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Research article

Framed Bertrand and Mannheim Curves in Three-Dimensional Space Forms of Non-zero Constant Curvatures

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Abstract

The purpose of this paper is to generalize definitions of Bertrand and Mannheim curves to non-null framed curves and to non-flat three-dimensional (Riemannian or Lorentzian) space forms. Denote by $\mathbb{M}^n_q(c)$ the n-dimensional space form of index $q = 0, 1$ and constant curvature $c \neq 0$. We introduce two types of framed Bertrand curves and framed Mannheim curves in $\mathbb{M}^3_q(c)$ by using two different moving frames: the general moving frame and the Frenet-type frame. We investigate geometric properties of these framed Bertrand and framed Mannheim curves in $\mathbb{M}^3_q(c)$ that may have singularities. We then give characterizations for a non-null framed curve to be a framed Bertrand curve or to be a framed Mannheim curve. We show that in special cases these characterizations reduce to the well-known classical formulas: $\lambda\kappa + \mu\tau = 1$ for Bertrand curves and $\lambda(\kappa^2 + \tau^2) = \kappa$ for Mannheim curves. We provide several examples to support our results, and we visualize these examples by using the Hopf map, the hyperbolic Hopf map, and the spherical projection.

Keywords: *Bertrand curves, Mannheim curves, framed curves, singularity, non-flat space forms, Hopf map.*

AMS Subject Classification (2020): *53A04; 53B30; 57R45; 58K05.*

1. Introduction

One of the popular curves that have been widely studied by geometers is the so-called Bertrand curve. These curves were introduced based on the question proposed by Saint-Venant [38]. The original proplem is to prove the existence of a curve $\bar{\gamma}$ on a ruled surface generated by principal normals of another curve γ such that γ and $\bar{\gamma}$ have the same principal normal vectors in the Euclidean 3-space. This problem was not answered until 1850, and eventually Bertrand [4] showed that such a curve indeed exits if the original curve is planar, or if the curvatures κ and τ of the original curve γ satisfy the linear relation: $\lambda\kappa + \mu\tau = 1$ for constants $\lambda \neq 0$ and μ. These curves have been widely studied [1, 2, 9, 11, 12, 21, 30, 31, 35, 37, 44]. There are also different approaches by generalizing the original definition of Bertrand curves [5, 28, 36]. Another kind of popular curve which is defined as very similar to the Bertrand curve is the so-called Mannheim curve, where the normal vectors of γ are parallel to the binormal vectors of another curve $\hat{\gamma}$ at corresponding points. There are several generalizations of Mannheim curves in the literature [11, 13, 15, 16, 23, 29, 43, 45, 48].

Many of the studies on Bertrand and Mannheim curves or on other similar curve pairs assumes that the original curve is regular. However, motivated by [14], recently the local differential geometry of certain families of singular curves has been studied. These curves have been investigated for many special curves (see for example [41]) and generalized to other ambient spaces and to higher dimensions [8, 17, 18, 24–27, 40, 42, 47]. Even though Bertrand and Mannheim curves of regular curves in the Riemannian or Lorentzian space forms have been extensively investigated, there are only a few papers investigating Bertrand and Mannheim curves of smooth curves that may have singularities: [19] for Bertrand and Mannheim curves with respect to the general

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moving frame in the Euclidean 3-space, [22] for Bertrand and Mannheim curves with respect to the Frenet-type frame in the Riemannian 3-space forms, [39] for Bertrand and Mannheim curves with respect to the general moving frame in the three-sphere, and [20] for Bertrand and Mannheim curves with respect to the general moving frame in the Euclidean 4-space. In this paper, we shall generalize the definitions of Bertrand and Mannheim curves to non-null framed curves in non-flat three-dimensional Riemannian or Lorentzian space forms, and then we investigate geometric and singularity properties of these curves. Hence the importance of this work is that it provides the most general characterizations for Bertrand and Mannheim curves of a certain family of smooth curves in non-flat Riemannian or Lorentzian 3-space forms. Besides we investigate two different types of framed Bertrand and Mannheim curves according to two different moving frames along the curve. Clearly the results of this paper will reduce to those of [22] and [39].

This paper is organized as follows. In Section 2 we briefly review the pseudo-Euclidean space $\mathbb{R}^{n+1}_v$ and introduce non-flat 3-dimensional space forms $\mathbb{M}^3_q(c)$ in $\mathbb{R}^{n+1}_v$. We then define framed curves in $\mathbb{M}^3_q(c)$ along with two orthonormal moving frames associated with these curves. We also remind three well-known maps, the Hopf map in the three sphere, the hyperbolic Hopf map in the anti-de Sitter 3-space, and the spherical projection in the hyperbolic 3-space. In Section 3, we go on to introduce Bertrand curves of non-null framed curves in $\mathbb{M}^3_q(c)$ with respect to the general moving frame and investigate geometric properties of these curves. We obtain a characterization for these curves. We then define Bertrand curves of non-null framed curves in $\mathbb{M}^3_q(c)$ with respect to the Frenet-type frame and study geometric properties of these curves. In Section 4, similar to Bertrand curves, we introduce two types of Mannheim curves of non-null framed curves in $\mathbb{M}^3_q(c)$ and investigate geometric properties of these curves. We finally give several examples of framed Bertrand and framed Mannheim curves and visualize them using the Hopf map, the hyperbolic Hopf map, and the spherical projection.

2. Preliminaries

Denote by $\mathbb{R}^{n+1}_v$ the $(n+1)$-dimensional pseudo-Euclidean space of index $v \geq 0$ endowed with a pseudo-scalar product defined by

$$\langle u, w\rangle = -\sum_{i=1}^{v} u_i w_i + \sum_{j=v+1}^{n+1} u_j w_j$$

where $u = (u_1, \dots, u_{n+1})$, $w = (w_1, \dots, w_{n+1}) \in \mathbb{R}^{n+1}$. Let $\mathbb{S}^n_q(c)$ denote the pseudo-Euclidean hyper-sphere with index $q \geq 0$ and constant curvature $c > 0$ given by

$$\mathbb{S}^n_q = \{u \in \mathbb{R}^{n+1}_q \,|\, \langle u, u\rangle = 1/c^2\},$$

and let $\mathbb{H}^n_q(c)$ denote the pseudo-Euclidean hyperbolic space with index $q \geq 0$ and constant curvature $c < 0$ given by

$$\mathbb{H}^n_q = \{u \in \mathbb{R}^{n+1}_{q+1} \,|\, \langle u, u\rangle = -1/c^2\}.$$

Without loss of generality we consider the unit pseudo-Euclidean hypersphere and the unit pseudo-Euclidean hyperbolic space, that is, we shall assume that $c = \pm 1$. For simplicity we will denote by $\mathbb{M}^n_q(c)$ these n-dimensional space forms of index $q \geq 0$ and constant curvature $c = \pm 1$, and we take $\mathbb{M}^n_q(1) = \mathbb{S}^n_q(1)$ and $\mathbb{M}^n_q(-1) = \mathbb{H}^n_q(-1)$. Note that for $c = 1$, $\mathbb{M}^n_q(1)$ lives in $\mathbb{R}^{n+1}_q$, and for $c = -1$, $\mathbb{M}^n_q(-1)$ lives in $\mathbb{R}^{n+1}_{q+1}$.

Clearly for $v = 0$, $\mathbb{R}^{n+1}_v$ reduces to the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For $v \neq 0$ vectors are classified with respect to the pseudo-scalar product defined above. Take a vector $u = (u_1, \dots, u_{n+1}) \in \mathbb{R}^{n+1}_v$. The vector u is called spacelike if $\langle u, u\rangle > 0$ or $u = 0$, timelike if $\langle u, u\rangle < 0$, or lightlike (null) if $\langle u, u\rangle = 0$ and $u \neq 0$. The pseudo-norm of the vector u is defined by $\|u\| = \sqrt{|\langle u, u\rangle|}$. A curve $\gamma : I \to \mathbb{R}^{n+1}_v$ is called spacelike, timelike, or lightlike (null) if the tangent vector of γ is always spacelike, timelike, or lightlike (null), respectively.

For n arbitrary vectors $u_1, \dots, u_n$ in $\mathbb{R}^{n+1}_v$, the vector product $u_1 \times \cdots \times u_n$ is defined as the unique vector in $\mathbb{R}^{n+1}_v$ satisfying the following relation for every $w \in \mathbb{R}^{n+1}_v$

$$\langle u_1 \times \cdots \times u_n, w\rangle = \det(u_1, \dots, u_n, w).$$

It is also possible to define a vector product $\wedge$ induced by $\times$ in $\mathbb{R}^{n+1}_v$ for vectors in the tangent space $T_p\mathbb{M}^n_q(c)$ at any point $p \in \mathbb{M}^n_q(c)$. Consider $n-1$ vectors $u_1, \dots, u_{n-1}$ in $T_p\mathbb{M}^n_q(c) \subset \mathbb{R}^{n+1}_v$. The vector product $u_1 \wedge \cdots \wedge u_{n-1} \in T_p\mathbb{M}^n_q(c)$ of these vectors is defined by

$$u_1 \wedge \cdots \wedge u_{n-1} = p \times u_1 \times \cdots \times u_{n-1}.$$

Denote respectively by $\bar{\nabla}$ and ∇^0 the Levi-Civita connections on $\mathbb{M}^n_q(c)$ and $\mathbb{R}^{n+1}_v$. For two vector fields U and V tangent to $\mathbb{M}^n_q(c)$, the Gauss formula reads

$$\nabla^0_U V = \bar{\nabla}_U V - c\langle U, V\rangle \phi, \tag{2.1}$$

where $\phi : \mathbb{M}^n_q(c) \to \mathbb{R}^{n+1}_v$ denotes the position vector.

Since we are interested in non-flat three-dimensional space forms, from now on we shall take $n = 3$.

Consider a regular non-null curve $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{M}^3_q(c)$ that is not a geodesic, and suppose that γ is a unit-speed curve, i.e., $\|\gamma'(s)\| = 1$ for all $s \in I$. Then we can easily define the Frenet frame $\{T, N, B\}$ along the curve γ such that

$$\nabla^0_T T = -\epsilon_1 c\gamma + \epsilon_2 \kappa N, \quad \nabla^0_T N = -\epsilon_1 \kappa T + \epsilon_3 \tau B, \quad \nabla^0_T B = \epsilon_2 \tau N, \tag{2.2}$$

where κ and τ stand for the curvature and the torsion of γ, and $\epsilon_1, \epsilon_2, \epsilon_3$ are the causal characters of the Frenet vectors T, N, B.

Obviously if the curve γ has singularities, then the Frenet frame defined above is not well-defined. Therefore, by generalizing regular curves, we introduce the concept of framed curves in $\mathbb{M}^n_q(c)$ and give well-defined frames along these curves. Let $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{M}^3_q(c)$ be a smooth immersed curve. Then $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is called a *framed curve* if $\langle \gamma(s), v_i(s)\rangle = 0 = \langle \gamma'(s), v_i(s)\rangle$ $(i = 1, 2)$ for all $s \in I$, where

$$\Delta = \{(u, w) \mid \langle u, w\rangle = 0\}.$$

We have the following cases for Δ:

1. For $\mathbb{M}^3_q(c) = \mathbb{S}^3_0 \subset \mathbb{R}^4$, we have $\Delta \subset \mathbb{S}^3_0 \times \mathbb{S}^3_0$.
2. For $\mathbb{M}^3_q(c) = \mathbb{S}^3_1 \subset \mathbb{R}^4_1$, we have the following cases:
   (a) $\Delta \subset \mathbb{S}^3_1 \times \mathbb{H}^3_0$ or $\Delta \subset \mathbb{H}^3_0 \times \mathbb{S}^3_1$. In this case γ is called a spacelike framed curve in $\mathbb{S}^3_1$.
   (b) $\Delta \subset \mathbb{S}^3_1 \times \mathbb{S}^3_1$. In this case γ is called a timelike framed curve in $\mathbb{S}^3_1$.
3. For $\mathbb{M}^3_q(c) = \mathbb{H}^3_0 \subset \mathbb{R}^4_1$, we have $\Delta \subset \mathbb{S}^3_1 \times \mathbb{S}^3_1$. In this case γ is called a spacelike framed curve in $\mathbb{H}^3_0$.
4. For $\mathbb{M}^3_q(c) = \mathbb{H}^3_1 \subset \mathbb{R}^4_2$, we have the following cases:
   (a) $\Delta \subset \mathbb{H}^3_1 \times \mathbb{S}^3_2$ or $\Delta \subset \mathbb{S}^3_2 \times \mathbb{H}^3_1$. In this case γ is called a spacelike framed curve in $\mathbb{H}^3_1$.
   (b) $\Delta \subset \mathbb{S}^3_2 \times \mathbb{S}^3_2$. In this case γ is called a timelike framed curve in $\mathbb{H}^3_1$.
5. For $\mathbb{M}^3_q(c) = \mathbb{S}^3_2 \subset \mathbb{R}^4_2$, we have the following cases:
   (a) $\Delta \subset \mathbb{H}^3_1 \times \mathbb{H}^3_1$. In this case γ is called a spacelike framed curve in $\mathbb{S}^3_2$.
   (b) $\Delta \subset \mathbb{H}^3_1 \times \mathbb{S}^3_2$ or $\Delta \subset \mathbb{S}^3_2 \times \mathbb{H}^3_1$. In this case γ is called a timelike framed curve in $\mathbb{S}^3_2$.

