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Enhancing Linear System Solving Through Third Refinement of Successive and Accelerated Over-Relaxation Methods

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Third refinements,
Linear System,
Irreducible matrix.

Abstract

One of the primary difficulties in linear algebra, considering its widespread application in many different domains, is solving linear system of equations. It is nevertheless apparent that there is a need for a quick, effective approach that can handle a variety of linear systems. In the realm of large and sparse systems, iterative methods play a crucial role in finding solutions. This research paper makes a significant contribution by introducing an enhancement to the current methodology Successive and Accelerated Over Relaxation methods, referred to as the "Third Refinement of Successive and Accelerated Over Relaxation Methods." This new iterative approach demonstrates its effectiveness when applied to coefficient matrices exhibiting properties such as M-matrix, irreducible diagonal dominance, positive definiteness and symmetry characteristics. Significantly, the proposed method substantially reduces the spectral radius, resulting in fewer iterations and notably enhancing the convergence rate. Numerical experiments were conducted to evaluate its performance compared to existing second refinement of Successive and Accelerated Over Relaxation methods. The outcomes underscore the "Third Refinement of Successive and Accelerated Over Relaxation" methods potentially to boost the efficiency of solving linear systems, thus rendering it a valuable asset within the arsenal of numerical methodologies utilized in scientific and engineering realms.

1. Introduction

In Numerical analysis, analyst is mainly focused on solutions to systems of Mathematics that develops, analyzes and implements algorithms for provision of numerical solution to mathematical problems, starting with an initial approximation (guess) to the solution of the problem. The applications of numerical analysis are evidently seen in all aspect of physical sciences and recently, some aspects of life sciences are experiencing the use of numerical linear algebra for evaluating data in scientific computation [1,2]. The utilization of two (2) parameters to accelerate convergence within the AOR method, rather than relying solely on a single parameter as is common in iterative methods, underscores the method's superior effectiveness compared to conventional approaches like the Successive Over-relaxation method. His theorem concerning irreducible weak diagonal matrices establishes that the AOR method tends to converge within specific parameter ranges, $0 \le \bar{r} \le 1$ and $0 < w \le 1$ particularly evident when the original matrix demonstrates irreducible weak diagonal dominance. Furthermore, harnessing these two parameters equips numerical solvers with a methodology that achieves quicker convergence rates than any other comparable method. whenever the original matrix is an irreducible weak diagonally dominant matrix. [3-6]. The development of the Accelerated Over-relaxation (AOR) method, pioneered by Hadjidimos, the Successive Over-relaxation (SOR) method etc., are also associated with the problem of rate of convergence. This has motivated many authors and researchers to examine the solution of system of linear equations by direct and indirect methods. These examinations gave birth to new developments and modifications of numerical methods by researchers and numerical analysts. In general, a linear system of equation is represented in the form;

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$$Pk = g \tag{1}$$

where $P \in \mathrm{R}^{n\times n}$, $g \in \mathrm{R}^{n}$ given throughout and $k \in \mathrm{R}^{n}$ is unknown and is used to represent physical problems. Iterative (indirect) and direct methods can be applied to generate solutions but however, if the problem is generally large, then iterative method is considered, and solutions can be obtained by decomposing P into;

$$P = Q - S \tag{2}$$

Also, if the coefficient matrix (P) is nonsingular, then it can be expressed in its diagonal section, exclusively low triangular part, and exclusively up triangular part as $P = D - L_P - U_P$ or $P = I - \underset{\sim}{L} - \underset{\sim}{U}$ where $\underset{\sim}{D}^{-1}D = \underset{\sim}{I}$, $\underset{\sim}{D}^{-1}L_P = \underset{\sim}{L}$, $\underset{\sim}{D}^{-1}U_P = \underset{\sim}{U}$. A consistent division of the square matrix P into $Q - S$ is necessary for iteratively solving equation (1). This division ensures that substituting (2) into (1) yields the subsequent expressions:

$$\begin{aligned} (\mathrm{Q}-\mathrm{S})\,\mathrm{k} &= g \\ Qk &= Sk + g \\ k &= Q^{-1}Sk + Q^{-1}g \end{aligned} \tag{3}$$

Putting (3) into iterative format, the stationary first-degree iterative method which can be used to solve form (1), can be expressed as;

$$\begin{aligned} k^{(n+1)} &= Q^{-1}Sk^{(n)} + Q^{-1}g \\ k^{(n+1)} &= \underline{C}k^{(n)} + \underline{d} \end{aligned} \tag{4}$$

where $\underline{C} = Q^{-1}S$ and $\underline{d} = Q^{-1}g$ [7]. Equation (4) is the generalized iterative format for solving linear problem such as (1). In this context, $\underline{C} = Q^{-1}S$ represents the iteration matrix that will be employed to evaluate spectral radius, and $\underline{d} = Q^{-1}g$ stands as column vector associated with the iterative technique. Thus, this investigation aims to establish a linear static iterative approach following a similar structure as described above. Consequently, due to frequent demands by scientist for numerical methods that are efficient and possess a high convergence rate, mathematicians have in more recent times discover the need to modify these existing methods to suit their requirements [7-9].

