International Electronic Journal of Geometry
Volume 12 No. 2 Page 507–518 (2024)
DOI: https://doi.org/10.36890/iejg.1461324
Research article

On Locally Conformal Kaehler Submersions

Çağrıhan Çimen, Beran Pirinççi* , Hakan Mete Taştan and Deniz Ulusoy

(Communicated by Kazım İlarslan)

Abstract

We study locally conformal Kaehler submersions, i.e., almost Hermitian submersions whose total manifolds are locally conformal Kaehler. We prove that the vertical distribution of a locally conformal Kaehler submersion is totally geodesic iff the Lee vector field of total manifold is vertical. We also obtain the O'Neill tensors $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{T}}$ with respect to the Weyl connection of a locally conformal Kaehler submersion. Then, we proved that the horizontal distribution of such a submersion is integrable iff $\tilde{\mathcal{A}} \equiv 0$. Finally, we get Chen-Ricci inequalities for locally conformal Kaehler space form submersions and Hopf space form submersions.

Keywords: *Locally conformal Kaehler manifold, almost Hermitian submersion, Hopf space form submersion, O'Neill tensors, horizontal distribution.*
AMS Subject Classification (2020): *Primary: 53B35 ; Secondary: 53C55; 53C18.*

1. Introduction

The most common type of almost Hermitian manifolds is Kaehler manifolds. Therefore, studying locally conformal Kaehler (birefly, l.c.K.) manifolds is both interesting and logical. The Gray-Hervella class of l.c.K. manifolds is W_4. Classical examples of l.c.K. manifolds are Hopf and Vaisman manifolds (see [4]). Locally conformal Kaehler manifolds were introduced and studied widely by Vaismann [11].

An almost Hermitian submersion is one of the most prevalent mappings between two almost Hermitian manifolds [13]. This is a Riemannian submersion [10] which is also an almost complex mapping. The fundamental property of such a submersion is that its vertical and horizontal distributions are invariant under the almost complex structure of the total manifold of that submersion. An almost Hermitian submersion can be called depending on its total manifold. For instance, if its total manifold is a l.c.K. manifold, then it is called a l.c.K. submersion. Locally conformal Kaehler submersions were studied in the papers [3], [8], [9] and the book [5].

In the present paper, we give some applications of l.c.K. submersions. The paper is organized as follows. In the second section, we introduce locally and globally conformal Kaehler manifolds and almost Hermitian submersions. In the third section, we obtain the O'Neill tensors of l.c.K. submersions. In conclusion, we give the necessary conditions for a vertical distribution of such a submersion to be totally geodesic. In the fourth section, we get the O'Neill tensors $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{T}}$ with respect to the Weyl connection of a l.c.K. submersion. Furthermore, we proved that the horizontal distribution is integrable and totally geodesic if and only if $\tilde{\mathcal{A}} = 0$. In the final section, we study optimal inequalities for l.c.K. space form submersions and Hopf space form submersions.

2. Preliminaries

This section provides information on locally and globally conformal Kaehler manifolds.

Received : *25-04-2024,* ***Accepted :*** *22-07-2024*
*** Corresponding author***

2.1. Locally and globally conformal Kaehler manifolds

Definition 2.1. [4] A Hermitian manifold (N_1, g_1, J_1) of dimension $2m$ is called *locally conformal Kaehler* (briefly l.c.K.) manifold , if N_1 has an open cover $\{\mathcal{O}_i\}_{i\in I}$ and $\forall i \in I$ with family of positive differentiable functions $\sigma_i : \mathcal{O}_i \to \mathbb{R}$ such that

$$g_{1|i} = exp(-\sigma_i)g_{1|\mathcal{O}_i}$$

is a Kaehler metric on $\mathcal{O}_i$. If $\mathcal{O}_i$ is equal to N_1, then (N_1, g_1, J_1) is said to a *globally conformal Kaehler* (briefly g.c.K.) manifold.

Now, we will give a well-known theorem for l.c.K. manifolds.