For a framed curve (γ, v_1, v_2) in $\mathbb{M}^3_q(c) \times \Delta$, set $\mu(s) = \gamma(s) \times v_1(s) \times v_2(s)$. Then there exists a smooth function $\alpha : I \to \mathbb{R}$ such that $\gamma'(s) = \alpha(s)\mu(s)$. Hence $\{v_1, v_2, \mu\}$ forms an orthonormal frame along γ in $\mathbb{M}^3_q$. Considering the Levi-Civita connection ∇^0 of $\mathbb{R}^4_v$, we have the following derivative formulas of this frame

$$\begin{aligned} \nabla^0_{\gamma'} v_1 &= \epsilon_2 \ell_1 v_2 + \epsilon_3 \ell_2 \mu, \\ \nabla^0_{\gamma'} v_2 &= -\epsilon_1 \ell_1 v_1 + \epsilon_3 \ell_3 \mu, \\ \nabla^0_{\gamma'} \mu &= -\epsilon_3 c\alpha\gamma - \epsilon_1 \ell_2 v_1 - \epsilon_2 \ell_3 v_2, \end{aligned} \tag{2.3}$$

where $\epsilon_i = \langle v_i, v_i\rangle$ $(i = 1, 2)$, $\epsilon_3 = \langle \mu, \mu\rangle$, $\alpha = \epsilon_3\langle \gamma', \mu\rangle$, $\ell_1 = \langle v_1', v_2\rangle$, $\ell_2 = \langle v_1', \mu\rangle$, and $\ell_3 = \langle v_2', \mu\rangle$. Note that for simplicity we shall also use the notation x' to represent the derivative $\nabla^0_{\gamma'} x$. The mapping $(\alpha, \ell_1, \ell_2, \ell_3) : I \to \mathbb{R}^4$ is said to be the curvature of (γ, v_1, v_2). It is easy to see that γ is singular at $s_0 \in I$ if and only if $\alpha(s_0) = 0$.

Note that these framed curves satisfy the existence and uniqueness. More specifically, for a smooth mapping $(\alpha, \ell_1, \ell_2, \ell_3) : I \to \mathbb{R}^4$, there exists a unique framed curve $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c)$ up to rigid motion such that α, ℓ_1, ℓ_2, and ℓ_3 are the curvatures of γ. This fact has been proved for different ambient spaces [14, 17, 40] and can be similarly proved for our cases.

We are also be able to form a Frenet-type frame along a framed curve. This frame is indeed like a moving frame defined above but the last component $\ell_3(s)$ of its curvature vanishes for all s. Therefore, we can construct this frame by keeping μ and by applying a suitable pseudo-Euclidean rotation of $\{v_1, v_2\}$ about μ. Denote this Frenet-type frame by $\{\mu, \omega_1, \omega_2\}$. Hence we have

$$\begin{pmatrix} \omega_1(s) \\ \omega_2(s) \end{pmatrix} = \begin{pmatrix} f(\sigma(s)) & -\epsilon_1\epsilon_2 g(\sigma(s)) \\ g(\sigma(s)) & f(\sigma(s)) \end{pmatrix} \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix},$$

where $(f(u), g(u)) = (\cos u, \sin u)$ if $\epsilon_1\epsilon_2 = 1$, $(f(u), g(u)) = (\cosh u, \sinh u)$ if $\epsilon_1\epsilon_2 = -1$. Differentiating ω_1 and ω_2 and using (2.3) we have

$$\begin{aligned} \omega_1'(s) &= \epsilon_2(-\epsilon_1\sigma'(s) + \ell_1(s))\omega_2 + \epsilon_3(\ell_2(s)f(\sigma(s)) - \epsilon_1\epsilon_2\ell_3(s)g(\sigma(s)))\mu(s), \\ \omega_2'(s) &= -\epsilon_1(-\epsilon_1\sigma'(s) + \ell_1(s))\omega_1 + \epsilon_3(\ell_2(s)g(\sigma(s)) + \ell_3(s)f(\sigma(s)))\mu(s). \end{aligned}$$

Let σ be a smooth function such that $\ell_2(s)g(\sigma(s)) + \ell_3(s)f(\sigma(s)) = 0$. Assume that $\ell_2(s) = p_2(s)f(\sigma(s))$ and $\ell_3(s) = -p_2(s)g(\sigma(s))$. Then we have

$$\mu'(s) = -\epsilon_3 c\alpha(s)\gamma(s) - \epsilon_1 p_2(s)\omega_1(s).$$

Then defining $p_1(s) := -\epsilon_1\sigma'(s) + \ell_1(s)$ we get the Frenet-Serret type formulas as follows

$$\begin{aligned} \nabla^0_{\gamma'}\omega_1 &= \varepsilon_2 p_1\omega_2 + \varepsilon_3 p_2\mu, \\ \nabla^0_{\gamma'}\omega_2 &= -\varepsilon_1 p_1\omega_1, \\ \nabla^0_{\gamma'}\mu &= -\varepsilon_3 c\alpha\gamma - \varepsilon_1 p_2\omega_1, \end{aligned} \tag{2.4}$$

where $\varepsilon_i = \langle\omega_i, \omega_i\rangle$ $(i = 1, 2)$, $\varepsilon_3 = \langle\mu, \mu\rangle$, $\alpha = \varepsilon_3\langle\gamma', \mu\rangle$. We call the mapping $(\alpha, p_1, p_2) : I \to \mathbb{R}^3$ the Frenet-type curvature of $(\gamma, \omega_1, \omega_2)$.

Next we introduce a geodesic curve in $\mathbb{M}^3_q(c) \subset \mathbb{R}^{n+1}_v$ as follows: let $\gamma(s)$ be a point of γ in $\mathbb{M}^3_q(c)$, and let $X(s)$ a point in $\mathbb{M}^3_r(d) \subset \mathbb{R}^{n+1}_v$. Then for $t \in \mathbb{R}$

$$\delta^{\gamma,X}_s(t) = \exp_\gamma(tX) = f(t)\gamma(s) + g(t)X(s) \tag{2.5}$$

is a geodesic curve in $\mathbb{M}^3_q(c)$, where f and g are functions given by $f(t) = \cos t$ and $g(t) = \sin t$ if $dc = 1$, by $f(t) = \cosh t$ and $g(t) = \sinh t$ if $dc = -1$. We shall consider two particular types of such geodesic curves. For a framed curve $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c)\Delta$, $\delta^{\gamma,v_1}_s(t)$ is called the generalized principal-normal geodesic starting at $\gamma(s)$, and $\delta^{\gamma,v_2}_s(t)$ is called the generalized binormal geodesic starting at $\gamma(s)$.

We recall the Hopf map given by [34]

$$\begin{aligned} \pi :&\mathbb{M}^3_0(1) \to S^2(1/2) \\ &(u_1, u_2, u_3, u_4) \mapsto \left(u_1u_3 + u_2u_4, u_2u_3 - u_1u_4, \frac{u_1^2 + u_2^2 - u_3^2 - u_4^2}{2}\right), \end{aligned} \tag{2.6}$$

where $S^2(1/2) = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1/4\}$ is the 2-sphere that is the surface of constant curvature $1/4$ in the three-dimensional Euclidean space $\mathbb{R}^3$.

We now recall the hyperbolic Hopf map given by [3]

$$\begin{aligned} \mathbf{h} :&\mathbb{M}^3_1(-1) \to H^2(1/2) \\ &(u_1, u_2, u_3, u_4) \mapsto \left(u_1u_3 + u_2u_4, u_1u_4 - u_2u_3, \frac{u_1^2 + u_2^2 + u_3^2 + u_4^2}{2}\right), \end{aligned} \tag{2.7}$$

where $H^2(1/2) = \{(y_1, y_2, y_3) \in \mathbb{R}^3_1 \mid y_1^2 + y_2^2 - y_3^2 = -1/4 \text{ and } y_3 > 0\}$ is the hyperbolic 2-space that is the surface of constant curvature $-1/4$ in the three-dimensional Minkowski space $\mathbb{R}^3_1$. By using this map we will get projections on $H^2(1/2)$ of curves in the anti-de Sitter 3-space $\mathbb{M}^3_1(-1)$ and visualize them.

We close this section by defining the spherical projection of a curve in $\mathbb{M}^3_0(-1) \subset \mathbb{R}^4_1$ [7]. Let $\gamma \in \mathbb{M}^3_0(-1)$ be a curve defined by $\exp_p(\rho(s)V(s))$, where $p \in \mathbb{M}^3_0(-1)$, $\rho(s) \neq 0$ is an arbitrary function, and $V(s)$ is a curve in $\mathbb{M}^2_0(1) = \mathbb{S}^2(1) \in T_p\mathbb{M}^3_0(-1)$. In this case, V is said to be the spherical projection of γ.

3. Non-null framed Bertrand curves in $\mathbb{M}^3_q(c)$

We, in this section, shall introduce two-types of Bertrand curves of non-null framed curves in $\mathbb{M}^3_q(c)$ with respect to two different frames: the general moving frame $\{v_1, v_2, \mu\}$ and the Frenet-type frame $\{\omega_1, \omega_2, \mu\}$.

3.1. Framed Bertrand curves in $\mathbb{M}^3_q(c)$ with respect to the general moving frame $\{v_1, v_2, \mu\}$

Definition 3.1. Suppose that $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve. Then (γ, v_1, v_2) is called a *Bertrand curve* if there exists another non-null framed curve $(\bar{\gamma}, \bar{v}_1, \bar{v}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ such that both curves (γ, v_1, v_2) and $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$ have common generalized principal normal geodesics at corresponding points. In this case, $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$ is said to be a *framed Bertrand mate (or conjugate)* of (γ, v_1, v_2), and as well (γ, v_1, v_2) and $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$ are called *a pair of framed Bertrand curves*.

By this definition, we immediately see that for a pair of framed Bertrand curves (γ, v_1, v_2) and $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$, there exists a differentiable function $\varphi(s)$ such that

$$\bar{\gamma}(s) = f(\varphi(s))\gamma(s) + g(\varphi(s))v_1(s). \tag{3.1}$$

Since $\bar{\gamma} \in \mathbb{M}^3_q(c)$, we have $f^2(\varphi(s)) + \epsilon_1 c g^2(\varphi(s)) = 1$. Note that from now on, we will simply say γ is a framed Bertrand curve instead of saying (γ, v_1, v_2) is a framed Bertrand curve. Furthermore, throughout the paper we will assume that $g(\varphi(s)) \not\equiv 0$ that is $\bar{\gamma} \neq \pm\gamma$.

Proposition 3.1. *Let γ be a framed Bertrand curve, and let $\bar{\gamma}$ be a Bertrand mate of this curve given by* (3.1)*. Then the function $\varphi(s)$ is constant.*

Proof. From Definition 3.1, γ and $\bar{\gamma}$ have common generalized principal normal geodesics at corresponding points. Thus we have

$$\left.\frac{d}{dt}\right|_{t=\varphi(s)} \delta_s^{\gamma, v_1}(t) = \bar{v}_1(s).$$

Since $f'(t) = -\epsilon_1 c g(t)$ and $g'(t) = f(t)$, we find that

$$\bar{v}_1(s) = -\epsilon_1 c g(\varphi(s))\gamma(s) + f(\varphi(s))v_1(s).$$

Differentiating (3.1) and using (2.3) yields

$$\begin{aligned}\bar{\gamma}'(s) = \bar{\alpha}(s)\bar{\mu}(s) = &-\epsilon_1 c\, \varphi'(s) g(\varphi(s))\gamma(s) + \left(f(\varphi(s))\alpha(s) + \epsilon_3 g(\varphi(s))\ell_2(s)\right)\mu(s)\\ &+ \varphi'(s) f(\varphi(s))v_1(s) + \epsilon_2 g(\varphi(s))\ell_1(s)v_2(s).\end{aligned}$$

Then we get

$$0 = \langle \bar{\gamma}'(s), \bar{v}_1(s)\rangle = \varphi'(s)(c g^2(\varphi(s)) + \epsilon_1 f^2(\varphi(s))) = \epsilon_1 \varphi'(s),$$

which directly yields $\varphi'(s) \equiv 0$. □

Proposition 3.2. *Let $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ be a non-null Bertrand curve with curvature $(\alpha, \ell_1, \ell_2, \ell_3)$. A Bertrand mate $(\bar{\gamma}, \bar{v}_1, \bar{v}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ of (γ, v_1, v_2) is also a non-null framed curve with curvature $(\bar{\alpha}, \bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3)$, where*

$$\begin{aligned}
\bar{v}_1(s) &= -\epsilon_1 c g(\varphi)\gamma(s) + f(\varphi)v_1(s),\\
\bar{v}_2(s) &= \xi(\theta(s))v_2(s) + \eta(\theta(s))\mu(s),\\
\bar{\mu}(s) &= \epsilon_2 \eta(\theta(s))v_2(s) - \epsilon_3 \xi(\theta(s))\mu(s),\\
\bar{\alpha}(s) &= \bar{\epsilon}_3 \epsilon_2 g(\varphi)\eta(\theta(s))\ell_1(s) - \bar{\epsilon}_3\left(f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s)\right)\xi(\theta(s)),\\
\bar{\ell}_1(s) &= f(\varphi)\xi(\theta(s))\ell_1(s) + \left(-\epsilon_1\epsilon_3 c g(\varphi)\alpha(s) + f(\varphi)\ell_2(s)\right)\eta(\theta(s)),\\
\bar{\ell}_2(s) &= \epsilon_2 f(\varphi)\eta(\theta(s))\ell_1(s) + \left(\epsilon_1 c g(\varphi)\alpha(s) - \epsilon_3 f(\varphi)\ell_2(s)\right)\xi(\theta(s)),\\
\bar{\ell}_3(s) &= -\bar{\epsilon}_3(\epsilon_3\theta'(s) + \ell_3(s)),
\end{aligned}$$

where $\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3$ are the causal characters of $\bar{v}_1, \bar{v}_2$, and $\bar{\mu}$, and $\theta : I \to \mathbb{R}$ is a smooth function. Moreover, the functions ξ and η are given by $(\xi(u), \eta(u)) = (\cos u, \sin u)$ if $\bar{\epsilon}_2 = \epsilon_2 = \epsilon_3$, by $(\xi(u), \eta(u)) = (\cosh u, \sinh u)$ if $\bar{\epsilon}_2 = \epsilon_2 = -\epsilon_3$, and by $(\xi(u), \eta(u)) = (\sinh u, \cosh u)$ if $\bar{\epsilon}_2 = -\epsilon_2 = \epsilon_3$. Thus, these two functions satisfy the relation $\bar{\epsilon}_2 = \epsilon_2\xi^2 + \epsilon_3\eta^2$.