For this research, emphasis shall be laid on the further refinements of the SOR method and AOR method. The first and second refinement of each of these methods has proven to improve effectiveness and converges faster compared to the standard approach earlier developed; hence, there exist a need for further modification.

The research in enhancing linear system solving through third refinement of successive and accelerated over-relaxation methods addresses a crucial gap in current methodologies. Despite existing advancements in iterative techniques, there remains a need for further refinement to achieve faster and more accurate solutions, particularly for large and sparse linear systems. Motivated by the increasing complexity of real-world problems requiring efficient computational solutions, this study aims to build upon prior research by introducing a novel refinement to successive and accelerated over-relaxation methods. The novelty of this approach lies in its strategic integration of both successive and accelerated over-relaxation techniques, leveraging their respective strengths to enhance convergence rates and solution accuracy. By systematically analyzing the performance of the proposed refinements, this study contributes to the optimization of iterative methods for linear system solving, offering valuable insights for practitioners and researchers in numerical analysis and computational mathematics.

A series of advancements in numerical approaches for addressing linear systems of equations have been developed and refined. These innovations include the creation of an Extended Accelerated Over Relaxation (EAOR) method [10], tailored to efficiently handle large and sparse linear systems. Furthermore, a refined version of this method, known as the Extended Refined Accelerate Over Relaxation (REAOR), has been introduced to enhance linear system solutions [11]. Simultaneously, enhancements have been made to the Classical Iterative

Algorithm, resulting in a more effective approach for solving linear equations [12]. Additionally, a Second (2nd) Refined Accelerate Over Relaxation Method has been proposed, aiming to optimize the resolution of linear systems [13]. The exploration of numerical techniques extends to methods such as the SOR Algorithm, providing insights into the numerical solution of linear equations [14]. Moreover, efforts have been directed towards refining traditional approaches like the Jacobi Method, with the introduction of a Third Refinement for resolving linear systems [15]. Similarly, an Accelerated Iterative Technique has been developed, specifically focusing on enhancing the Gauss-Seidel Algorithm for linear system resolution [16]. The scope of advancements encompasses addressing complex equations, as evidenced by the introduction of an Accelerated Over-Relaxation Partitioning Technique tailored for symmetric tensor equations [17]. Furthermore, advancements include refinements in the Accelerated Over Relaxation Method [18] and the introduction of Iterative Methods in Numerical Analysis [19]. Throughout these developments, a common theme emerges: a dedication to refining numerical methods for more efficient and accurate solutions to linear systems of equations.

The paper begins with an Introduction section, providing background information on the problem under investigation and outlining the research objectives. Following this, the Methodology section introduces the numerical methods under study, namely the Third Refine Successive Overrelaxation (TRSOR) Method and the Third Refinement of Accelerated Overrelaxation (TRAOR) Method. Detailed derivations of these methods are presented, along with a comprehensive convergence analysis and descriptions of computational algorithms employed. The subsequent section, Numerical Examples presents practical applications of TRSOR and TRAOR methods through various numerical examples. The results are thoroughly analyzed, including comparisons with existing methods where relevant, and the implications of the findings are discussed in detail. Lastly, the concluding segment provides a comprehensive overview of the key findings of the study, along with a discussion of the significance and acknowledges limitations.

2. Methodology

2.1. Derivation of Third Refinement Successive Over-relaxation (TRSOR) Technique

Considering the SOR technique [1];

$$k^{(n+1)} = \tilde{k}^{(n+1)} + \left(I - w\hat{L}\right)^{-1}\left[w\left(g - \hat{P}\tilde{k}^{(n+1)}\right)\right] \tag{6}$$

Further refinement of the above method gives a method called Refinement of SOR (RSOR) [4]

$$k^{(n+1)} = L^2{}_w k^{(n)} + \left[I + L_w\right]w\left(I - w\hat{L}\right)^{-1} g \tag{7}$$

Also the below equation is known as Second Refinement of SOR (SRSOR)

$$\tilde{\tilde{k}}^{(n+1)} = L^3{}_w k^{(n)} + \left[I + L_w + L^2{}_w\right]w\left(I - w\hat{L}\right)^{-1} g \tag{8}$$

Remodeling (6) by replacing $\tilde{\tilde{k}}^{(n+1)}$ in (8) to obtain;

$$\tilde{\tilde{\tilde{k}}}^{(n+1)} = L^3{}_w k^{(n)} + \left[I + L_w + L^2{}_w\right]w\left(I - w\hat{L}\right)^{-1} g + \left(I - w\hat{L}\right)^{-1}\left[w\left(g - \hat{P}\left\{L^3{}_w k^{(n)} + \left[I + L_w + L^2{}_w\right] w\left(I - w\hat{L}\right)^{-1} g\right\}\right)\right]$$

$$\tilde{\tilde{\tilde{k}}}^{(n+1)} = L^4{}_w k^{(n)} + \left[I + L_w + L^2{}_w + L^3{}_w\right]w\left(I - w\hat{L}\right)^{-1} g \tag{9}$$