Theorem 2.1. *[4] Let E, F be arbitrary vector fields on a Hermitian manifold (N_1, g_1, J_1) and Φ be a 2−form defined by $\Phi(E, F) = g_1(E, J_1F)$. Then (N_1, g_1, J_1) is a l.c.K. manifold iff there exists a closed 1-form ω_1 globally defined on N_1 satisfying $d\Phi = \omega_1 \wedge \Phi$.*

The 1−form ω_1 is called the *Lee form* of (N_1, g_1, J_1) . If ω_1 is also exact, then (N_1, g_1, J_1) is a g.c.K. manifold. A l.c.K. manifold is reduced to a *Kaehler manifold* when $\omega_1 = 0$ identically.

Suppose that ∇ is the Riemannian connection of (N_1, g_1, J_1) and $\mathcal{D}^i$ is the Riemannian connection of local Kaehler metrics $g_{1|i}$, $\forall i \in I$. Then, a linear connection $\mathcal{D}$ [4] on N_1 is defined by gluing together the connections $\mathcal{D}^i$ as follows:

$$\mathcal{D}_X X' = \nabla_X X' - \frac{1}{2}\Big\{\omega_1(X)X' + \omega_1(X')X - g_1(X, X')B\Big\}, \tag{2.1}$$

where B is the g_1−dual vector field of ω_1, and is called the *Lee vector field* of N_1. The connection $\mathcal{D}$ is called *Weyl connection* of the l.c.K. manifold (N_1, g_1, J_1) . It is well-known that the Weyl connection $\mathcal{D}$ is torsion-free and satisfies $\mathcal{D}J_1 = 0$. Thus, using this fact and (2.1), it follows that

$$(\nabla_X J_1)X' = \frac{1}{2}\Big\{\omega_1(J_1X')X - \omega_1(X')J_1X - g_1(X, J_1X')B + g_1(X, X')J_1B\Big\}. \tag{2.2}$$

Remark 2.1. We will denote a locally conformal Kaehler manifold (N_1, g_1, J_1) with its Lee form ω_1 as $(N_1, g_1,\ J_1, \omega_1)$.

The curvature tensor field of l.c.K space form is given by

$$\begin{aligned}
4R(X, X', Y, Y') =& c\Big\{g_1(X, Y')g_1(X', Y) - g_1(X, Y)g_1(X', Y') + g_1(J_1X, Y')g_1(J_1X', Y) \\
&- g_1(J_1X, Y)g_1(J_1X', Y') - 2g_1(J_1X, X')g_1(J_1Y, Y')\Big\} \\
&+ 3\Big\{\Omega(X, Y')g_1(X', Y) - \Omega(X, Y)g_1(X', Y') + g_1(X, Y')\Omega(X', Y) \\
&- g_1(X, Y)\Omega(X', Y')\Big\} \\
&- \tilde{\Omega}(X, Y')g_1(J_1X', Y) + \tilde{\Omega}(X, Y)g_1(J_1X', Y') - g_1(J_1X, Y')\tilde{\Omega}(X', Y) \\
&+ g_1(J_1X, Y)\tilde{\Omega}(X', Y') \\
&+ 2\Big\{\tilde{\Omega}(X, X')g_1(J_1Y, Y') + g_1(J_1X, X')\tilde{\Omega}(Y, Y')\Big\},
\end{aligned} \tag{2.3}$$

where

$$\Omega(X, X') = -(\nabla_X \omega_1)X' - \omega_1(X)\omega_1(X') + \frac{1}{2}||B||^2 g_1(X, X'), \tag{2.4}$$

and

$$\tilde{\Omega}(X, X') = \Omega(J_1X, X'). \tag{2.5}$$

2.2. Almost Hermitian Submersions

Suppose that (N_1, g_1) and (N_2, g_2) are Riemannian manifolds. A mapping Ψ of N_1 onto N_2 is called a *Riemannian submersion* [6, 10] if the following two conditions hold:

(I) For each $p \in N_1$, the rank of derivative map Ψ_* of Ψ at p is equal to $dim(N_2)$.