Proof. Since $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$ is a Bertrand mate of (γ, v_1, v_2), by (3.1) and Proposition 3.1 we directly have

$$\begin{aligned}\bar{\gamma}(s) &= f(\varphi)\gamma(s) + g(\varphi)v_1(s),\\ \bar{v}_1(s) &= -\epsilon_1 c g(\varphi)\gamma(s) + f(\varphi)v_1(s).\end{aligned}$$

Notice that $\bar{\epsilon}_1 = \langle \bar{v}_1, \bar{v}_1 \rangle = \epsilon_1$. The derivative of $\bar{\gamma}(s)$ with respect to s is given by

$$\bar{\gamma}'(s) = \bar{\alpha}(s)\bar{\mu}(s) = \left(f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s)\right)\mu(s) + \epsilon_2 g(\varphi)\ell_1(s)v_2(s).$$

Therefore there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\bar{v}_2(s) = \xi(\theta(s))v_2(s) + \eta(\theta(s))\mu(s),$$

where ξ and η are functions so that $\bar{\epsilon}_2 = \langle \bar{v}_2, \bar{v}_2 \rangle = \epsilon_2\xi^2 + \epsilon_3\eta^2$. Therefore, based on the causal characters $\bar{\epsilon}_2, \epsilon_2$, and ϵ_3, the pair $(\xi(u), \eta(u))$ can be chosen as one of the following pairs of functions; $(\cos u, \sin u)$, $(\cosh u, \sinh u)$, or $(\sinh u, \cosh u)$. Notice that $\xi'(u) = -\epsilon_2\epsilon_3\eta(u)$ and $\eta'(u) = \xi(u)$. Then we find that $\langle \bar{\gamma}, \bar{v}_1 \rangle = \langle \bar{\gamma}, \bar{v}_2 \rangle = 0$ and $\langle \bar{\gamma}', \bar{v}_1 \rangle = \langle \bar{\gamma}', \bar{v}_2 \rangle = 0$, and thus $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$ is a non-null framed curve in $\mathbb{M}^3_q(c) \times \Delta$. It is easy to find that $\bar{\mu}(s) = \epsilon_2\eta(\theta(s))v_2(s) - \epsilon_3\xi(\theta(s))\mu(s)$.

To find the curvature of this framed curve, we shall need the following derivatives:

$$\begin{aligned}
\bar{v}_1'(s) &= \epsilon_2 f(\varphi)\ell_1(s)v_2(s) + \big(\epsilon_3 f(\varphi)\ell_2(s) - \epsilon_1 c g(\varphi)\alpha(s)\big)\mu(s),\\
\bar{v}_2'(s) &= -\epsilon_3 c\alpha(s)\eta(\theta(s))\gamma(s) - \epsilon_1(\ell_1(s)\xi(\theta(s)) + \ell_2(s)\eta(\theta(s)))v_1(s)\\
&\quad - \epsilon_2(\epsilon_3\theta'(s) + \ell_3(s))\eta(\theta(s))v_2(s) + (\theta'(s) + \epsilon_3\ell_3(s))\xi(\theta(s))\mu(s).
\end{aligned}$$

We know that $\bar{\alpha}(s) = \bar{\epsilon}_3\langle \bar{\gamma}'(s), \bar{\mu}(s) \rangle$, $\bar{\ell}_1(s) = \langle \bar{v}_1'(s), \bar{v}_2(s) \rangle$, $\bar{\ell}_2(s) = \langle \bar{v}_1'(s), \bar{\mu}(s) \rangle$, and $\bar{\ell}_3(s) = \langle \bar{v}_2'(s), \bar{\mu}(s) \rangle$. The rest of the proof follows straightforwardly from the equations given above. ☐

Notice that if γ has a singularity at s_0, that is, $\alpha(s_0) = 0$. Then its framed Bertrand conjugate $\bar{\gamma}$ has a singularity at s_0 if and only if

$$\epsilon_2\eta(\theta(s_0))\ell_1(s_0) = \epsilon_3\xi(\theta(s_0))\ell_2(s_0). \tag{3.2}$$

Proposition 3.3. *Let γ and $\bar{\gamma}$ be a pair of non-null framed Bertrand curves in $\mathbb{M}^3_q(c)$. Then there exist a constant φ and a smooth function $\theta : I \to \mathbb{R}$ such that*

(a) $\left(f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s)\right)\eta(\theta(s)) = -\epsilon_3 g(\varphi)\xi(\theta(s))\ell_1(s)$,

(b) $\epsilon_2\left(f(\varphi)\bar{\alpha}(s) - \bar{\epsilon}_3 g(\varphi)\bar{\ell}_2(s)\right)\eta(\theta(s)) = \bar{\epsilon}_2 g(\varphi)\xi(\theta(s))\bar{\ell}_1(s)$,

(c) $\left(f(s)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s)\right)\left(f(s)\bar{\alpha}(s) - \bar{\epsilon}_3 g(\varphi)\bar{\ell}_2(s)\right) = \epsilon_3\bar{\epsilon}_3\alpha(s)\bar{\alpha}(s)\xi^2(\theta(s))$,

(d) $-\epsilon_3\alpha(s)\bar{\alpha}(s)\eta^2(\theta(s)) = \ell_1(s)\bar{\ell}_1(s)g^2(\varphi)$.

Proof. (a) Bearing in mind that φ is a constant, we get the derivative of (3.1)

$$\bar{\gamma}'(s) = \left(f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s)\right)\mu(s) + \epsilon_2 g(\varphi)\ell_1(s)v_2(s).$$

We also know that $\bar{\gamma}'(s) = \bar{\alpha}(s)\bar{\mu}(s)$. Now introducing $\bar{\mu}(s)$ given in Proposition 3.2 into this equation gives us the following relation:

$$\bar{\gamma}'(s) = \bar{\alpha}(s)\left(\epsilon_2\eta(\theta(s))v_2(s) - \epsilon_3\xi(\theta(s))\mu(s)\right).$$

From the last two equations, we find that

$$\bar{\alpha}(s)\eta(\theta(s)) = g(\varphi)\ell_1(s), \tag{3.3}$$
$$-\epsilon_3\bar{\alpha}(s)\xi(\theta(s)) = f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s). \tag{3.4}$$

Claim (a) follows directly from these equations.

(b) To prove this statement we need to write the moving frame $\{\gamma, v_1, v_2, \mu\}$ in terms of $\{\bar{\gamma}, \bar{v}_1, \bar{v}_2, \bar{\mu}\}$. This is simple from the equations given in Proposition 3.2:

$$\begin{aligned}
\gamma(s) &= f(\varphi)\bar{\gamma}(s) - g(\varphi)\bar{v}_1(s),\\
v_1(s) &= \epsilon_1 c g(\varphi)\bar{\gamma}(s) + f(\varphi)\bar{v}_1(s),\\
v_2(s) &= \bar{\epsilon}_3(\epsilon_3\xi(\theta(s))\bar{v}_2(s) + \eta(\theta(s))\bar{\mu}(s)),\\
\mu(s) &= \bar{\epsilon}_3(\epsilon_2\eta(\theta(s))\bar{v}_2(s) - \xi(\theta(s))\bar{\mu}(s)).
\end{aligned} \tag{3.5}$$

Following similar steps to those in (a), we obtain

$$\epsilon_2\bar{\epsilon}_3\alpha(s)\eta(\theta(s)) = -\bar{\epsilon}_2 g(\varphi)\bar{\ell}_1(s), \tag{3.6}$$
$$-\bar{\epsilon}_3\alpha(s)\xi(\theta(s)) = f(\varphi)\bar{\alpha}(s) - \bar{\epsilon}_3 g(\varphi)\bar{\ell}_2(s). \tag{3.7}$$

From these equations we get (b).

(c) The proof directly follows from (3.4) and (3.7).

(d) The proof directly follows from (3.3) and (3.6).

□

Suppose that γ has a singularity. In this case, the framed Bertrand mate $\bar{\gamma}$ of γ might be a regular curve or might have singularities. Let $\alpha(s_0) = 0$. Then from (3.6) and (3.7) we immediately have $\bar{\ell}_1(s_0) = 0$ and $f(\varphi)\bar{\alpha}(s_0) = \bar{\epsilon}_3 g(\varphi)\bar{\ell}_2(s_0)$. Now suppose that $\bar{\gamma}$ has singularity at s_0, that is, $\bar{\alpha}(s_0) = 0$. Then from (3.3) and (3.4) we directly get $\ell_1(s_0) = 0$ and $f(\varphi)\alpha(s_0) = -\epsilon_3 g(\varphi)\ell_2(s_0)$. Therefore, if both γ and $\bar{\gamma}$ has singularity at s_0, then we find that $\ell_1(s_0) = 0$, $\ell_2(s_0) = 0$, $\bar{\ell}_1(s_0) = 0$, $\bar{\ell}_2(s_0) = 0$. The converse of this proposition does not have to be true.

Proposition 3.4. *Let γ and $\bar{\gamma}$ be a pair of non-null framed Bertrand curves in $\mathbb{M}^3_q(c)$. Then there exist a constant φ and a smooth function $\theta : I \to \mathbb{R}$ such that*

(a) $\ell_1(s)f(\varphi) = \epsilon_2\bar{\epsilon}_2\bar{\ell}_1(s)\xi(\theta(s)) + \bar{\epsilon}_3\bar{\ell}_2(s)\eta(\theta(s))$ *and* $\bar{\ell}_1(s)f(\varphi) = \ell_1(s)\xi(\theta(s)) + \ell_2(s)\eta(\theta(s))$,

(b) $-\epsilon_1 c\alpha(s)g(\varphi) + \epsilon_3\ell_2(s)f(\varphi) = \bar{\epsilon}_2\bar{\ell}_1(s)\eta(\theta(s)) - \epsilon_3\bar{\epsilon}_3\bar{\ell}_2(s)\xi(\theta(s))$ *and* $\epsilon_2\ell_1(s)\eta(\theta(s)) - \epsilon_3\ell_2(s)\xi(\theta(s)) = \epsilon_1\bar{\epsilon}_3 c\bar{\alpha}(s)g(\varphi) + \bar{\ell}_2(s)f(\varphi)$.

Proof. From (2.3) and Proposition 3.2 we have

$$\begin{aligned}\bar{v}_1'(s) =& \bar{\epsilon}_2\bar{\ell}_1(s)\bar{v}_2(s) + \bar{\epsilon}_3\bar{\ell}_2(s)\bar{\mu}(s)\\ =& \left(\bar{\epsilon}_2\bar{\ell}_1(s)\xi(\theta(s)) + \epsilon_2\bar{\epsilon}_3\bar{\ell}_2(s)\eta(\theta(s))\right)v_2(s)\\ &+ \left(\bar{\epsilon}_2\bar{\ell}_1(s)\eta(\theta(s)) - \epsilon_3\bar{\epsilon}_3\bar{\ell}_2(s)\xi(\theta(s))\right)\mu(s).\end{aligned}$$

On the other hand, differentiating $\bar{v}_1(s)$ given in Proposition 3.2 yields

$$\bar{v}_1'(s) = \epsilon_2\ell_1(s)f(\varphi)v_2(s) + \left(-\epsilon_1 c\alpha(s)g(\varphi) + \epsilon_3\ell_2(s)f(\varphi)\right)\mu(s).$$

Using the last two equations we prove the first relations in (a) and (b).

For the proof of the second relations, a similar procedure is followed by using relations in (3.5). □

Notice that if γ has singularity at s_0, then from the second equality of Proposition 3.4(a) we have $\xi(\theta(s_0))\ell_1(s_0) + \eta(\theta(s_0))\ell_2(s_0) = 0$ and from the first equality of Proposition 3.4(b) we have $\ell_2(s_0)f(\varphi) = -\bar{\epsilon}_3\bar{\ell}_2(s_0)\xi(\theta(s_0))$ since $\bar{\ell}_1(s_0) = 0$.

In the following proposition, we shall consider a special case where $\ell_1(s) = 0$ for all $s \in I$.