Equation (9) is called Third refinement of SOR (TRSOR) method, where

$$L^4{}_w = \left[\left(I - w\hat{L}\right)^{-1}\left[(1-w)I + w\hat{U}\right]\right]^4$$

$$L^3{}_w = \left[\left(I - w\hat{L}\right)^{-1}\left[(1-w)I + w\hat{U}\right]\right]^3$$

$$L^2{}_w = \left[\left(I - w\hat{L}\right)^{-1}\left[\left(1-w\right)I + w\hat{U}\right]\right]^2$$

$$L_w = \left[\left(I - w\hat{L}\right)^{-1}\left[\left(1-w\right)I + w\hat{U}\right]\right]$$

2.2. Derivation of Third Refinement Accelerated Over-relaxation (TRAOR) Technique

The classical Accelerated Over Relaxation (AOR) technique governed by

$$k^{(n+1)} = M_{\mathrm{w},\bar{r}}k^{(n)} + w\left(I - \bar{r}\hat{L}\right)^{-1} g \tag{10}$$

Then it results to its refinement by the relation equation (11) [3];

$$k^{(n+1)} = \tilde{k}^{(n+1)} + w\left(I - \bar{r}L\right)^{-1}\left(g - \hat{P}\tilde{k}^{(n+1)}\right) \tag{11}$$

Which can be expressed as

$$\tilde{k} = M^2{}_{\mathrm{w},\bar{r}}k^{(n)} + \omega\left(I + M_{\mathrm{w},\bar{r}}k^{(n)}\right)\left(I - \bar{r}\hat{L}\right)^{-1} g \tag{12}$$

Remodeling (10) gives

$$\tilde{\tilde{k}}^{(n+1)} = M^3{}_{\mathrm{w},\bar{r}}k^{(n)} + \left[I + M_{\mathrm{w},\bar{r}}k^{(n)} + M^2{}_{\mathrm{w},\bar{r}}k^{(n)}\right]w\left(I - \bar{r}\hat{L}\right)^{-1} g \tag{13}$$

Equation (13) becomes Second Refinement of AOR (SRAOR) [6], substitute (13) in equation (11) to obtain;

$$\tilde{\tilde{\tilde{k}}}^{(n+1)} = M^3{}_{\mathrm{w},\bar{r}}k^{(n)} + \left[I + M_{\mathrm{w},\bar{r}}k^{(n)} + M^2{}_{\mathrm{w},\bar{r}}k^{(n)}\right]\omega\left(I - \bar{r}\hat{L}\right)^{-1} g + w\left(I - \bar{r}L\right)^{-1}\left(g - \hat{A}\right.$$
$$\left.\left[M^3{}_{\mathrm{w},\bar{r}}k^{(n)} + \left[I + M_{\mathrm{w},\bar{r}}k^{(n)} + M^2{}_{\mathrm{w},\bar{r}}k^{(n)}\right]w\left(I - \bar{r}\hat{L}\right)^{-1} g\right]\right) \tag{14}$$

Further mathematical algebraic simplification gives;

$$\tilde{\tilde{\tilde{k}}}^{(n+1)} = M^4{}_{\mathrm{w},\bar{r}}k^{(n)} + \left\{\left[I + M_{\mathrm{w},\bar{r}}k^{(n)} + M^2{}_{\mathrm{w},\bar{r}}k^{(n)} + M^3{}_{\mathrm{w},\bar{r}}k^{(n)}\right]\right\} w\left(I - \bar{r}\hat{L}\right)^{-1} g \tag{15}$$

Equation (15) is called Third Refinement of AOR (TRAOR) method**,** where

$$M^4{}_{\mathrm{w},\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[\left(1-w\right)I + \left(w - \bar{r}\right)\hat{L} + w\hat{U}\right]\right]^4$$

$$M^3{}_{\mathrm{w},\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[\left(1-w\right)I + \left(w - \bar{r}\right)\hat{L} + w\hat{U}\right]\right]^3$$

$$M^2{}_{\mathrm{w},\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[\left(1-w\right)I + \left(w - \bar{r}\right)\hat{L} + w\hat{U}\right]\right]^2$$

$$M_{\mathrm{w},\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[\left(1-w\right)I + \left(w - \bar{r}\right)\hat{L} + w\hat{U}\right]\right]$$

2.3. Convergence Analysis

2.3.1. Convergence of TRSOR

Theorem 1: If P is irreducible matrix with limited diagonal control, then Successive Refinement of SOR technique achieves convergence regardless of the starting estimate selected.

Proof: Utilizing the concept introduced in [20], the proof is examined in the following manner:

Let k^* represents the actual solution and assume $\bar{k}^{n+1}$ to represent the $\left(n+1\right)^{th}$ estimate concerning the true solution to with respect to $pk = g$ by the method of refinement format then we have

$$\|\bar{\mathrm{k}}^{n+1} - k^*\| = \|\mathrm{k}^{n+1} + w(\mathrm{I} - w\hat{\mathrm{L}})^{-1}(\mathrm{g} - \hat{P}k) - k^*\|$$

$\leq \| k^{n+1} - k^* \| \| g - \hat{P}k \| \| w(I - w\hat{L})^{-1} \|$

If we observe $\| k^{n+1} - k^* \| \to 0$ and $\| (g - \hat{P}k) \| \to 0$ then $\| (k^{n+1} - k^* \| \to 0$

Hence, the Successive Refinement of SOR (RSOR) method converges to the solution of the linear system.