This is equivalent to say that Ψ_* at p is surjective; hence for each $q \in N_2$, $\Psi^{-1}(q)$ is a $dim(N_1) - dim(N_2)$ dimensional closed submanifold of N_1 and called a *fiber* of Ψ. A vector field in the tangent space of N_1 at p, is called *vertical* (resp. *horizontal*) if it is tangent (resp. orthogonal) to fiber $\Psi^{-1}(\Psi(p))$. Vertical (resp. horizontal) vectors at p are in $ker\Psi_{*p}$ (resp. $(ker\Psi_{*p})^\perp$). A vector field Y on N_1 which is horizontal and Ψ-related to a vector field Y^* on N_2 is called a *basic* vector field.

(II) $\forall Y, Y' \in \Gamma(ker\Psi_*)^\perp$, $p \in \Psi^{-1}(q), (q \in N_2)$, we have

$$g_{1|p}(Y, Y') = g_{2|\Psi(p)}(\Psi_* Y, \Psi_* Y') \ .$$

This condition says that for each $p \in \Psi^{-1}(q)$, the derivative map Ψ_{*p} restricted to $(ker\Psi_*)^\perp$, is a linear isometry. We call the manifold (N_1, g_1) (rep. (N_2, g_2)) as *total* (resp. *base*) *manifold* of the submersion Ψ.

A vector field E on N_1 can be written uniquely as $E = E^\nu + E^\hbar$, where E^ν is the vertical part of E, $E^\hbar$ is the horizontal part of E. Then, the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ of type $(1, 2)$ characterizing Riemannian submersions are defined as follows:

$$\mathcal{T}_E F = (\nabla_{E^\nu} F^\hbar)^\nu + (\nabla_{E^\nu} F^\nu)^\hbar \ , \tag{2.6}$$

$$\mathcal{A}_E F = (\nabla_{E^\hbar} F^\hbar)^\nu + (\nabla_{E^\hbar} F^\nu)^\hbar, \tag{2.7}$$

where $E, F \in \Gamma(N_1)$ any vector fields and ∇ is the Riemannian connection of the metric g_1. One can see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of N_1 and reverse the vertical and horizontal distributions. Moreover, we have

$$\mathcal{T}_V V' = \mathcal{T}_{V'} V, \tag{2.8}$$

$$\mathcal{A}_Y Y' = -\mathcal{A}_{Y'} Y = \frac{1}{2}[Y, Y']^\nu. \tag{2.9}$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V V' = \mathcal{T}_V V' + (\nabla_V V')^\nu, \tag{2.10}$$

$$\nabla_V Y = \mathcal{T}_V Y + (\nabla_V Y)^\hbar, \tag{2.11}$$

$$\nabla_Y V = \mathcal{A}_Y V + (\nabla_Y V)^\nu, \tag{2.12}$$

$$\nabla_Y Y' = (\nabla_Y Y')^\hbar + \mathcal{A}_Y Y', \tag{2.13}$$

where $V, V' \in \Gamma(ker\Psi_*)$ and $Y, Y' \in \Gamma(ker\Psi_*)^\perp$. (2.10) says that the tensor $\mathcal{T}$ is the second fundamental form of the fibers. We refer to the papers [6, 10] and the book [5] for more details on the theory of Riemannian submersions.

Now, suppose that (N_1, g_1, J_1) and (N_2, g_2, J_2) be almost Hermitian manifolds and Ψ is a Riemannian submersion from N_1 to N_2. A Riemannian submersion Ψ which satisfies $\Psi_* \circ J_1 = J_2 \circ \Psi_*$, is called an *almost Hermitian submersion* [13]. For an almost Hermitian submersion Ψ, we have $J_1(ker\Psi_*) = ker\Psi_*$ and $J_1(ker\Psi_*^\perp) = ker\Psi_*^\perp$, i.e., vertical and horizontal distributions are $J-$invariant.