Proposition 3.5. (i) *A non-null framed curve $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ with curvature $(\alpha, 0, \ell_2, \ell_3)$ is a framed Bertrand curve, and this curve has infinite Bertrand conjugates $\bar{\gamma}$ such that $\bar{\ell}_1(s) = 0$ for all $s \in I$.*

(ii) *Let γ and $\bar{\gamma}$ be a pair of non-null framed Bertrand curves with curvatures $(\alpha, \ell_1, \ell_2, \ell_3)$ and $(\bar{\alpha}, \bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3)$. If $\bar{\ell}_1(s) = 0$ for all $s \in I$, then $\ell_1(s) = 0$ for all $s \in I$.*

Proof. (i) Let γ be a non-null framed curve such that $\ell_1(s) = 0$ for all $s \in I$. For each real number $\varphi \in (-\varepsilon, \varepsilon)$, consider the curve in $\mathbb{M}^3_q(c)$ defined by

$$\bar{\gamma}_\varphi(s) = f(\varphi)\gamma(s) + g(\varphi)v_1(s). \tag{3.8}$$

We will show that for all $\varphi \in (-\varepsilon, \varepsilon)$, $\bar{\gamma}_\varphi$ is a framed Bertrand conjugate of γ. Differentiating (3.8) and using (2.3), we may take

$$\bar{\mu}_\varphi(s) = \mu(s), \tag{3.9}$$

$$\bar{\alpha}_\varphi(s) = f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s). \tag{3.10}$$

Therefore we get

$$\bar{v}_{1_\varphi}(s) = -\epsilon_1 c g(\varphi)\gamma(s) + f(\varphi)v_1(s). \tag{3.11}$$

Then the generalized principal normal geodesic starting at $\bar{\gamma}_\varphi(s_0)$ is given by

$$\delta(t) = f(t)\bar{\gamma}_\varphi(s_0) + g(t)\bar{v}_{1_\varphi}(s_0) = f(t+\varphi)\gamma(s_0) + g(t+\varphi)v_1(s_0).$$

Clearly this is just a reparametrization of the generalized principal normal geodesic starting at $\gamma(s_0)$. Therefore $\bar{\gamma}_\varphi$ is a non-null framed Bertrand conjugate of γ.

Differentiating (3.11) and using (3.8), (3.9), and (3.10) yields

$$\bar{v}_{2_\varphi}(s) = v_2(s), \quad \bar{\ell}_{1_\varphi}(s) = 0, \quad \bar{\ell}_{2_\varphi}(s) = \epsilon_2 c g(\varphi)\alpha(s) + f(\varphi)\ell_2(s), \quad \bar{\ell}_{3_\varphi}(s) = \ell_3(s). \tag{3.12}$$

(ii) Assume that $\bar{\ell}_1(s) = 0$ for all $s \in I$. From Proposition 3.3(d) we find that $\eta(\theta(s)) = 0$ for all $s \in I$. Now introducing these relations into Proposition 3.3(a), we get $g(\varphi)\ell_1(s) = 0$ for all $s \in I$. Then we have two cases: (1) if $g(\varphi) = 0$, then $f^2(\varphi) = 1$ and so $\gamma = \pm\bar{\gamma}$. Hence $\bar{\ell}_1(s) = \ell_1(s) = 0$ for all $s \in I$. (2) if $\ell_1(s) = 0$ for all $s \in I$, there is nothing to prove.

□

We next give the main theorem of this section that characterizes framed Bertrand curves in $\mathbb{M}^3_q(c)$. But before this theorem, we recall the following conditions from the above discussion:

C1. f and g are smooth functions such that $f^2 + \epsilon_1 c g^2 = 1$, $f' = -\epsilon_1 c g$, and $g' = f$.

C2. $\xi(u)$ and $\eta(u)$ are smooth functions such that

$$(\xi, \eta) \in \{(\cos u, \sin u), (\cosh u, \sinh u), (\sinh u, \cosh u)\}.$$

Theorem 3.1. *Suppose that $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve with curvature $(\alpha, \ell_1, \ell_2, \ell_3)$, where $\ell_1(s) \neq 0$ for all $s \in I$. Then γ is a framed Bertrand curve if and only if there exist a constant φ such that $g(\varphi) \neq 0$ and a smooth function $\theta : I \to \mathbb{R}$ such that for all $s \in I$*

$$\epsilon_3 g(\varphi)\xi(\theta(s))\ell_1(s) + (f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)\ell_2(s))\,\eta(\theta(s)) = 0, \tag{3.13}$$

where the functions f and g satisfy condition C1, and the functions ξ and η satisfy condition C2.

Proof. The sufficiency part is clear from Proposition 3.3(a).

Now suppose that (3.13) is satisfied for a certain function $\theta(s)$ and a constant φ. Consider the curve

$$\bar{\gamma}(s) = f(\varphi)\gamma(s) + g(\varphi)v_1(s). \tag{3.14}$$

We will show that this curve is a framed Bertrand conjugate of γ. Then using (3.13) it is easy to show that $(\bar{\gamma}, \bar{v}_1, \bar{v}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve with

$$\bar{v}_1(s) = -\epsilon_1 c g(\varphi)\gamma(s) + f(\varphi)v_1(s), \tag{3.15}$$

$$\bar{v}_2(s) = \xi(\theta(s))v_2(s) + \eta(\theta(s))\mu(s). \tag{3.16}$$

Using (3.14) and (3.15), we see that the generalized principal normal geodesic starting at a point $\bar{\gamma}(s_0)$ is

$$\begin{aligned}\delta(t) &= f(t)\bar{\gamma}(s_0) + g(t)\bar{v}_1(s_0)\\ &= f(t+\varphi)\gamma(s_0) + g(t+\varphi)v_1(s_0).\end{aligned}$$

This gives a reparametrization of the generalized principal normal geodesic at $\gamma(s_0)$. Therefore $\bar{\gamma}$ is a framed Bertrand conjugate of γ. □

Remark 3.1. Equation (3.13) characterizes framed Bertrand curves with $\ell_1(s) \neq 0$ in $\mathbb{M}^3_q(c)$. But this characterization looks different from its counterparts for regular Bertrand curves. Let $\lambda = g/f$, and assuming $\eta(\theta(s)) \neq 0$ for all $s \in I$, define $\rho(s) = \lambda\xi(\theta(s))/\eta(\theta(s))$. Hence (3.13) becomes

$$\lambda\ell_2(s) + \rho(s)\ell_1(s) = -\epsilon_3\alpha(s). \tag{3.17}$$

This is quite familiar since we know that regular non-null Bertrand curves in $\mathbb{M}^3_q(c)$ are characterized by the formula: $\lambda\kappa + \mu\tau = 1$, where $\lambda \neq 0$ and μ are constants, and κ and τ stand for the curvature and torsion of the curve. Equation (3.17) is clearly a generalization of this classical formula.

3.2. Framed Bertrand curves in $\mathbb{M}^3_q(c)$ with respect to the Frenet-type frame

Definition 3.2. Suppose that $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve with the Frenet-type frame. Then $(\gamma, \omega_1, \omega_2)$ is called a *Bertrand curve* with respect to the Frenet-type frame if there exists another non-null framed curve $(\bar{\gamma}, \bar{\omega}_1, \bar{\omega}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ such that both curves $(\gamma, \omega_1, \omega_2)$ and $(\bar{\gamma}, \bar{\omega}_1, \bar{\omega}_2)$ have common generalized principal normal geodesics with respect to the Frenet-type frame at corresponding points. In this case, $(\bar{\gamma}, \bar{\omega}_1, \bar{\omega}_2)$ is said to be a *framed Bertrand mate (or conjugate)* of $(\gamma, \omega_1, \omega_2)$, and as well $(\gamma, \omega_1, \omega_2)$ and $(\bar{\gamma}, \bar{\omega}_1, \bar{\omega}_2)$ are called *a pair of framed Bertrand curves.*

For a pair of framed Bertrand curves $(\gamma, \omega_1, \omega_2)$ and $(\bar{\gamma}, \bar{\omega}_1, \bar{\omega}_2)$, we can write

$$\bar{\gamma}(s) = f(\varphi)\gamma(s) + g(\varphi)\omega_1(s), \tag{3.18}$$

where $f^2(\varphi) + \varepsilon_1 c g^2(\varphi) = 1$, and φ is a constant such that $g(\varphi) \neq 0$.

We will not give all of the results of Section 3.1 again for the Frenet-type frame along γ. But nevertheless we discuss some properties of these framed Bertrand curves. Actually a trick for carrying the results of Section 3.1 to this section is simply letting $\epsilon_i = \varepsilon_i$ $(i = 1, 2)$, $\ell_1 = p_1$, $\ell_2 = p_2$, $\ell_3 = 0$, $\bar{\ell}_1 = \bar{p}_1$, $\bar{\ell}_2 = \bar{p}_2$, $\bar{\ell}_3 = 0$ in all equations of Section 3.1. Note that what is meant here is not that the Frenet-type frame is a special case of the general moving frame for $\ell_3(s) = 0$. But using this trick will save us making many calculations. Therefore letting $\ell_3 = \bar{\ell}_3 = 0$ in the equalities in Proposition 3.2 we have $\theta'(s) = 0$ and so $\theta(s) = \theta$ is a constant function. Thus we have

$$\begin{aligned}
\bar{\omega}_1(s) &= -\varepsilon_1 c g(\varphi)\gamma(s) + f(\varphi)\omega_1(s),\\
\bar{\omega}_2(s) &= \xi(\theta)\omega_2(s) + \eta(\theta)\mu(s),\\
\bar{\mu}(s) &= \varepsilon_2\eta(\theta)\omega_2(s) - \varepsilon_3\xi(\theta)\mu(s),\\
\bar{\alpha}(s) &= \bar{\varepsilon}_3\varepsilon_2 g(\varphi)\eta(\theta)p_1(s) - \bar{\varepsilon}_3\left(f(\varphi)\alpha(s) + \varepsilon_3 g(\varphi)p_2(s)\right)\xi(\theta),\\
\bar{p}_1(s) &= f(\varphi)\xi(\theta)p_1(s) + \left(-\varepsilon_1\varepsilon_3 c g(\varphi)\alpha(s) + f(\varphi)p_2(s)\right)\eta(\theta),\\
\bar{p}_2(s) &= \varepsilon_2 f(\varphi)\eta(\theta)p_1(s) + \left(\varepsilon_1 c g(\varphi)\alpha(s) - \varepsilon_3 f(\varphi)p_2(s)\right)\xi(\theta),
\end{aligned} \tag{3.19}$$

where $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3$ are the causal characters of $\bar{\omega}_1, \bar{\omega}_2$, and $\bar{\mu}$, and θ is a constant. Moreover, the functions ξ and η are given by $(\xi(u), \eta(u)) = (\cos u, \sin u)$ if $\bar{\varepsilon}_2 = \varepsilon_2 = \varepsilon_3$, by $(\xi(u), \eta(u)) = (\cosh u, \sinh u)$ if $\bar{\varepsilon}_2 = \varepsilon_2 = -\varepsilon_3$, and by $(\xi(u), \eta(u)) = (\sinh u, \cosh u)$ if $\bar{\varepsilon}_2 = -\varepsilon_2 = \varepsilon_3$. Thus, these two functions satisfy the relation $\bar{\varepsilon}_2 = \varepsilon_2\xi^2 + \varepsilon_3\eta^2$.

Propositions 3.3, 3.4, and 3.5 can also be obtained in a similar way. Similar to Theorem 3.1 we can derive the equation that characterizes framed Bertrand curves with respect to the Frenet-type frame. Suppose that $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve with the Frenet-type curvature (α, p_1, p_2), where $p_1(s) \neq 0$ for all $s \in I$. Then γ is a framed Bertrand curve with respect to the Frenet-type frame if and only if there exist two constants φ and θ with $g(\varphi) \neq 0$ such that for all $s \in I$

$$\varepsilon_3 g(\varphi)\xi(\theta)p_1(s) + \left(f(\varphi)\alpha(s) + \epsilon_3 g(\varphi)p_2(s)\right)\eta(\theta) = 0. \tag{3.20}$$

Letting $\lambda = g/f$ and $\rho = \lambda\xi(\theta)/\eta(\theta)$ with $\eta(\theta) \neq 0$, this equation will directly lead to the following theorem.

Theorem 3.2. *$(\gamma, \omega_1, \omega_2)$ is a framed Bertrand curve with respect to the Frenet-type frame if and only if there exist two constants $\lambda \neq 0$ and ρ such that*

$$\lambda p_2(s) + \rho p_1(s) = -\varepsilon_3\alpha(s). \tag{3.21}$$

4. Non-null framed Mannheim curves in $\mathbb{M}^3_q(c)$

Similar to the previous section, we, in this section, define two-types of Mannheim curves of non-null framed curves in $\mathbb{M}^3_q(c)$ with respect to two different frames: the general moving frame $\{v_1, v_2, \mu\}$ and the Frenet-type frame $\{\omega_1, \omega_2, \mu\}$.

4.1. *Framed Mannheim curves in $\mathbb{M}^3_q(c)$ with respect to the general moving frame $\{v_1, v_2, \mu\}$*

Definition 4.1. Let $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ be a non-null framed curve. Then (γ, v_1, v_2) is called a *Mannheim curve* if there exists another non-null framed curve $(\hat{\gamma}, \hat{v}_1, \hat{v}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ such that generalized principal normal geodesics of (γ, v_1, v_2) and generalized binormal geodesics of $(\hat{\gamma}, \hat{v}_1, \hat{v}_2)$ at corresponding points are coincident. In this case, $(\hat{\gamma}, \hat{v}_1, \hat{v}_2)$ is called a *Mannheim mate* of (γ, v_1, v_2), and (γ, v_1, v_2) and $(\hat{\gamma}, \hat{v}_1, \hat{v}_2)$ are called *a pair of framed Mannheim curves*.