Theorem 2: if P is irreducible matrix with limited diagonal control, then $\| \bar{C} \|_\infty = \| \bar{C} \|_\infty^4 < 1$

Verification: Contemplate

$$\| \bar{C} \|_\infty = \| L_r^4 \|_\infty$$
$$= \| L_r \|_\infty^4 = \| L_r^2 \|_\infty . \| L_r^2 \|_\infty$$
$$= \| C \|_\infty^4 < 1$$

Theorem 3: If P is i irreducible matrix with limited diagonal control t, then $\| \bar{C} \|_\infty < \| C \|_\infty$

Verification: According to proposition (theorem) 3, it is observed that

$$\| C \|_\infty = \| C \|_\infty^4$$
$$= \| C \|_\infty^2 . \| C \|_\infty^2$$
$$< \| C \|_\infty^2 < \| C \|_\infty$$

Theorem 4: The Third Refinement Successive Over Relaxation (TRSOR) technique converges quicker compared to the improvement of Successive Over Relaxation (RSOR) technique when refinement of SOR technique achieves convergence.

Proof

Let $\bar{k}$ is the solution of $Pk = g$ obtained by Third Refinement Successive Over Relaxation (TRSOR) technique and k^* be the solution obtained by $Pk = g$ from

$$k^{n+1} = L_w^4 k^n + r(1 + L_w + L^2{}_w + L^3{}_w)\, w(1 - w\hat{L})^{-1} g$$

We have

$$\bar{k} = \bar{C}k^* + \bar{d}$$
$$\bar{k} = L_\omega^4 k^* + \bar{d}$$

Considering

$$\bar{k}^{n+1} - \bar{k} = L_\omega^4 k + \bar{d} - \bar{k}$$
$$= L_w^4 (k^n - k^*) + \bar{d} - \bar{k} + L_w^4 k^*$$
$$= L_w^4 (k^n - k^*) - \bar{k} + (L_w^4 k^* + \bar{d})$$
$$= L_w^4 (k^n - k^*) - \bar{k} + \bar{k}$$
$$= L_w^4 (k^n - k^*)$$

$$\| \bar{k}^{n+1} - \bar{k} \|_\infty = \| L_w^4 (k^n - k^*) \|_\infty$$
$$\leq \| L_w^4 \|_\infty \| (k^n - k^*) \|_\infty$$
$$\leq \| L_w \|_\infty^4 \| (k^n - k^*) \|_\infty$$

Hence the theorem (3) and (4) shows that Third Refinement of Successive Over Relaxation (TSOR) method converges faster than Refinement of Successive Over Relaxation (SOR) method.

2.3.2. Convergence of TRAOR

Theorem 5: If P is irreducible matrix with weak diagonal dominance, then the accelerated Refinement of RAOR method converges for any arbitrary choice of the initial approximation.

Proof:

Let k^* be the real solution and let $\bar{k}^{n+1}$ be the $(n+1)^{th}$ approximation to the solution of $pk = g$ by the method of refinement format then we have

$$\|\bar{\mathrm{k}}^{n+1} - k^*\| = \|\mathrm{k}^{n+1} + \bar{r}(\mathrm{I} + \mathrm{M}_{\bar{r},\mathrm{w}})(\mathrm{I} - w\hat{\mathrm{L}})^{-1}(\hat{\mathrm{g}} - \hat{P}k) - k^*\|$$

$$\le \|\mathrm{k}^{n+1} - k^*\| \|\|\hat{\mathrm{g}} - \hat{P}k\|\|\|\bar{r}(\mathrm{I} + M_{\bar{r},\mathrm{w}})(\mathrm{I} - w\hat{L})^{-1}\|$$

We know $\|\mathrm{k}^{n+1} - k^*\| \to 0$ and $\|(\hat{\mathrm{g}} - \hat{P}k)\| \to 0$ then $\|(\mathrm{k}^{n+1} - k^*\| \to 0$

Hence, the Accelerated Refinement of AOR (ARAOR) method converges to the solution of the linear system.

Theorem 6: if P is irreducible matrix with weak diagonal dominance, then $\|\bar{\mathrm{C}}\|_\infty = \|\bar{\mathrm{C}}\|_\infty^4 < 1$

Proof

Consider

$$\|\bar{C}\|_\infty = \|\mathrm{M}_{\bar{r},\mathrm{w}}^4\|_\infty$$

$$= \|\mathrm{M}_{\bar{r},\mathrm{w}}^2\|_\infty \cdot \|\mathrm{M}_{\bar{r},\mathrm{w}}^2\|_\infty$$

$$= \|\mathrm{M}_{\bar{r},\mathrm{w}}\|_\infty^4 = \|\mathrm{C}\|_\infty^4 < 1$$

Theorem 7: If P is irreducible matrix with weak diagonal dominant, then $\|\bar{C}\|_\infty < \|C\|_\infty$

Proof

By theorem 7 we have

$$\|C\|_\infty = \|C\|_\infty^4$$

$$= \|C\|_\infty^2 \cdot \|C\|_\infty^2$$

$$< \|C\|_\infty^2 < \|C\|_\infty$$

Theorem 8: The Third Refinement of Accelerated Over Relaxation (TRAOR) method converges faster than the Refinement of Accelerated Over Relaxation (AOR) method when refinement of AOR method convergent.

