dots+i_{d+1}=n}} \binom{i_1+i_2+\dots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}} r_1^{i_1}(x,y) r_2^{i_2}(x,y) \cdots r_{d+1}^{i_{d+1}}(x,y) \left[\sum_{j=1}^{d+1} C_j(x,y)\alpha_j^{n-(i_1+i_2+\dots+i_{d+1})}(x,y)\right] \\
&= \sum_{\substack{i_1,i_2,\dots,i_{d+1} \\ (d+1)i_1+di_2+\dots+i_{d+1}=n}} \binom{i_1+i_2+\dots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}} r_1^{i_1}(x,y) r_2^{i_2}(x,y) \cdots r_{d+1}^{i_{d+1}}(x,y) \left[\sum_{j=1}^{d+1} C_j(x,y)\alpha_j^{di_1+(d-1)i_2+\dots+i_d}(x,y)\right] \\
&= C_1(x,y) \sum_{\substack{i_1,i_2,\dots,i_{d+1} \\ (d+1)i_1+di_2+\dots+i_{d+1}=n}} \binom{i_1+i_2+\dots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}} \left[r_1(x,y)\alpha_1^d(x,y)\right]^{i_1} \left[r_2(x,y)\alpha_1^{d-1}(x,y)\right]^{i_2} \cdots \left[r_{d+1}(x,y)\right]^{i_{d+1}} \\
&\quad + \dots + C_{d+1}(x,y) \sum_{\substack{i_1,i_2,\dots,i_{d+1} \\ (d+1)i_1+di_2+\dots+i_{d+1}=n}} \binom{i_1+i_2+\dots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}} \left[r_1(x,y)\alpha_{d+1}^d(x,y)\right]^{i_1} \left[r_2(x,y)\alpha_{d+1}^{d-1}(x,y)\right]^{i_2} \cdots \left[r_{d+1}(x,y)\right]^{i_{d+1}} \\
&= C_1(x,y)\left[r_1(x,y)\alpha_1^d(x,y) + r_2(x,y)\alpha_1^{d-1}(x,y) + \dots + r_{d+1}(x,y)\right]^n \\
&\quad + \dots + C_{d+1}(x,y)\left[r_1(x,y)\alpha_{d+1}^d(x,y) + r_2(x,y)\alpha_{d+1}^{d-1}(x,y) + \dots + r_{d+1}(x,y)\right]^n \\
&= \sum_{j=1}^{d+1} C_j(x,y)\alpha_j^{(d+1)n}(x,y) \\
&= F^{(d)}_{(d+1)n}(x,y).
\end{aligned}$$

□

Theorem 2.10. *For $n \geq 0$, then we get*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \left[-2r_{d+1}(x,y)\right]^k F^{(d)}_{(d+1)(n-k)}(x,y) \\
&= \sum_{\substack{i_1,i_2,\dots,i_{d+1} \\ (d+1)i_1+di_2+\dots+i_{d+1}=n}} \binom{i_1+i_2+\dots+i_{d+1}}{i_1,i_2,\dots,i_{d+1}} r_1^{i_1}(x,y) r_2^{i_2}(x,y) \cdots \left(-r_{d+1}^{i_{d+1}}(x,y)\right) F^{(d)}_{n-(i_1+i_2+\dots+i_{d+1})}(x,y).
\end{aligned} \tag{2.8}$$

Proof. We denote the right hand side of (2.8) by $T(x,y)$. Next, considering the proof of Theorem 2.9, we find that

$$\begin{aligned}
T(x,y) &= C_1(x,y)\left[r_1(x,y)\alpha_1^d(x,y) + \dots + r_d(x,y)\alpha_1(x,y) - r_{d+1}(x,y)\right]^n \\
&\quad + \dots + C_{d+1}(x,y)\left[r_1(x,y)\alpha_{d+1}^d(x,y) + \dots + r_d(x,y)\alpha_{d+1}(x,y) - r_{d+1}(x,y)\right]^n \\
&= \sum_{j=1}^{d+1} C_j(x,y)\left[\alpha_j^{d+1}(x,y) - 2r_{d+1}(x,y)\right]^n \\
&= \sum_{j=1}^{d+1} C_j(x,y) \sum_{k=0}^{n} \binom{n}{k} \alpha_j^{(d+1)(n-k)}(x,y)\left[-2r_{d+1}(x,y)\right]^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \left[-2r_{d+1}(x,y)\right]^k F^{(d)}_{(d+1)(n-k)}(x,y).
\end{aligned}$$

□

3. The Infinite d-Bivariate Fibonacci Polynomials Matrix

In this section, we introduce a new infinite matrix called the infinite d-bivariate Fibonacci polynomials matrix. Then, we present two factorizations of infinite Pascal matrix.