For a pair of framed Mannheim curves (γ, v_1, v_2) and $(\hat{\gamma}, \hat{v}_1, \hat{v}_2)$, there exists a differentiable function $\phi(s)$ such that

$$\hat{\gamma}(s) = f(\phi(s))\gamma(s) + g(\phi(s))v_1(s). \tag{4.1}$$

Since $\hat{\gamma} \in \mathbb{M}^3_q(c)$, we have $f^2(\phi(s)) + \epsilon_1 c g^2(\phi(s)) = 1$. We will assume that $g(\phi(s)) \not\equiv 0$ that is $\hat{\gamma} \neq \pm\gamma$. Definition 4.1 states that principal normal geodesics of γ coincide with binormal geodesics of $\hat{\gamma}$ at corresponding points. Then

$$\left.\frac{d}{dt}\right|_{t=\phi(s)} \delta^{\gamma, v_1}_s(t) = \hat{v}_2(s).$$

We know that $f'(t) = -\epsilon_1 c g(t)$ and $g'(t) = f(t)$. Then we get

$$\hat{v}_2(s) = -\epsilon_1 c g(\phi(s))\gamma(s) + f(\phi(s))v_1(s).$$

Similar to Proposition 3.1, the proof of the following proposition follows from this equation.

Proposition 4.1. *Let γ be a framed Mannheim curve, and let $\hat{\gamma}$ be a Mannheim mate of this curve given by (4.1). The function $\phi(s)$ is constant.*

Proposition 4.2. *Let $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ be a non-null Mannheim curve with curvature $(\alpha, \ell_1, \ell_2, \ell_3)$. Then a Mannheim mate $(\hat{\gamma}, \hat{v}_1, \hat{v}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ of (γ, v_1, v_2) is also a non-null framed curve with curvature $(\hat{\alpha}, \hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3)$, where*

$$\begin{aligned}
\hat{v}_1(s) &= \xi(\beta(s))v_2(s) + \eta(\beta(s))\mu(s),\\
\hat{v}_2(s) &= -\epsilon_1 c g(\phi)\gamma(s) + f(\phi)v_1(s),\\
\hat{\mu}(s) &= -\epsilon_2\eta(\beta(s))v_2(s) + \epsilon_3\xi(\beta(s))\mu(s),\\
\hat{\alpha}(s) &= -\hat{\epsilon}_3\epsilon_2 g(\phi)\eta(\beta(s))\ell_1(s) + \hat{\epsilon}_3\left(f(\phi)\alpha(s) + \epsilon_3 g(\phi)\ell_2(s)\right)\xi(\beta(s)),\\
\hat{\ell}_1(s) &= -f(\phi)\xi(\beta(s))\ell_1(s) + \left(\epsilon_1\epsilon_3 c g(\phi)\alpha(s) - f(\phi)\ell_2(s)\right)\eta(\beta(s)),\\
\hat{\ell}_2(s) &= \epsilon_3\hat{\epsilon}_3\beta'(s) + \hat{\epsilon}_3\ell_3(s),\\
\hat{\ell}_3(s) &= -\epsilon_2 f(\phi)\eta(\beta(s))\ell_1(s) - \left(\epsilon_1 c g(\phi)\alpha(s) - \epsilon_3 f(\phi)\ell_2(s)\right)\xi(\beta(s)),
\end{aligned}$$

where $\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3$ are the causal characters of $\hat{v}_1, \hat{v}_2$, and $\hat{\mu}$, and $\beta : I \to \mathbb{R}$ is a smooth function. Moreover, the functions ξ and η are given by $(\xi(u), \eta(u)) = (\cos u, \sin u)$ if $\hat{\epsilon}_1 = \epsilon_2 = \epsilon_3$, by $(\xi(u), \eta(u)) = (\cosh u, \sinh u)$ if $\hat{\epsilon}_1 = \epsilon_2 = -\epsilon_3$, and by $(\xi(u), \eta(u)) = (\sinh u, \cosh u)$ if $\hat{\epsilon}_1 = -\epsilon_2 = \epsilon_3$. Thus, these two functions satisfy the relation $\hat{\epsilon}_1 = \epsilon_2\xi^2 + \epsilon_3\eta^2$.

From (4.1) and the equations given in Proposition 4.2, it is easy to show that

$$\begin{aligned}
\gamma(s) &= f(\phi)\hat{\gamma}(s) - g(\phi)\hat{v}_2(s),\\
v_1(s) &= \epsilon_1 c g(\phi)\hat{\gamma}(s) + f(\phi)\hat{v}_2(s),\\
v_2(s) &= \hat{\epsilon}_3(\epsilon_3\xi(\beta(s))\hat{v}_1(s) - \eta(\beta(s))\hat{\mu}(s)),\\
\mu(s) &= \hat{\epsilon}_3(\epsilon_2\eta(\beta(s))\hat{v}_1(s) + \xi(\beta(s))\hat{\mu}(s)).
\end{aligned}$$

We will not provide proofs for the following results since they follow similar to the results given in Section 3.

Proposition 4.3. *Let γ and $\hat{\gamma}$ be a pair of non-null framed Mannheim curves in $\mathbb{M}^3_q(c)$. Then there exist a constant ϕ and a smooth function $\beta : I \to \mathbb{R}$ such that*

(a) $\left(f(\phi)\alpha(s) + \epsilon_3 g(\phi)\ell_2(s)\right)\eta(\beta(s)) = -\epsilon_3 g(\phi)\xi(\beta(s))\ell_1(s)$,

(b) $\epsilon_2\left(f(\phi)\hat{\alpha}(s) - \hat{\epsilon}_3 g(\phi)\hat{\ell}_3(s)\right)\eta(\beta(s)) = \hat{\epsilon}_1 g(\phi)\xi(\beta(s))\hat{\ell}_1(s)$,

(c) $\left(f(s)\alpha(s) + \epsilon_3 g(\phi)\ell_2(s)\right)\left(f(s)\hat{\alpha}(s) - \hat{\epsilon}_3 g(\phi)\hat{\ell}_3(s)\right) = \epsilon_3\hat{\epsilon}_3\alpha(s)\hat{\alpha}(s)\xi^2(\beta(s))$,

(d) $-\epsilon_3\alpha(s)\hat{\alpha}(s)\eta^2(\beta(s)) = \ell_1(s)\hat{\ell}_1(s)g^2(\phi)$.

Proposition 4.4. *Let γ and $\hat{\gamma}$ be a pair of non-null framed Mannheim curves in $\mathbb{M}^3_q(c)$. Then there exist a constant ϕ and a smooth function $\beta : I \to \mathbb{R}$ such that*

(a) $\ell_1(s)f(\phi) = -\epsilon_2\hat{\epsilon}_1\hat{\ell}_1(s)\xi(\beta(s)) - \hat{\epsilon}_3\hat{\ell}_3(s)\eta(\beta(s))$ *and* $\hat{\ell}_1(s)f(\phi) = -\ell_1(s)\xi(\beta(s)) - \ell_2(s)\eta(\beta(s))$,

(b) $-\epsilon_1 c\alpha(s)g(\phi) + \epsilon_3\ell_2(s)f(\phi) = -\hat{\epsilon}_1\hat{\ell}_1(s)\eta(\beta(s)) + \epsilon_3\hat{\epsilon}_3\hat{\ell}_3(s)\xi(\beta(s))$ *and*
$-\epsilon_2\ell_1(s)\eta(\beta(s)) + \epsilon_3\ell_2(s)\xi(\beta(s)) = \epsilon_1\hat{\epsilon}_3 c\hat{\alpha}(s)g(\phi) + \hat{\ell}_3(s)f(\phi)$,

Proof. The proofs of the first relations in (a) and (b) directly follow from the equality of the left-hand sides of the following relations:

$$\begin{aligned}
\hat{v}_2'(s) =& -\hat{\epsilon}_1\hat{\ell}_1(s)\hat{v}_1(s) + \hat{\epsilon}_3\hat{\ell}_3(s)\hat{\mu}(s)\\
=& \left(-\hat{\epsilon}_1\hat{\ell}_1(s)\xi(\beta(s)) - \epsilon_2\hat{\epsilon}_3\hat{\ell}_3(s)\eta(\beta(s))\right)v_2(s)\\
&+ \left(-\hat{\epsilon}_1\hat{\ell}_1(s)\eta(\beta(s)) + \epsilon_3\hat{\epsilon}_3\hat{\ell}_3(s)\xi(\beta(s))\right)\mu(s),
\end{aligned}$$

and

$$\hat{v}_2'(s) = \epsilon_2\ell_1(s)f(\phi)v_2(s) + \left(-\epsilon_1 c\alpha(s)g(\phi) + \epsilon_3\ell_2(s)f(\phi)\right)\mu(s).$$

On the other hand, the proofs of the second relations in (a) and (b) follow from the equality of the left-hand sides of the following relations:

$$\begin{aligned} v_1'(s) =& \epsilon_2\ell_1(s)v_2(s) + \epsilon_3\ell_2(s)\mu(s) \\ =& \hat{\epsilon}_1\left(\ell_1(s)\xi(\beta(s)) + \ell_2(s)\eta(\beta(s))\right)\hat{v}_1(s) \\ &+ \hat{\epsilon}_3\left(-\epsilon_2\ell_1(s)\eta(\beta(s)) + \epsilon_3\hat{\ell}_2(s)\xi(\beta(s))\right)\hat{\mu}(s), \end{aligned}$$

and

$$v_1'(s) = -\hat{\epsilon}_1\hat{\ell}_1(s)f(\phi)\hat{v}_1(s) + \left(\epsilon_1 c\hat{\alpha}(s)g(\phi) + \hat{\epsilon}_3\hat{\ell}_3(s)f(\phi)\right)\hat{\mu}(s).$$

□

Proposition 4.5. (i) *A non-null framed curve $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ with curvature $(\alpha, 0, \ell_2, \ell_3)$ is a framed Mannheim curve, and this curve has infinite Mannheim conjugates $\hat{\gamma}$ such that $\hat{\ell}_1(s) = 0$ for all $s \in I$.*

(ii) *Let γ and $\hat{\gamma}$ be a pair of non-null framed Mannheim curves with curvatures $(\alpha, \ell_1, \ell_2, \ell_3)$ and $(\hat{\alpha}, \hat{\ell}_1, \hat{\ell}_2, \hat{\ell}_3)$. If $\hat{\ell}_1(s) = 0$ for all $s \in I$, then $\ell_1(s) = 0$ for all $s \in I$.*

The main theorem of this section that characterizes framed Mannheim curves is given below.

Theorem 4.1. *Suppose that $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve with curvature $(\alpha, \ell_1, \ell_2, \ell_3)$, where $\ell_1(s) \neq 0$ for all $s \in I$. Then γ is a framed Mannheim curve if and only if for a real constant ϕ such that $g(\phi) \neq 0$, for a smooth function $\beta : I \to \mathbb{R}$, and for all $s \in I$*

$$g(\phi)\xi(\beta(s))\ell_1(s) + \epsilon_3\left(f(\phi)\alpha(s) + \epsilon_3 g(\phi)\ell_2(s)\right)\eta(\beta(s)) = 0, \tag{4.2}$$

where f and g satisfy condition C1, and ξ and η satisfy condition C2 given in Section 3.

Next we give a theorem that relates framed Bertrand curves and framed Mannheim curves. The proof of this theorem is clear from Theorem 3.1 and Theorem 4.1.

Theorem 4.2. *Let $(\gamma, v_1, v_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ be a non-null framed curve. Then (γ, v_1, v_2) is a framed Bertrand curve if and only if (γ, v_1, v_2) is a framed Mannheim curve.*

Remark 4.1. The previous theorem may seem the reader a bit odd since this theorem clearly states that every framed Bertrand curve is a framed Mannheim curve and vice versa. However, we may see this fact with the following reasoning. Consider a framed Bertrand curve (γ, v_1, v_2) and its framed Bertrand mate $(\bar{\gamma}, \bar{v}_1, \bar{v}_2)$. Then the generalized principal normal geodesics of these two framed curves are coincident. We can easily show that for $\tilde{v}_1 = \bar{v}_2$ and $\tilde{v}_2 = \bar{v}_1$, $(\bar{\gamma}, \tilde{v}_1, \tilde{v}_2)$ is also a framed curve. But in this case $(\bar{\gamma}, \tilde{v}_1, \tilde{v}_2)$ is a framed Mannheim mate of (γ, v_1, v_2) since the generalized principal normal geodesics of (γ, v_1, v_2) are coincident with the generalized binormal geodesics of $(\bar{\gamma}, \tilde{v}_1, \tilde{v}_2)$. The converse of this statement can similarly be proved.

4.2. *Framed Mannheim curves in $\mathbb{M}^3_q(c)$ with respect to the Frenet-type frame*

Definition 4.2. Let $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ be a non-null framed curve with the Frenet-type frame. Then $(\gamma, \omega_1, \omega_2)$ is called a *Mannheim curve* with respect to the Frenet-type frame if there exists another non-null framed curve $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ such that generalized principal normal geodesics with respect to the Frenet-type frame of $(\gamma, \omega_1, v\omega_2)$ and generalized binormal geodesics with respect to the Frenet-type frame of $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2)$ at corresponding points are coincident. In this case, $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2)$ is called a *Mannheim mate* of $(\gamma, \omega_1, \omega_2)$, and $(\gamma, \omega_1, \omega_2)$ and $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2)$ are called *a pair of framed Mannheim curves*.