Proof

Let $\bar{k}$ is the solution of $\hat{P}k = \hat{g}$ obtained by RTAOR method and k^* be the solution obtained by $Pk = \hat{g}$ from

$$k^{n+1} = M_{\bar{r},\mathrm{w}}^4 k^n + \bar{r}(1 + \mathrm{M}_{\bar{r},\mathrm{w}})(1 + M_{\bar{r},\mathrm{w}}^2)(1 - w\hat{\mathrm{L}})\hat{\mathrm{g}}$$

We have

$$\bar{k} = \bar{C}k^* + \bar{d}$$

$$\bar{k} = M_{\bar{r},\mathrm{w}}^4 k^* + \bar{d}$$

Considering

$$\bar{k}^{n+1} - \bar{k} = M_{\bar{r},\mathrm{w}}^4 k + \bar{d} - \bar{k}$$

$$= M_{\bar{r},\mathrm{w}}^4(\mathrm{k}^n - k^*) + \bar{\mathrm{d}} - \bar{k} + \mathrm{M}_{\bar{r},\mathrm{w}}^4 k^*$$

$$= M_{\bar{r},\mathrm{w}}^4(\mathrm{k}^n - k^*) - \bar{k} + (\mathrm{M}_{\bar{r},\mathrm{w}}^4 k^* + \bar{\mathrm{d}})$$

$$= \mathrm{M}_{\bar{r},\mathrm{w}}^4(k^n - k^*) - \bar{k} + \bar{k}$$

$$= M_{\bar{r},\mathrm{w}}^4(k^n - k^*)$$

$$\|\bar{k}^{n+1} - \bar{k}\|_\infty = \|M_{\bar{r},\mathrm{w}}^4(k^n - k^*)\|_\infty$$

$$\le \|M_{\bar{r},\mathrm{w}}^4\|_\infty \|(k^n - k^*)\|_\infty$$

$$\le \|M_{\bar{r},\mathrm{w}}\|_\infty^4 \|(k^n - k^*)\|_\infty$$

Hence the theorem (7) and (8) shows that RTAOR method converges faster than RAOR method**.**

2.4. Computational Algorithm

2.4.1. Algorithm of TRSOR

1. Key in matrix P, opt for an initial estimate $k^{(0)}$.
2. Select an appropriate step size for w with the range $(0 < w < 2)$
3. Derive L, U and D from matrix P and $D^{-1}P$
4. Compute $\tilde{\tilde{k}}^{(n+1)} = L^4{}_w + \left[I + L_w + L^2{}_w + L^3{}_w \right] w\left(I - w\hat{L}\right)^{-1} g$ and obtain

$$L^4{}_w = \left[\left(I - w\hat{L}\right)^{-1}\left[(1-w)I + w\hat{U}\right]\right]^4$$

$$L^3{}_w = \left[\left(I - w\hat{L}\right)^{-1}\left[(1-w)I + w\hat{U}\right]\right]^3$$

$$L^2{}_w = \left[\left(I - w\hat{L}\right)^{-1}\left[(1-w)I + w\hat{U}\right]\right]^2$$

$$L_w = \left[\left(I - w\hat{L}\right)^{-1}\left[(1-w)I + w\hat{U}\right]\right]$$

5. Compute $R = \left[I + L_w + L^2{}_w + L^3{}_w \right] w\left(I - w\hat{L}\right)^{-1} g$
6. Compute $S = L^4{}_w$
7. Using maple compute

```
K := S . (k[i]);
for i from 0 to N do
  k[i + 1] := (S . (k[i])) + R;
end do;
```

8. Obtain the desired result after the tolerance

2.4.2. Algorithm of TRAOR

1. Enter matrix P, select an initial guess $k^{(0)}$.
2. Select an appropriate step size for $\bar{r}$ with the range $(0 < \bar{r} \leq 1)$
3. Obtain L, U and D from matrix P and $D^{-1}P$
4. Compute $\tilde{\tilde{k}}^{(n+1)} = M^4{}_{w,\bar{r}}k^{(n)} + \left\{\left[I + M_{w,\bar{r}}k^{(n)} + M^2{}_{w,\bar{r}}k^{(n)} + M^3{}_{w,\bar{r}}k^{(n)} \right]\right\} w\left(I - \bar{r}\hat{L}\right)^{-1} g$ and obtain

$$M^4{}_{w,\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[(1-w)I + (w-\bar{r})\hat{L} + w\hat{U}\right]\right]^4$$

$$M^3{}_{w,\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[(1-w)I + (w-\bar{r})\hat{L} + w\hat{U}\right]\right]^3$$

$$M^2{}_{w,\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[(1-w)I + (w-\bar{r})\hat{L} + w\hat{U}\right]\right]^2$$

$$M_{w,\bar{r}} = \left[\left(I - \bar{r}\hat{L}\right)^{-1}\left[(1-w)I + (w-\bar{r})\hat{L} + w\hat{U}\right]\right]$$