We recall the following curvature formulas of a Riemannian submersion by

$$R(U, U', V, V') = \hat{R}(U, U', V, V') + g_1(\mathcal{T}_U V', \mathcal{T}_{U'} V) - g_1(\mathcal{T}_{U'} V', \mathcal{T}_U V), \tag{2.14}$$

$$\begin{aligned} R(X, X', Y, Y') = \ & R^*(X, X', Y, Y') - 2g_1(\mathcal{A}_X X', \mathcal{A}_Y Y') \\ & + g_1(\mathcal{A}_{X'} Y, \mathcal{A}_X Y') - g_1(\mathcal{A}_X Y, \mathcal{A}_{X'} Y'), \end{aligned} \tag{2.15}$$

$$\begin{aligned} R(X, V, X', V') = \ & g_1((\nabla_X \mathcal{T})(V, V'), X') + g_1((\nabla_V \mathcal{A})(X, X'), V') \\ & - g_1(\mathcal{T}_V X, \mathcal{T}_{V'} X') + g_1(\mathcal{A}_{X'} V', \mathcal{A}_X V), \end{aligned} \tag{2.16}$$

where $U, V, U', V' \in \Gamma(ker\Psi_*)$, $X, Y, X', Y' \in \Gamma(ker\Psi_*)^\perp$ and $R, R', \hat{R}$, R^* the Riemannian curvatures of Riemannian manifolds N_1, N_2, the vertical distribution $ker\Psi_*$, the horizontal distribution $(ker\Psi_*)^\perp$, respectively.

Moreover, for an orthonormal basis $\{U_1, ..., U_k\}$ of $ker\Psi_*$

$$H = \frac{1}{k}\sum_{i=1}^{k} \mathcal{T}_{U_i}U_i, \tag{2.17}$$

is called the *mean curvature vector field* of any fiber of Ψ.

3. Locally conformal Kaehler submersions

We first examine how a l.c.K. structure effects on the O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ of an almost Hermitian submersion $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$. We know that the base manifold $(N_2, g_2, J_2, \omega_2)$ also carries a l.c.K. structure from Proposition 3.35 of [5]:

Proposition 3.1. *[5] Let $\Psi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be an almost Hermitian submersion and let X_1, Y_1 be basic vector fields on N_1, Ψ-related to X_2, Y_2 on N_2. Then, we have*
1) J_1X_1 is the basic vector field Ψ-related to J_2X_2;
2) $(N_{J_1}(X_1, Y_1))^{\hbar}$ is the basic vector field Ψ-related to $(N_{J_2}(X_2, Y_2))^{\hbar}$;
3) $((\nabla^1_{X_1}J_1)Y_1)^{\hbar}$ is the basic vector field Ψ-related to $((\nabla^2_{X_2}J_2)Y_2)^{\hbar}$,
where N_{J_1} and N_{J_2} are Nijenhuis tensors of J_1 and J_2, respectively.

Lemma 3.1. *Let $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a l.c.K. submersion. Then, we have*

$$\mathcal{T}_U J_1V = J_1\mathcal{T}_U V + \frac{1}{2}\{g_1(U, V)(J_1B)^{\hbar} - g_1(U, J_1V)B^{\hbar}\}, \tag{3.1}$$

$$\mathcal{T}_V J_1X = J_1\mathcal{T}_V X + \frac{1}{2}\{\omega_1(J_1X)V - \omega_1(X)J_1V\}, \tag{3.2}$$

$$\mathcal{A}_X J_1V = J_1\mathcal{A}_X V + \frac{1}{2}\{\omega_1(J_1V)X - \omega_1(V)J_1X\}, \tag{3.3}$$

$$\mathcal{A}_X J_1Y = J_1\mathcal{A}_X Y + \frac{1}{2}\{g_1(X, Y)(J_1B)^{\nu} - g_1(X, J_1Y)B^{\nu}\}, \tag{3.4}$$

where $U, V \in \Gamma(ker\Psi_*)$ and $X, Y \in \Gamma(ker\Psi_*)^{\perp}$, and B is the Lee vector field of N_1.