For a pair of framed Mannheim curves $(\gamma, \omega_1, \omega_2)$ and $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2)$, we have

$$\hat{\gamma}(s) = f(\phi)\gamma(s) + g(\phi)\omega_1(s), \tag{4.3}$$

where $f^2(\phi) + \varepsilon_1 c g^2(\phi) = 1$, and ϕ is a constant such that $g(\phi) \neq 0$.

A similar trick to that in Section 3.2 can be also considered for this section. That is, setting $\epsilon_i = \varepsilon_i$ $(i = 1, 2)$, $\ell_1 = p_1$, $\ell_2 = p_2$, $\ell_3 = 0$, $\bar{\ell}_1 = \bar{p}_1$, $\bar{\ell}_2 = \bar{p}_2$, $\bar{\ell}_3 = 0$ in all equations of Section 4.1, we can obtain similar results to those given in Section 4.1. However, we still want to discuss some of these results in detail.

Let $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ be a non-null Mannheim curve with the Frenet-type curvature (α, p_1, p_2). Then a framed Mannheim mate $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ of $(\gamma, \omega_1, \omega_2)$ is also a non-null framed curve with the following Frenet-type frame and Frenet-type curvature

$$\begin{aligned}
\hat{\omega}_1(s) &= \xi(\beta(s))\omega_2(s) + \eta(\beta(s))\mu(s), \\
\hat{\omega}_2(s) &= -\varepsilon_1 c g(\phi)\gamma(s) + f(\phi)\omega_1(s), \\
\hat{\mu}(s) &= -\varepsilon_2\eta(\beta(s))\omega_2(s) + \varepsilon_3\xi(\beta(s))\mu(s), \\
\hat{\alpha}(s) &= -\hat{\varepsilon}_3\varepsilon_2 g(\phi)\eta(\beta(s))p_1(s) + \hat{\varepsilon}_3\left(f(\phi)\alpha(s) + \varepsilon_3 g(\phi)p_2(s)\right)\xi(\beta(s)), \\
\hat{p}_1(s) &= -f(\phi)\xi(\beta(s))p_1(s) + \left(\varepsilon_1\varepsilon_3 c g(\phi)\alpha(s) - f(\phi)p_2(s)\right)\eta(\beta(s)), \\
\hat{p}_2(s) &= \varepsilon_3\hat{\varepsilon}_3\beta'(s).
\end{aligned}$$

We also have the relation

$$-\varepsilon_2 f(\phi)\eta(\beta(s))p_1(s) - \left(\varepsilon_1 c g(\phi)\alpha(s) - \varepsilon_3 f(\phi)p_2(s)\right)\xi(\beta(s)) = 0. \tag{4.4}$$

We consider the following two cases.

1. $f(\phi) \neq 0$. We know that

$$\hat{\alpha}(s) = \hat{\varepsilon}_3 g(\phi)(-\varepsilon_2\eta(\beta(s))p_1(s) + \varepsilon_3\xi(\beta(s))p_2(s)) + \hat{\varepsilon}_3 f(\phi)\alpha(s)\xi(\beta(s)).$$

Introducing (4.4) into this equation yields

$$\hat{\alpha}(s) = \hat{\varepsilon}_3\alpha(s)\xi(\beta(s))/f(\phi). \tag{4.5}$$

Similarly considering (4.4) in $\hat{p}_1(s)$ given above yields

$$\hat{p}_1(s) = \varepsilon_3\alpha(s)\eta(\beta(s))/g(\phi), \tag{4.6}$$

because we already assumed that $g(\phi) \neq 0$. Notice that from (4.5) and (4.6), if γ has a singularity at s_0, then its framed Mannheim mate with respect to the Frenet-type frame too is singular at s_0, and also $\hat{p}_1(s_0) = 0$. Conversely if the framed Mannheim mate $\hat{\gamma}$ of γ has a singularity at s_0, then γ has a singularity at s_0 or $\xi(\beta(s_0)) = 0$.

2. $f(\phi) = 0$. In this case we may assume that $g(\phi) = 1$. From (4.4) we have $\xi(\beta(s)) = 0$ for all s, and therefore we may assume that $\eta(\beta(s)) = 1$. Then after some computation we find that $p_2(s) = 0$,

$$\begin{aligned}
&\hat{\gamma}(s) = \omega_1(s), \quad \hat{\omega}_1(s) = \mu(s), \quad \hat{\omega}_2(s) = \gamma(s), \quad \hat{\mu}(s) = -\varepsilon_2\omega_2(s), \\
&\hat{\alpha}(s) = -p_1(s), \quad \hat{p}_1(s) = -\varepsilon_3\alpha(s), \quad \hat{p}_2(s) = 0.
\end{aligned}$$

Propositions 4.3, 4.4, and 4.5 can also be obtained in a similar way. Similar to Theorem 4.1 we can derive the equation that characterizes framed Mannheim curves with respect to the Frenet-type frame. Suppose that $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_q(c) \times \Delta$ is a non-null framed curve with the Frenet-type curvature (α, p_1, p_2), where $p_1(s) \neq 0$ for all $s \in I$. Similar to (4.2) we also have

$$g(\phi)\xi(\beta(s))p_1(s) + \varepsilon_3\left(f(\phi)\alpha(s) + \varepsilon_3 g(\phi)p_2(s)\right)\eta(\beta(s)) = 0. \tag{4.7}$$

Using (4.4) and (4.7), and letting $\lambda = f(\phi)g(\phi)/(\varepsilon_1 c g^2(\phi) - f^2(\phi))$, we get the following theorem.

Theorem 4.3. *$(\gamma, \omega_1, \omega_2)$ is a framed Mannheim curve with respect to the Frenet-type frame if and only if either (i) $\varepsilon_1 c g^2(\phi) - f^2(\phi) \neq 0$ and there exists a constant λ such that*

$$\lambda(\varepsilon_2 p_1^2(s) + \varepsilon_3 p_2^2(s) - \varepsilon_1\varepsilon_3 c\alpha^2(s)) = \alpha(s)p_2(s),$$

or (ii) $\varepsilon_1 c g^2(\phi) - f^2(\phi) = 0$ and $\varepsilon_2 p_1^2(s) + \varepsilon_3 p_2^2(s) = \varepsilon_1\varepsilon_3 c\alpha^2(s)$.

Proof. The sufficiency part is easy enough. So we will do only the necessity part of the proof.

(i) Assume that $\varepsilon_1 c g^2(\phi) - f^2(\phi) \neq 0$ and there exists a constant λ such that

$$\lambda(\varepsilon_2 p_1^2(s) + \varepsilon_3 p_2^2(s) - \varepsilon_1\varepsilon_3 c\alpha^2(s)) = \alpha(s)p_2(s). \tag{4.8}$$

We need to find a Frenet-type framed curve $(\hat{\gamma}, \hat{\omega}_1, \hat{\omega}_2)$ such that γ and $\hat{\gamma}$ give a pair of framed Mannheim curves relative to the Frenet-type frame. Define

$$\hat{\gamma}(s) = f(\phi)\gamma(s) + g(\phi)\omega_1(s), \tag{4.9}$$

and let $\lambda = \frac{f(\phi)g(\phi)}{\varepsilon_1 cg^2(\phi) - f^2(\phi)}$. Differentiating (4.9) twice yields

$$\hat{\alpha}(s)\hat{\mu}(s) = \varepsilon_2 g(\phi)\omega_2(s) + (f(\phi)\alpha(s) + \varepsilon_3 g(\phi)p_2(s))\mu(s), \tag{4.10}$$

$$\begin{aligned}\hat{\alpha}'(s)\hat{\mu}(s) - \hat{\varepsilon}_3 c\hat{\alpha}^2(s)\hat{\gamma}(s) - \hat{\varepsilon}_1\hat{\alpha}\hat{p}_2(s)\hat{\omega}_1(s) &= \varepsilon_2 g(\phi)p_1'(s)\omega_2(s)\\ &+ (f(\phi)\alpha(s) + \varepsilon g(\phi)p_2(s))'\mu(s) - \varepsilon_3 c\alpha(s)(f(\phi)\alpha(s) + \varepsilon_3 g(\phi)p_2(s))\gamma(s)\\ &- \varepsilon_1\left(g(\phi)\big(\varepsilon_2 p_1^2(s) + \varepsilon_3 p_2^2(s)\big) + f(\phi)\alpha(s)p_2(s)\right)\omega_1(s).\end{aligned} \tag{4.11}$$

From the vector product of (4.9), (4.10), and (4.11), we find that

$$\begin{aligned}-\hat{\varepsilon}_1\hat{\alpha}^2\hat{p}_2\hat{\omega}_2 &= \varepsilon_1\varepsilon_2\varepsilon_3 g\big(p_1(f\alpha + \varepsilon_3 gp_2)' - p_1'(f\alpha + \varepsilon_3 gp_2)\big)(-\varepsilon_1 cg\gamma + f\omega_1)\\ &+ \varepsilon_1\varepsilon_2\left[fg\big(-(\varepsilon_2 p_1^2 + \varepsilon_3 p_2^2) + \varepsilon_1\varepsilon_3 c\alpha^2\big) + \alpha p_2(\varepsilon_1 cg^2 - f^2)\right]\big(\varepsilon_3(f\alpha + \varepsilon_3 gp_2)\omega_2 - gp_1\mu\big),\end{aligned}$$

where for simplicity we omitted the parameter s and the constant ϕ. Now introducing (4.8) in the last equation, we see that the coefficients of ω_2 and μ on the right-hand side of this equation vanish. Then we find that

$$\hat{\omega}_2 = -\varepsilon_1 cg\gamma + f\omega_1. \tag{4.12}$$

Using (4.9) and (4.12), we find that the generalized principal normal geodesic starting at a point $\hat{\gamma}(s_0)$ is

$$\delta(t) = f(t)\hat{\gamma}(s_0) + g(t)\hat{\omega}_1(s_0) = f(t+\varphi)\gamma(s_0) + g(t+\varphi)\omega_1(s_0),$$

which is just a reparametrization of the generalized binormal geodesic at $\gamma(s_0)$. Therefore $\hat{\gamma}$ is a framed Mannheim conjugate of γ with respect to the Frenet-type frame.

(ii) The proof of this case follows quite similar to the proof of (i). □

Remark 4.2. Notice that the formula in Theorem 4.3 looks quite familiar, and indeed it provides a generalization of the classical formulas: $\lambda(\kappa^2(s) + \tau^2(s)) = \kappa(s)$ that characterizes Mannheim curves in the Euclidean 3-space [29] and $\lambda(\varepsilon_1\kappa^2(s) + \varepsilon_3\tau^2(s) - \varepsilon_1\varepsilon_2 c) = \lambda\kappa(s)$ that characterizes Mannheim curves in non-flat 3-dimensional space forms [48].

5. Examples

Example 5.1. Take the smooth curve $\gamma : I \to \mathbb{M}_1^3(-1) = H_1^3 \subset \mathbb{R}_2^4$ defined by

$$\gamma(s) = \frac{2}{\sqrt{3}}\left(s\sin s + \cos s, -s\cos s + \sin s, s\sin(2s) + \frac{1}{2}\cos(2s), -\cos(2s) + \frac{1}{2}\sin(2s)\right).$$

The derivative of this equation is

$$\gamma'(s) = \frac{2s}{\sqrt{3}}\left(\cos s, \sin s, 2\cos(2s), 2\sin(2s)\right).$$

Then γ has a singularity at $s = 0$. Let $v_1 : I \to H_1^3$ and $v_2 : I \to S_2^3$

$$\begin{aligned}v_1(s) &= \frac{1}{\sqrt{3(5+4s^2)}}\big(4(\sin s - s\cos s), -4(\cos s + s\sin s), -2s\cos(2s) + \sin(2s),\\ &\qquad -\cos(2s) - 2s\sin(2s)\big),\\ v_2(s) &= \frac{1}{\sqrt{3(5+4s^2)}}\big(-\sin s - 4s(\cos s + s\sin s), \cos s + 4s(s\cos s - \sin s),\\ &\qquad -2s\cos(2s) - 4(1+s^2)\sin(2s), 4(1+s^2)\cos(2s) - 2s\sin(2s)\big).\end{aligned}$$

We have $\langle v_1, \gamma\rangle = 0$, $\langle v_2, \gamma\rangle = 0$, $\langle v_1, \gamma'\rangle = 0$, and $\langle v_2, \gamma'\rangle = 0$. Hence, $(\gamma, v_1, v_2) : I \to H^3_1 \times \Delta$ is a timelike framed curve in H^3_1. From $\gamma \times v_1 \times v_2$ we get

$$\mu(s) = -\frac{1}{\sqrt{3}}\left(\cos s, \sin s, 2\cos(2s), 2\sin(2s)\right) \in S^3_2.$$

The curvature of (γ, v_1, v_2) is given by $(\alpha, \ell_1, \ell_2, \ell_3)$ so that

$$\alpha(s) = -2s, \quad \ell_1(s) = \frac{-4s}{5+4s^2}, \quad \ell_2(s) = 0, \quad \ell_3(s) = \sqrt{5+4s^2}.$$

Notice that if γ is a framed Bertrand curve, then any framed Bertrand conjugate of γ will be in the form $\bar{\gamma} = \cos\varphi\gamma + \sin\varphi v_1$ since $\epsilon_1 c = 1$. Furthermore we will have $\bar{v}_2 = \cos(\theta(s))v_2 + \sin(\theta(s))\mu$. Thus, we see that for the smooth function $\rho(s) = -2s^2 - 5/2$ and for any $\lambda \in \mathbb{R}$, (3.17) is satisfied. This means that γ is a framed Bertrand curve having infinite framed Bertrand mates. From (3.2), all of these conjugate curves have singularity at $s = 0$. Moreover, from Theorem 4.2 γ is also a Mannheim curve. Using the hyperbolic Hopf map (2.7), the projections of the framed Bertrand conjugates of γ for $\varphi = \pi/4$ and for $\varphi = \pi/3$ on the hyperbolic space $H^2(1/2)$ are visualized in Figure 1. Notice that these framed Bertrand mates have also singularity at $s = 0$.