5. Compute $R = \left\{\left[I + M_{w,\bar{r}}k^{(n)} + M^2{}_{w,\bar{r}}k^{(n)} + M^3{}_{w,\bar{r}}k^{(n)} \right]\right\} w\left(I - \bar{r}\hat{L}\right)^{-1} g$
6. Compute $S = M^4_{w,\bar{r}}$

7. Using maple compute

```
K := S . (k[i]);
for i from 0 to N do
   k[i + 1] := (S . (k[i])) + R;
end do;
```

8. Obtain the desired result after the tolerance.

3. Numerical Examples

Example 1: Consider the system of linear equation of irreducible diagonal dominance matrix with weak diagonal dominance, $Pk = g$

$$\begin{pmatrix} 7.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & 0.0 \\ -1.0 & 7.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & -1.0 & 7.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 \\ 1.0 & 0.0 & -1.0 & 7.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & -1.0 & 7.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 \\ -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 7.0 & -1.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 7.0 & -1.0 & 0.0 & -1.0 \\ -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 7.0 & -1.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 7.0 & -1.0 \\ 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 7.0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \\ k_9 \\ k_{10} \end{pmatrix} = \begin{pmatrix} 7.00 \\ 3.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \\ 2.00 \end{pmatrix}$$

Example 2: Consider the system of linear equation of irreducible matrix with limited diagonal control analyzed through $Pk = g$

$$\begin{pmatrix} 1 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & 0 \\ \frac{-1}{7} & 1 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 \\ 0 & \frac{-1}{7} & 1 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} \\ \frac{-1}{7} & 0 & \frac{-1}{7} & 1 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 \\ 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 1 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} \\ \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 1 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 \\ 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 1 & \frac{-1}{7} & 0 & \frac{-1}{7} \\ \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 1 & \frac{-1}{7} & 0 \\ 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 1 & \frac{-1}{7} \\ 0 & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 0 & \frac{-1}{7} & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \\ k_9 \\ k_{10} \end{pmatrix} = \begin{pmatrix} 11.90 \\ 9.32 \\ 8.09 \\ 9.32 \\ 8.09 \\ 8.32 \\ 8.09 \\ 8.32 \\ 8.09 \\ 8.32 \end{pmatrix}$$

Example 3: Consider the system of linear equation of irreducible matrix with weak diagonal dominance [17], $Pk = g$

$$\begin{pmatrix} 5.0 & -1.0 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & -1.0 \\ -1.0 & 5.0 & -1.0 & 0.0 & 0.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & -1.0 & 5.0 & -1.0 & 0.0 & 1.0 & -1.0 & 0.0 \\ 1.0 & 0.0 & -1.0 & 5.0 & -1.0 & 0.0 & 0.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 0.0 & 5.0 & -1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & -1.0 & -1.0 & 0.0 & 5.0 & 1.0 & -1.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & 5.0 & -1.0 \\ -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 5.0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{pmatrix} = \begin{pmatrix} -2.0 \\ -1.0 \\ 4.0 \\ 13.0 \\ 4.0 \\ 2.0 \\ 9.0 \\ 12.0 \end{pmatrix}$$

Example 4: Consider the system of linear equation of irreducible diagonal dominance matrix with weak diagonal dominance as considered by [18]. $Pk = g$

$$\begin{pmatrix} 8 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 8 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 8 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 8 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 8 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 8 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 8 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -1 & 8 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 8 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 8 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 8 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 8 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \\ k_9 \\ k_{10} \\ k_{11} \\ k_{12} \end{pmatrix} = \begin{pmatrix} 9 \\ -9 \\ 21 \\ 18 \\ 9 \\ 13 \\ 23 \\ 33 \\ -13 \\ -18 \\ 7 \\ 19 \end{pmatrix}$$

The numerical applications (problem 1, 2, 3 and problem 4) were performed using Maple 2023 software, with the outcomes depicted in the subsequent tables

Table 1: Spectral Radii of SRSOR and TRSOR for Example 1

w	Existing Method $\rho(SRSOR)$	Proposed Method $\rho(TRSOR)$
0.2	0.8248211990	0.7735346145
0.3	0.7368601407	0.6655490781
0.4	0.6491147923	0.5620332486
0.5	0.5620442796	0.4638320019
0.6	0.4762257416	0.3718922736
0.7	0.3923879455	0.2872681930
0.8	0.3114576831	0.2111213065
0.9	0.2346261799	0.1447111382
1.0	0.1634493312	0.08936707228
1.1	0.07132445823	0.02957907864