Definition 3.1. *The infinite d-bivariate Fibonacci polynomials matrix is defined by*

$$\mathscr{F}_d(x,y) = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
r_1(x,y) & 1 & 0 & \cdots \\
r_1^2(x,y) + r_2(x,y) & r_1(x,y) & \ddots & \cdots \\
r_1^3(x,y) + 2r_1(x,y)r_2(x,y) + r_3(x,y) & r_1^2(x,y) + r_2(x,y) & \ddots & \cdots \\
\vdots & \ddots & \ddots & \cdots
\end{bmatrix},$$

where $[\mathscr{F}_d(x,y)]_{i,1} = F_i^{(d)}(x,y)$ *for* $i \in \mathbb{Z}^+$.

From Definition 3.1, we can write the infinite d-bivariate Fibonacci polynomials matrix as follows:

$$\mathscr{F}_d(x,y) = \begin{bmatrix} F_1^{(d)}(x,y) & 0 & 0 & \cdots \\ F_2^{(d)}(x,y) & F_1^{(d)}(x,y) & 0 & \cdots \\ F_3^{(d)}(x,y) & F_2^{(d)}(x,y) & \ddots & \cdots \\ F_4^{(d)}(x,y) & F_3^{(d)}(x,y) & \ddots & \cdots \\ \vdots & \ddots & \ddots & \cdots \end{bmatrix}.$$

Therefore the matrix $\mathscr{F}_d(x,y)$ is a Riordan matrix. As the first column of $\mathscr{F}_d(x,y)$ is

$$\left(1, r_1(x,y), r_1^2(x,y)+r_2(x,y), r_1^3(x,y)+2r_1(x,y)r_2(x,y)+r_3(x,y), \dots\right)^T$$

by Theorem 2.3, we can derive the following corollary.

Corollary 3.2. *The generating function of the first column of the matrix* $\mathscr{F}_d(x,y)$ *is*

$$g_{\mathscr{F}_d(x,y)}(u) = \frac{1}{1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1}}.$$

In the matrix $\mathscr{F}_d(x,y)$, for $n \geq 1$ and $j \in \mathbb{Z}^+$, we have

$$[\mathscr{F}_d(x,y)]_{n+1,j} = r_1(x,y)\,[\mathscr{F}_d(x,y)]_{n,j} + r_2(x,y)\,[\mathscr{F}_d(x,y)]_{n-1,j} + \cdots + r_{d+1}(x,y)\,[\mathscr{F}_d(x,y)]_{n-d,j}$$

by Definition 2.1. So if we take $f_{\mathscr{F}_d(x,y)}(u) = u$, for the matrix $\mathscr{F}_d(x,y)$ we get the following corollary.

Corollary 3.3. *The infinite* d*-bivariate Fibonacci polynomials matrix* $\mathscr{F}_d(x,y)$ *is*

$$\begin{aligned} \mathscr{F}_d(x,y) &= \left(g_{\mathscr{F}_d(x,y)}(u), f_{\mathscr{F}_d(x,y)}(u)\right) \\ &= \left(\frac{1}{1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1}}, u\right). \end{aligned}$$

For $i, j \in \mathbb{Z}^+$, we define the infinite matrix $\Gamma_d(x,y)$ such that

$$[\Gamma_d(x,y)]_{i,j} = \sum_{k=0}^{d+1} -r_k(x,y)\binom{i-k-1}{j-1},$$

where $r_0(x,y) = -1$. Then we get

$$\Gamma_d(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-r_1(x,y) & 1 & 0 & 0 & \cdots \\ 1-r_1(x,y)-r_2(x,y) & 2-r_1(x,y) & 1 & \ddots & \cdots \\ 1-r_1(x,y)-r_2(x,y)-r_3(x,y) & 3-2r_1(x,y)-r_2(x,y) & 3-r_1(x,y) & \ddots & \cdots \\ \vdots & \ddots & & \ddots & \cdots \end{bmatrix}. \tag{3.1}$$

Now we provide the first factorization of the infinite Pascal matrix in the next theorem.

Theorem 3.4. *Let* $\mathscr{F}_d(x,y)$ *be the infinite* d*-bivariate Fibonacci polynomials matrix and* $\Gamma_d(x,y)$ *be the infinite matrix as in* (3.1)*, then we have*

$$P = \mathscr{F}_d(x,y) * \Gamma_d(x,y),$$

where P *is the infinite Pascal matrix as in* (1.1).