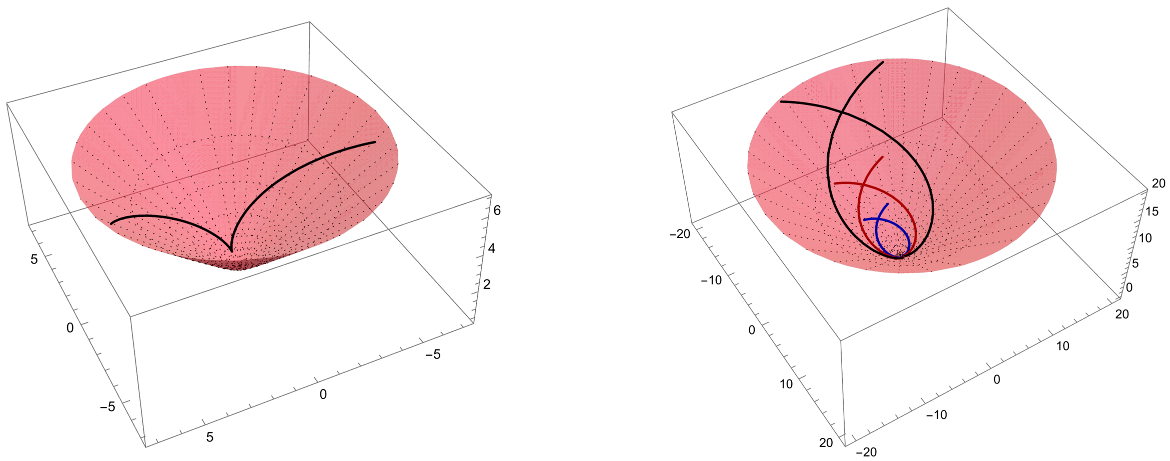

Figure 1. Left: the projection of γ on $H^2(1/2)$. Right: the projections on $H^2(1/2)$ of γ (black) and its framed Bertrand conjugates for $\varphi = \pi/4$ (red) and for $\varphi = \pi/3$ (blue).

Example 5.2. Consider the smooth curve $\gamma : [0, 2\pi) \to \mathbb{M}^3_0(-1) = H^3_0 \subset \mathbb{R}^4_1$ defined by

$$\gamma(s) = \left(\frac{\sqrt{17+7\cos(4s)}}{2\sqrt{2}}, \cos^3 s, \sin^3 s, \cos(2s)\right).$$

Differentiating this curve

$$\gamma'(s) = \sin s \cos s\left(-\frac{28\cos(2s)}{\sqrt{34+14\cos(4s)}}, -3\cos s, 3\sin s, -4\right).$$

Then γ has singularities at $s = 0, \pi/2, \pi, 3\pi/2$. Define $v_1 : I \to S^3_1$ and $v_2 : I \to S^3_1$ by

$$\begin{aligned} v_1(s) = \frac{40\sqrt{2}}{\sqrt{4143 - 836\cos(4s) + 21\cos(8s)}}&\Bigg(\frac{\cos(2s)\sqrt{17+7\cos(4s)}}{10\sqrt{2}}, \cos s, -\sin s, \\ &\frac{1}{40}(-23+7\cos(4s))\Bigg), \\ v_2(s) = \frac{\sqrt{229 - 21\cos(4s)}}{\sqrt{4143 - 836\cos(4s) + 21\cos(8s)}}&\Bigg(\sin s\cos s\sqrt{34+14\cos(4s)}, \\ &\frac{1}{4}(18\sin s + 3\sin(3s) + \sin(5s)), \frac{1}{8}(18\cos s - 3\cos(3s) + \cos(5s)), \sin(4s)\Bigg). \end{aligned}$$

Then we have that $\langle v_1,\gamma\rangle=0$, $\langle v_2,\gamma\rangle=0$, $\langle v_1,\gamma'\rangle=0$, and $\langle v_2,\gamma'\rangle=0$. Hence, $(\gamma,v_1,v_2):[0,2\pi)\to H^3_0\times\Delta$ is a spacelike framed curve. From the triple vector product $\gamma\times v_1\times v_2$

$$\mu(s)=\frac{\sqrt{17+7\cos(4s)}}{\sqrt{229-21\cos(4s)}}\left(-\frac{28\cos(2s)}{\sqrt{34+14\cos(4s)}},-3\cos s,3\sin s,-4\right)\in S^3_1.$$

The curvature of (γ,v_1,v_2) is given by $(\alpha,\ell_1,\ell_2,\ell_3)$, where

$$\begin{aligned}
\alpha(s)&=\frac{\sin(2s)\sqrt{229-21\cos(4s)}}{2\sqrt{17+7\cos(4s)}},\\
\ell_1(s)&=\frac{-160\sqrt{458-42\cos(4s)}}{4143-836\cos(4s)+21\cos(8s)},\\
\ell_2(s)&=\frac{280\sqrt{2}(8265\sin(4s)-836\sin(8s)+21\sin(12s))}{\sqrt{229-21\cos(4s)}\sqrt{17+7\cos(4s)}(4143-836\cos(4s)+21\cos(8s))^{3/2}},\\
\ell_3(s)&=\frac{(-229+21\cos(4s))(5230+14395\cos(4s)-1214\cos(8s)+21\cos(12s))}{4\sqrt{17+7\cos(4s)}(4143-836\cos(4s)+21\cos(8s))^{3/2}}.
\end{aligned}$$

We have $\epsilon_1 c=-1$. Hence if γ is a framed Bertrand curve, then any framed Bertrand conjugate of γ will be in the form $\bar\gamma=\cosh\varphi\gamma+\sinh\varphi v_1$. Then we see that (3.17) is satisfied for the constant $\lambda=3/5$ and the smooth function

$$\begin{aligned}
\rho(s)=&\frac{(1344\sqrt{2}\cos(2s))(19\sin(2s)-\sin(6s))}{320\sqrt{34+14\cos(4s)}\sqrt{4143-836\cos(4s)+21\cos(8s)}}\\
&+\frac{(229-21\cos(4s))(19\sin(2s)-\sin(6s))}{320\sqrt{34+14\cos(4s)}}.
\end{aligned}$$

Therefore γ is a framed Bertrand curve and $\bar\gamma=(5/4)\gamma+(3/4)v_1$. Notice that γ has singularities at $s_0=0,\pi/2,\pi,3\pi/2$, however $\bar\gamma$ is not singular at these points. This can be easily seen: if $\bar\gamma$ was singular at these points, then (3.2), or equivalently, the equation $\lambda\ell_1(s_0)=\rho(s_0)\ell_2(s_0)$ would be satisfied. Eventhough $\rho(s)$ vanishes at all of these points, $\ell_1(s)\neq 0$. Therefore $\bar\gamma$ is regular at these points. Moreover, from Theorem 4.2 γ is also a Mannheim curve. It is easy to show that we can rewrite the curve γ in the form $\gamma(s)=\exp_p(q(s)V(s))$, where $p=(1,0,0,0)$,

$$q(s)=\operatorname{arccosh}\left(\frac{\sqrt{17+7\cos(4s)}}{2\sqrt{2}}\right),\quad V(s)=\frac{1}{\sinh(q(s))}\big(0,\cos^3 s,\sin^3(s),\cos(2s)\big).$$

Here $\sinh(q(s))=\sqrt{18+14\cos(4s)}/4$. The curve $V(s)$ clearly lies on $S^2(1)\subset T_p\mathbb{H}^3_0(-1)\subset\mathbb{R}^4_1$. This is the spherical projection of γ which is visualized in Figure 2(left). It is also possible to write $\bar\gamma(s)$ in the form $\bar\gamma(s)=\exp_p\left(\bar q(s)\bar V(s)\right)$, where $p=(1,0,0,0)$,

$$\begin{aligned}
\bar q(s)&=\operatorname{arccosh}\left(\frac{\sqrt{17+7\cos(4s)}}{16}\left(5\sqrt{2}+\frac{48\cos(2s)}{\sqrt{4143-836\cos(4s)+21\cos(8s)}}\right)\right),\\
\bar V(s)&=\frac{1}{\sinh(\bar q(s))}\left(0,\frac{5\cos^3(s)}{4}+\frac{30\sqrt{2}\cos(s)}{\sqrt{4143-836\cos(4s)+21\cos(8s)}},\right.\\
&\qquad\frac{5\sin^3(s)}{4}-\frac{30\sqrt{2}\sin s}{\sqrt{4143-836\cos(4s)+21\cos(8s)}},\\
&\qquad\left.\frac{5\cos(2s)}{4}-\frac{3\sqrt{2}(23-7\cos(4s))}{4\sqrt{4143-836\cos(4s)+21\cos(8s)}}\right).
\end{aligned}$$

Notice that $\bar V(s)$ is a curve lying on $S^2(1)\subset T_p\mathbb{H}^3_0(-1)\subset\mathbb{R}^4_1$. This is the spherical projection of $\bar\gamma$ which is visualized in Figure 2(right).

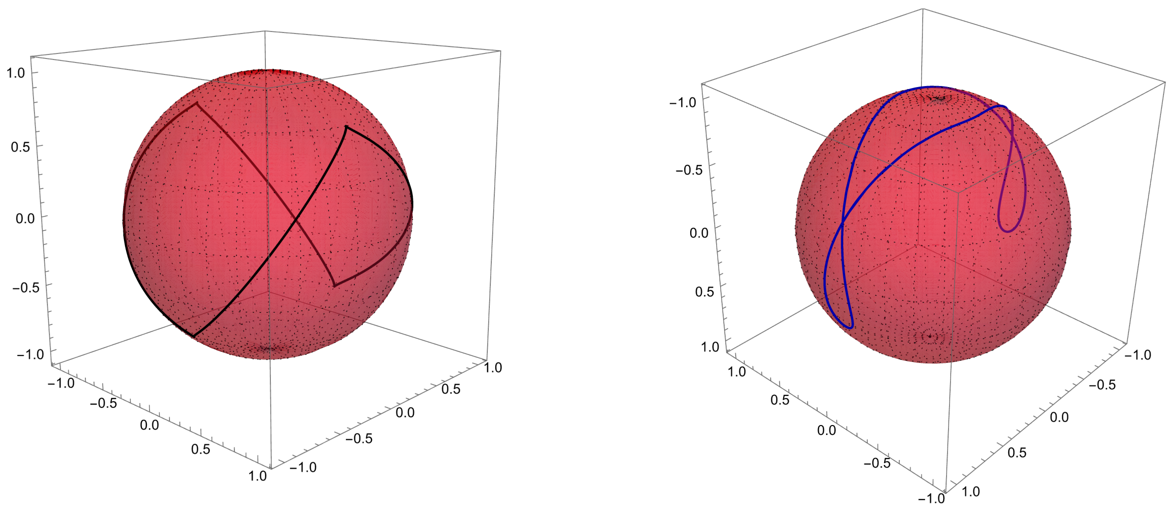

Figure 2. Left: the spherical projection of γ on $S^2(1)$. Right: the spherical projection of the framed Bertrand curve $\bar{\gamma}$ for $\varphi = 3/5$.

Example 5.3. We know that given the curvature (α, p_1, p_2), a framed curve with a Frenet-type frame can be determined uniquely up to rigid motion. We will use this fact in this example to construct a framed Mannheim curve with respect to the Frenet-type frame in the de Sitter 3-space. Suppose that a framed curve $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_1(1) \times \Delta$ such that $\varepsilon_1 = \varepsilon_2 = 1$ has the curvature $(s, s\sinh s, s\cosh s)$. We can choose $f(t) = \cos t$, $g(t) = \sin t$, and $\phi = \pi/4$. Then it is easy to see that Theorem 4.3(ii) is satisfied. That is, γ is a framed Mannheim curve with respect to the Frenet-type frame. The Mannheim mate $\hat{\gamma}$ of γ is given by

$$\hat{\gamma}(s) = (\gamma(s) + \omega_1(s))/\sqrt{2}.$$

Notice that equations (4.4) and (4.7) are satisfied for $\beta(s) = -s/2$, $\eta(u) = \sinh u$, and $\xi(u) = \cosh u$. Hence $\hat{\epsilon}_1 = -1$. Using these relations, we find that

$$\hat{\alpha}(s) = -\sqrt{2}s\sinh(s/2), \quad \hat{p}_1(s) = -\sqrt{2}s\cosh(s/2), \quad \hat{p}_2(s) = 1/2.$$

From the uniqueness of a framed curve whose curvature is given, we can construct the framed Mannheim curve γ and its Mannheim mate $\hat{\gamma}$ by using numerical methods. We can get the projections of these curves on the de Sitter 2-space by choosing $p = (0, 0, 1, 0)$ and by following a similar procedure to Example 5.2. We visualize these projections in Figure 3.