Table 2: SRAOR vs. TRAOR Spectral Radii for Example 1

R	w	Existing Method $\rho(SRAOR)$	Proposed Method $\rho(TRAOR)$
0.10	0.20	0.8186652779	0.7658466695
0.20	0.30	0.7275048028	0.6543063770
0.30	0.40	0.6365155358	05475351578
0.40	0.50	0.5461999305	0.4464801829
0.50	0.60	0.4571977830	0.3522130862
0.60	0.70	0.3703286835	0.2659396758
0.70	0.80	0.2866525676	0.1890055026
0.80	0.90	0.2075602145	0.1228925288
0.90	1.00	0.1349180507	0.06919797899
1.00	1.10	0.0010000000	0.00010000000

Table 3: Convergence Summary Result for Example 1

Iterational Approaches	Iterations	Computational Time (seconds)	Convergence Rate
SRAOR	10	1.534	3.000000000
TRAOR	7	0.935	4.000000000
SRSOR	17	2.206	1.146761518
TRSOR	11	1.783	1.529015358

Table 4: SRSOR vs. TRSOR Spectral Radii for Example 2

W	Existing Method $\rho(SRSOR)$	Proposed Method $\rho(TRSOR)$
0.20	0.819304825	0.7666444908
0.30	0.7288723568	0.6559468327
0.40	0.6389006199	0.5502724198
0.50	0.5498986546	0.4505159896
0.60	0.4625033028	0.3576732420
0.70	0.3775153634	0.2728430153
0.80	0.2959495588	0.1972227132
0.90	0.2191063124	0.1320909873
1.00	0.1486809386	0.7876629038
1.10	0.08694597348	0.03851819840

Table 5: Spectral Radii of SRAOR and TRAOR for Example 2

r	w	Existing Method $\rho(SRAOR)$	Proposed Method $\rho(TRAOR)$
0.1	0.2	0.8129455196	0.7587206817
0.2	0.3	0.7192270676	0.6443987627
0.3	0.4	0.6259407953	0.5354402438
0.4	0.5	0.5336453017	0.4328495000
0.5	0.6	0.4430485409	0.3377549693
0.6	0.7	0.3550538836	0.2514156781
0.7	0.8	0.2708261105	0.1752214931
0.8	0.9	0.1918908784	0.1106808386
0.9	1.0	0.1202964126	0.05938410771
1.0	1.1	0.0589080905	0.02292098414

Table 6: Convergence Summary Outcome for Example 2

Iterational Approaches	Iterations	Computational Time (seconds)	Convergence Rate
SRAOR	20	1.563	1.229825055
TRAOR	11	0.795	1.639766739
SRSOR	32	2.533	1.060750526
TRSOR	27	1.703	1.414334034

Table 7: Spectral Radii of SRSOR and TRSOR for Example 3

w	Existing Method $\rho(SRSOR)$	Proposed Method $\rho(TRSOR)$
0.2	0.8072962516	0.7676420671
0.3	0.7117884597	0.6571924881
0.4	0.6174788469	0.5515906177
0.5	0.5249791833	0.4517308433
0.6	0.4350421866	0.35861629932
0.7	0.3485994450	0.2453403168
0.8	0.2668134036	0.1717685003
0.9	0.1911515211	0.1101125984
1.0	0.1234986926	0.06150113726
1.1	0.06634634169	0.02685896715

Table 8: Spectral Radii of SRAOR and TRAOR for Example 3

r	w	Existing Method $\rho(SRAOR)$	Proposed Method $\rho(TRAOR)$
0.1	0.2	0.8006618980	0.7434735952
0.2	0.3	0.7017857461	0.6236477944
0.3	0.4	0.6041288789	0.5107081582
0.4	0.5	0.5083654000	0.4057276876
0.5	0.6	0.4153354581	0.3098836618
0.6	0.7	0.3260945782	0.2244526673
0.7	0.8	0.2419842642	0.1507935738
0.8	0.9	0.1647388795	0.09030839909
0.9	1.0	0.09666217010	0.04436175524
1.0	1.1	0.04096184747	0.01412015231

Table 9: Convergence Summary Result for Example 3

Iterative Approaches	Iterations	Computational Time (Seconds)	Convergence Rate
SRAOR	48	2.085	1.387620464
TRAOR	26	1.538	1.850160619
SRSOR	60	2.946	1.178183019
TRSOR	49	2.193	1.570910692

Table 10: Spectral Radii of SRSOR and TRSOR for Example 4

w	Existing Method $\rho(SRSOR)$	Proposed Method $\rho(TRSOR)$
0.2	0.7741859128	0.7108757281
0.3	0.6450597945	0.5816180606
0.4	0.5618774393	0.4636484290
0.5	0.4626042074	0.3577772908
0.6	0.3691197529	0.2647827664
0.7	0.2825094927	0.1853719679
0.8	0.2040426492	0.1201234891
0.9	0.1352141216	0.06940052165
1.0	0.07780478502	0.03321557253
1.1	0.03398394145	0.01100774421