Proof. By the definition of the infinite Pascal matrix, we know that

$$P = \left(\frac{1}{1-u}, \frac{u}{1-u}\right). \tag{3.2}$$

The generating function of the first column of the matrix $\Gamma_d(x,y)$ is

$$\begin{aligned} g_{\Gamma_d(x,y)}(u) &= 1+(1-r_1(x,y))u+(1-r_1(x,y)-r_2(x,y))u^2+(1-r_1(x,y)-r_2(x,y)-r_3(x,y))u^3+\cdots \\ &= (1+u+u^2+u^3+\cdots) - r_1(x,y)\left(u+u^2+u^3+\cdots\right) - r_2(x,y)\left(u^2+u^3+u^4+\cdots\right) \\ &\quad -\cdots - r_{d+1}(x,y)\left(u^{d+1}+u^{d+2}+u^{d+3}+\cdots\right) \\ &= \frac{1}{1-u} - \frac{r_1(x,y)u}{1-u} - \frac{r_2(x,y)u^2}{1-u} - \cdots - \frac{r_{d+1}(x,y)u^{d+1}}{1-u} \\ &= \frac{1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1}}{1-u}. \end{aligned} \tag{3.3}$$

On the other hand, the generating function of the second column of the matrix $\Gamma_d(x,y)$ is

$$\begin{aligned} g_{\Gamma_d(x,y)}(u)f_{\Gamma_d(x,y)}(u) &= u+(2-r_1(x,y))u^2+(3-2r_1(x,y)-r_2(x,y))u^3+\cdots \\ &= (u+2u^2+3u^3+\cdots)-r_1(x,y)u(u+2u^2+3u^3+-r_2(x,y)u^2(u+2u^2+3u^3+\cdots) \\ &\quad -\cdots-r_{d+1}(x,y)u^{d+1}(u+2u^2+3u^3+\cdots) \\ &= (1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1})(u+2u^2+3u^3+\cdots) \\ &= \left(\frac{1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1}}{1-u}\right)\left(\frac{u}{1-u}\right), \end{aligned}$$

and so from the equation (3.3), we have

$$f_{\Gamma_d(x,y)}(u)=\frac{u}{1-u}. \tag{3.4}$$

Hence by the equations (3.3) and (3.4), we obtain

$$\begin{aligned} \Gamma_d(x,y) &= \left(g_{\Gamma_d(x,y)}(u), f_{\Gamma_d(x,y)}(u)\right) \\ &= \left(\frac{1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1}}{1-u}, \frac{u}{1-u}\right). \end{aligned} \tag{3.5}$$

By Corollary 3.3, (3.2) and (3.5), we can obtain $P=\mathscr{F}_d(x,y)*\Gamma_d(x,y)$. □

For $i,j\in\mathbb{Z}^+$, we define the infinite matrix $\Delta_d(x,y)$ such that

$$[\Delta_d(x,y)]_{i,j}=\sum_{k=0}^{d+1}-r_k(x,y)\binom{i-1}{j+k-1},$$

where $r_0(x,y)=-1$. Then we get

$$\Delta_d(x,y)=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1-r_1(x,y) & 1 & 0 & 0 & \cdots \\ 1-2r_1(x,y)-r_2(x,y) & 2-r_1(x,y) & 1 & \ddots & \cdots \\ 1-3r_1(x,y)-3r_2(x,y)-r_3(x,y) & 3-3r_1(x,y)-r_2(x,y) & 3-r_1(x,y) & \ddots & \cdots \\ \vdots & \ddots & & \ddots & \cdots \end{bmatrix}. \tag{3.6}$$

We will present the second factorization of the infinite Pascal matrix in the next corollary.

Corollary 3.5. *Let $\mathscr{F}_d(x,y)$ be the infinite d-bivariate Fibonacci polynomials matrix and $\Delta_d(x,y)$ be the infinite matrix as in (3.6), then we have*

$$P=\Delta_d(x,y)*\mathscr{F}_d(x,y),$$

where P is the infinite Pascal matrix as in (1.1).

We can simply obtain the inverse of $\mathscr{F}_d(x,y)$ in Corollary 3.3 via the definition of the reverse element of the Riordan group in the following corollary.

Corollary 3.6. *The inverse of the infinite d-bivariate Fibonacci polynomials matrix $\mathscr{F}_d(x,y)$ is*

$$\mathscr{F}_d^{-1}(x,y)=\left(1-r_1(x,y)u-r_2(x,y)u^2-\cdots-r_{d+1}(x,y)u^{d+1},u\right).$$

4. Conclusions

In this work, we generalize the known bivariate Fibonacci polynomials, and call these polynomials as d-bivariate Fibonacci polynomials $F_n^{(d)}(x,y)$. We present the generating function, Binet's formula, combinatorial identities and summation formulas of $F_n^{(d)}(x,y)$. We introduce the new matrix $Q_d(x,y)$, whose powers generate $F_n^{(d)}(x,y)$. Also, we define the infinite d-bivariate Fibonacci polynomials matrix $\mathscr{F}_d(x,y)$, which is a Riordan matrix. In order to factorize the infinite Pascal matrix P, we use the Riordan method, and so find that two factorizations of P including the matrix $\mathscr{F}_d(x,y)$. Finally, we give the Riordan representation for the inverse of the matrix $\mathscr{F}_d(x,y)$.

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