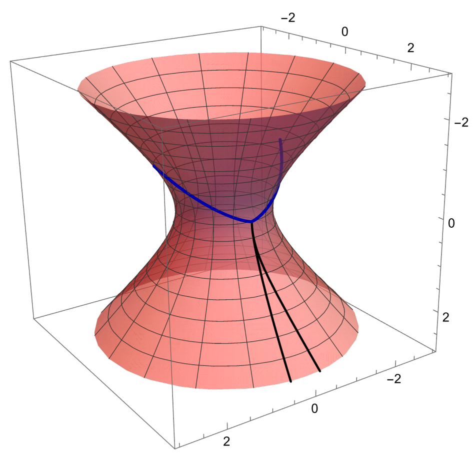

Figure 3. The projections on $S^2_1(1)$ of γ (black) and its framed Mannheim mate for $\phi = \pi/4$ (blue).

Example 5.4. Suppose that a framed curve $(\gamma, \omega_1, \omega_2) : I \to \mathbb{M}^3_0(1) \times \Delta$ has the curvature $(\alpha(s), p_1(s), p_2(s)) = (-\sin s, \cos s, \sin s)$. Then we see that Theorem 3.2 is satisfied for $\lambda = 1$ and $\rho = 0$. Hence γ is a framed Bertrand curve with respect to the Frenet-type frame. We have $\varphi = \pi/4$, $f(t) = \cos t$, $g(t) = \sin t$, $\theta = \pi/2$, $\xi(u) = \cos u$, and $\eta(u) = \sin u$. The Bertrand mate $\bar{\gamma}$ of γ is given by $\bar{\gamma}(s) = (\gamma(s) + \omega_1(s))/\sqrt{2}$. From the relations in (3.19), we find that

$$\bar{\alpha}(s) = (\cos s)/\sqrt{2}, \quad \bar{p}_1(s) = \sqrt{2}\sin s, \quad \bar{p}_2(s) = (\cos s)/\sqrt{2}.$$

Similar to Example 5.3 we can construct the framed Bertrand curve γ and its Bertrand mate $\bar{\gamma}$ by using numerical methods. We can get the projections of these curves on the 2-sphere $\mathbb{S}^2(1/2)$ by using the Hopf map given in (2.6). We visualize these projections in Figure 4.

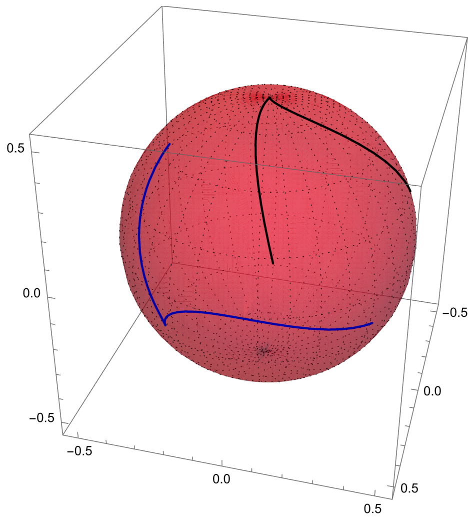

Figure 4. The projections on $S^2(1)$ of γ (black) and its framed Bertrand mate for $\phi = \pi/4$ (blue).

6. Conclusions

We investigated geometric properties of Bertrand and Mannheim curves of non-null framed curves with respect to the general moving frame and the Frenet-type frame in non-flat three-dimensional Riemannian and Lorentzian space forms. We showed that Bertrand and Mannheim curves of framed curves are also framed curves, and we obtained their curvatures in terms of the curvatures of the original curve. Then we provided some results involving the curvatures of a framed curve and its Bertrand or Mannheim mate. We also mentioned singularities of these framed curves and presented some relations of the curvatures of both the framed curve and its Bertrand or Mannheim mate at singular points. We gave characterizations for a non-null framed curve to be a framed Bertrand curve or a framed Mannheim curve. We concluded that framed Bertrand curves and framed Mannheim curves with respect to the general moving frame in non-flat three-dimensional space forms are equivalent. We provided important characterizations for framed Bertrand curves and framed Mannheim curves with respect to the Frenet-type frame, and we showed that these characterizations are generalizations of the classical characterizations for regular Bertrand and Mannheim curves. Finally we provided several examples of these curves and visualized them by using the Hopf map, the hyperbolic Hopf map, and the spherical projection.

There are several fruitful research directions we could pursue in the future. One possible direction is to consider a similar problem for another kind of well-known curves, rectifying curves. A rectifying curve is defined by the property that its position vector lies always in the rectifying plane of the curve with respect to the Frenet frame [6]. These regular rectifying curves have been generalized to rectifying curves with singularities [46]. Regular rectifying curves have also been considered in the 3-sphere [32] and the hyperbolic 3-space [33], but in these papers the rectifying curves are assumed to be regular. This suggests that we could also generalize these rectifying curves to rectifying curves with singularities. Therefore next we aim to investigate framed rectifying curves in non-flat three-dimensional space forms.

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Competing interests

The authors declare that they have no competing interests.

References

[1] Altın Erdem, H., İlarslan, K.: *Spacelike Bertrand curves in Minkowski 3-space revisited*. An. Şt. Univ. Ovidius Constanta **31** (3), 87-109 (2023).
[2] Balgetir, H., Bektaş, M., Inoguchi, J.: *Null Bertrand curves in Minkowski 3-space and their characterizations*. Note Mat. **23** (1), 7-13 (2004/05).
[3] Benyounes, M., Loubeau, E., Nishikawa, S.: *Generalized Cheeger-Gromoll metrics and the Hopf map*. Diff. Geom. Appl. **39**, 187-213 (2011).
[4] Bertrand, J.: *Mémoire sur la théorie des courbes é double courbure*. Comptes Rendus **36** (1850); Journal de Mathématiques Pures et Appliquées **15**, 332-350 (1850).
[5] Camcı, Ç., Uçum, A., İlarslan, K.: *A new approach to Bertrand curves in Euclidean 3-space*. J. Geom **111**, 49, (2020).
[6] Chen, B. Y.: *When does the position vector of a space curve always lie in its rectifying plane?*. Amer. Math. Monthly **110**, 147-152 (2003).
[7] Chen, B. Y., Dillen, F.: *Rectifying curves as centrodes and extremal curves*. Bull. Inst. Math. Acad. Sin. **33**, 77–90 (2005).
[8] Chen, L., Takahashi, M.: *Dualities and evolutes of fronts in hyperbolic and de Sitter space*. J. Math. Anal. Appl. **437**, 133-159 (2016).
[9] Cheng, Y.-M., Lin, C.-C.: *On the generalized Bertrand curves in Euclidean N-spaces*. Note Mat. **29** (2), 33-39 (2009).
[10] Choi, J. H., Kang, T. H., Kim, Y. H.: *Bertrand curves in 3-dimensional space forms*. Appl. Math. Comput. **219**, 1040–1046 (2012).
[11] Choi, J. H., Kang, T. H., Kim, Y. H.: *Mannheim curves in 3-dimensional space forms*. Bull. Korean Math. Soc. **50**, 1099–1108 (2013).
[12] Ekmekci, N., Ilarslan, K.: *On Bertrand curves and their characterization*. Differ. Geom. Dyn. Syst. **3** (2), 17-24 (2001).
[13] Ersoy, S., Tosun, M., Matsuda, H.: *Generalized Mannheim curves in Minkowski space-time E_1^4*. Hokkaido Math. J. **41** (3), 441-461 (2012).
[14] Fukunaga, T., Takahashi, M.: *Existence and Uniqueness for Legendre Curves*. J. Geom. **104**, 297-307 (2013).
[15] Grbović, M., İlarslan, K., Nešović, E.: *On null and pseudo null Mannheim curves in Minkowski 3-space*. J. Geom. **105**, 177–183 (2014).
[16] Gök, İ, Okuyucu, O.Z., Ekmekci, N., Yaylı, Y.: *On Mannheim partner curves in three dimensional Lie groups*. Miskolc Math. **15** (2), 467-479 (2014).
[17] Honda, S., Takahashi, M.: *Framed curves in the Euclidean space*. Adv. Geom. **16** (3), 265–276 (2016).
[18] Honda, S., Takahashi, M.: *Evolutes and focal surfaces of framed immersions in the Euclidean space*. Proc. Roy. Soc. Edinburgh Sect. A **150**, 497-516 (2020).
[19] Honda, S., Takahashi, M.: *Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space*. Turk. J. Math. **44**, 883–899 (2020).
[20] Honda, S., Takahashi, M., Haiou, Y.: *Bertrand and Mannheim curves of framed curves in the 4-dimensional Euclidean space*. J. Geom **114**:12, (2023).
[21] Huang, J., Chen, L., Izumiya, S., Pei, D.: *Geometry of special curves and surfaces in 3-space form*. J. Geom. Phys. **136**, 31-38 (2019).
[22] Huang, J., Pei, D.: *Singular Special Curves in 3-Space Forms*. Mathematics **8**, 846, (2020).
[23] İlarslan, K., Kılıç Aslan, N.: *On generalized null Mannheim curves in E_2^4*. Math. Meth. Appl. Sci. **44** (9), 7588-7600 (2021).
[24] Li, Y., Pei, D.: *Pedal Curves of Fronts in the sphere*. J. Nonlinear Sci. Appl. **9**, 836-844 (2016).
[25] Li, P., Pei, D., Zhao, X.: *Spacelike Framed Curves with Lightlike Components and Singularities of Their Evolutes and Focal Surfaces in Minkowski 3-space*. Acta Math. Sin. Engl. Ser., DOI: 10.1007/s10114-023-1672-2, (2023).
[26] Li, Y., Sun, Q-Y.: *Evolutes of fronts in the Minkowski plane*. Math. Meth. Appl. Sci. **42** (16), 5416-5426 (2019).
[27] Li, Y., Tuncer, O.O.: *On (contra)pedals and (anti)orthotomics of frontals in de Sitter 2-space*. Math. Meth. Appl. Sci. **46** (9), 11157-11171 (2023).
[28] Li, Y., Uçum, A., İlarslan, K., Camcı, Ç.: *A New Class of Bertrand Curves in Euclidean 4-Space*. Symmetry **14** (6), 1191, (2022).
[29] Liu, H., Wang, F.: *Mannheim partner curves in 3-space*. J. Geom. **88**, 120-126 (2008).
[30] Lucas, P., Ortega-Yagües, J.A.: *Bertrand curves in the three-dimensional sphere*. J. Geom. Phys. **62**, 1903-1914 (2012).
[31] Lucas, P., Ortega-Yagües, J. A.: *Bertrand curves in non-flat 3-dimensional (Riemannian or Lorentzian) space forms*. Bull. Korean Math. Soc. **50**, 1109-1126 (2013).
[32] Lucas, P., Ortega-Yagües, J. A.: *Rectifying curves in the three-dimensional sphere*. J. Math. Anal. Appl. **421** (2), 1855-1868 (2015).
[33] Lucas, P., Ortega-Yagües, J. A.: *Rectifying Curves in the Three-Dimensional Hyperbolic Space*. Mediterr. J. Math. **13**, 2199-2214 (2016).
[34] Lyons, D. W.: *An elementary introduction to the Hopf fibration*. Math. Mag. **76** (2), 87-98 (2003).
[35] Matsuda, H., Yorozu, S.: *Notes on Bertrand curves*. Yokohama Math. J. **50** (1-2), 41-58 (2003).
[36] Okuyucu, O. Z., Gök, İ, Yaylı, Y., Ekmekci, N.: *Bertrand curves in three dimensional Lie groups*. Miskolc Math. **17** (2), 999–1010 (2017).
[37] Pears, L. R.: *Bertrand Curves in Riemannian Space*. J. London Math. Soc. **10**, 180-183 (1935).
[38] Saint-Venant, J. C.: *Mémoire sur les lignes courbes non planes*. Journal d'Ecole Polytechnique **30**, 1-76 (1845).
[39] Takahashi, M., Yu, H.: *Bertrand and Mannheim Curves of Spherical Framed Curves in a Three-Dimensional Sphere*, Mathematics **10**, 1292, (2022).
[40] Tuncer, O. O.: *Singularities of focal sets of pseudo-spherical framed immersions in the three-dimensional anti-de Sitter space*. arXiv:2304.08045, (2023).
[41] Tuncer, O. O., Ceyhan, H., Gök, İ, Ekmekci, F.N.: *Notes on pedal and contrapedal curves of fronts in the Euclidean plane*. Math. Meth. Appl. Sci. **41**, 5096-5111 (2018).
[42] Tuncer, O. O., Gök, İ: *Hyperbolic caustics of light rays reflected by hyperbolic front mirrors*. Eur. Phys. J. Plus **138**:266 (2023).
[43] Uçum, A., Camcı, Ç., İlarslan, K.: *A New Approach to Mannheim Curve in Euclidean 3-Space*. Tamkang J. Math. **54**, (2021).
[44] Uçum, A., İlarslan, K.: *On timelike Bertrand curves in Minkowski 3-space*. Honam Math. J. **38** (3), 467-477 (2016).
[45] Wang, Y., Chang, Y.: *Mannheim curves and spherical curves*. Int. J. Geom. Methods Mod. Phys. **17** (7), 2050101, (2020).
[46] Wang, Y., Pei, D., Gao, R.: *Generic Properties of Framed Rectifying Curves*. Mathematics **7** (1), 37, (2019).
[47] Yu, H., Pei, D., Cui, X.: *Evolutes of Fronts on Euclidean 2-sphere*. J. Nonlinear Sci. Appl. **8**, 678-686 (2015).
[48] Zhao, W., Pei, D., Cao, X.: *Mannheim Curves in Nonflat 3-Dimensional Space Forms*. Adv. Math. Phys. **2015**, Article ID 319046, (2015).

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