Table 11: Spectral Radii of SRAOR and TRAOR for Example 4

r	w	Existing Method $\rho(SRAOR)$	Proposed Method $\rho(TRAOR)$
0.1	0.2	0.7673877686	0.7025649780
0.2	0.3	0.6559695871	0.5699607370
0.3	0.4	0.5487872193	0.4493023099
0.4	0.5	0.4467258764	0.3414979854
0.5	0.6	0.3508227329	0.2474288309
0.6	0.7	0.2623011440	0.1679062696
0.7	0.8	0.1826178354	0.1826178354
0.8	0.9	0.1135333246	0.1135333246
0.9	1.0	0.05723259634	0.02205589052
1.0	1.1	0.01658687617	0.00423012112

Table 12: Convergence Summary Result for Example 4

Iterative Approaches	Iterations	Computational Time (Seconds)	Convergence Rate
SRAOR	27	1.096	1.780235398
TRAOR	18	0.653	3.373647197
SRSOR	31	1.502	1.468726253
TRSOR	29	1.149	1.958301671

3.2. Discussion of Results

Spectral Radii Outcome: Tables 1, 2, 4, 5, 7, 8, 10, and 11 collectively demonstrate the spectral radii of various iterative methods across different examples (1-4). The consistent observation across these tables is All methods exhibit spectral radii below 1, indicating their convergence properties. This implies that regardless of the specific example or method employed, all iterative approaches are convergent. However, the key factor influencing the speed of convergence lies in the proximity of their spectral radii to zero. As emphasized by the established fact, as the spectral radius approaches zero, the convergence speed increases. Notably, it's noticed that the spectral radii for the derived TRAOR and TRSOR methods are smaller compared to spectral radii of the existing SRSOR and SRAOR. This is attributed to the fact that their spectral radii are closer to zero, suggesting a potentially faster convergence rate. Therefore, while these tables affirm the convergent nature of the methods, they also underscore the importance of spectral radius proximity to zero in determining the speed of convergence. Tables 1, 2, 4, 5, 7, 8, 10, and 11 serve as valuable references for understanding the convergence behavior of iterative methods across different scenarios, aiding in the selection of the most efficient approach based on spectral radius proximity to zero.

Convergence Outcome (Tables 3, 6, 9 and 12): The tables provide a comprehensive comparison of iterative methods across multiple examples, focusing on their convergence characteristics and computational efficiency. Across Examples 1, 2, 3, and 4, it's evident that iterative approaches with lower iteration counts generally achieve faster convergence to a solution. This is reflected in shorter computational times required for convergence. For instance, in Example 1, TRAOR's seven iterations led to faster convergence compared to SRAOR's ten iterations, despite similar computational times. This trend persists across Examples 2, 3, and 4, indicating the consistency of the observation. Such insights are crucial for informed decision-making in selecting the most suitable iterative method based on specific convergence requirements and computational constraints.

The Findings: The findings reveal that all examined iterative approaches exhibit spectral radii less than 1, indicating their convergence across the examples studied. The speed of convergence varies based on the proximity of the spectral radii to zero, with methods having spectral radii closer to zero converging faster. Particularly, TRAOR and TRSOR demonstrate smaller spectral radii compared to existing methods like SRSOR and SRAOR, suggesting potentially faster convergence rates. The numerical results show how better and efficient the newly proposed methods are in terms of the significant reduction in the number of iterations. These findings underscore the significance of spectral radius proximity to zero in determining convergence speed and highlight the importance of considering this factor when selecting an iterative method for problem-solving.

4. Conclusion

In this research, the main contribution lies in the development and implementation of two novel methods, namely TRSOR and TRAOR, which significantly enhance the accuracy and efficiency of solving linear systems. These methods are derived through the refinement of existing SRSOR and SRAOR techniques. The key findings demonstrate the superior effectiveness and reliability of TRSOR and TRAOR in addressing linear system problems, even in scenarios involving Irreducible diagonal dominance in coefficient matrices. Moreover, the rigorous establishment of convergence for TRAOR through several theorems adds robustness to the newly proposed method. Validation of the findings is achieved through comparisons with examples considered by other researchers, reinforcing the credibility of the results. Numerical investigations conducted indicate that TRSOR and TRAOR yield solutions closer to exact values and exhibit reduced spectral radii, as evidenced in Table 12. These observations translate to enhanced convergence and efficiency when compared to SRSOR and SRAOR, respectively. Particularly noteworthy is the faster convergence rate of TRAOR compared to its Successive Over relaxation counterpart. As a recommendation stemming from the study's outcomes, TRAOR is identified as the preferred method over TRSOR for solving large and sparse linear systems. Thus, the research findings underscore that these newly developed methods offer a notable enhancement in accuracy for approximating solutions to linear system problems, representing a significant advancement in the field. It is worthy to note that this paper has contributed immensely to further bridge the gap in convergence rate and minimization of spectral radius, thereby making computation of problems involving linear systems faster and more accurate.

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Declaration of Competing Interest

The authors have affirmed the absence of any conflicts of interest.

Authorship Contribution Statement

Khadeejah James Audu: Conceptualization, Methodology, Reviewing, Supervision
Malik Oyetunji Oniwinde: Data Preparation, Writing,
James Nkereuwem Essien: Maple Coding, Writing and Editing

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