Proof. For any $X, Y \in \Gamma(ker\Psi_*)^{\perp}$, we have

$$\nabla_X J_1Y = J_1\nabla_X Y + \tfrac{1}{2}\Big\{\omega_1(J_1Y)X - \omega_1(Y)J_1X - g_1(X, J_1Y)B + g_1(X, Y)J_1B\Big\},$$

from (2.2). Using (2.11), we obtain

$$\begin{aligned}\mathcal{A}_X J_1Y + (\nabla_X J_1Y)^{\hbar} &= J_1\mathcal{A}_X Y + J_1(\nabla_X Y)^{\hbar}\\ &\quad + \frac{1}{2}\Big\{\omega_1(J_1Y)X - \omega_1(Y)J_1X - g_1(U, J_1V)B + g_1(X, Y)J_1B\Big\}.\end{aligned}$$

Taking the vertical parts of both sides of the above equation, we get (3.2). By a similar method, we can obtain the other assertions. □

Theorem 3.1. *Let $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a l.c.K. submersion. Then, the vertical distribution $ker\Psi_*$ is totally geodesic iff for any $U, V \in \Gamma(ker\Psi_*)$*

$$\mathcal{T}_U J_1V = \frac{1}{2}\{g_1(U, J_1V)B^{\hbar} - g_1(U, V)J_1B^{\hbar}\}, \tag{3.5}$$

is satisfied.

Proof. The vertical distribution $ker\Psi_*$ is totally geodesic iff for any $U, V \in \Gamma(ker\Psi_*)$, $\nabla_U V \in \Gamma(ker\Psi_*)$. From (2.2) and (2.10), we obtain

$$\begin{aligned}\nabla_U V = &- J_1\Big(T_U J_1V + (\nabla_U J_1V)^{\nu} - \frac{1}{2}\Big\{\omega_1(J_1V)U\\ &- \omega_1(V)J_1U - g_1(U, J_1V)(B^{\nu} + B^{\hbar}) + g_1(U, V)(J_1B^{\nu} + J_1B^{\hbar})\Big\}\Big).\end{aligned}$$

In this equation, if horizontal terms vanish, we get (3.5). □

As a result of this theorem, we can give the following corollary.

Corollary 3.1. *A fiber of a l.c.K. submersion* $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *is totally geodesic iff the Lee vector field* B *is vertical.*

Vilms [12] showed that a Riemannian submersion is a totally geodesic map iff both O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ are zero, identically. But, we know that the fibers of a Riemannian submersion are totally geodesic iff $\mathcal{T}$ is zero, identically. Thus, using these facts and Theorem (3.1), we obtain that :

Corollary 3.2. *A l.c.K. submersion* $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *cannot be a totally geodesic map.*

4. O'Neill's tensors with respect to Weyl connection

For a l.c.K. submersion $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$, if we take the Weyl connection $\mathcal{D}$ instead of ∇ in (2.6) and (2.7), we define two tensors of types $(1, 2)$. Let us denote these tensors by $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$, respectively.

Lemma 4.1. *Let* $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a l.c.K. submersion. For any* $U, V \in \Gamma(ker\Psi_*)$ *and* $X, Y \in \Gamma(ker\Psi_*)^\perp$, *we have*

$$\tilde{\mathcal{T}}_U V = \mathcal{T}_U V + \frac{1}{2} g_1(U, V) B^\hbar, \tag{4.1}$$

$$\tilde{\mathcal{T}}_V X = \mathcal{T}_V X - \frac{1}{2} \omega_1(X) V, \tag{4.2}$$

$$\tilde{\mathcal{A}}_X V = \mathcal{T}_X V - \frac{1}{2} \omega_1(V) X, \tag{4.3}$$

$$\tilde{\mathcal{A}}_X Y = \mathcal{A}_X Y + \frac{1}{2} g_1(X, Y) B^\nu. \tag{4.4}$$

Proof. Using (2.1), and (2.10) $\sim$ (2.13), we get all assertions. □

First, we summarize the properties of $\tilde{\mathcal{T}}$.

Lemma 4.2. *Let* $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a l.c.K. submersion. For any* $U, V \in \Gamma(ker\Psi_*)$, *we have*
(a) $\tilde{\mathcal{T}}_V(.)$, *reverse the vertical and horizontal distribution,*
(b) $\tilde{\mathcal{T}}$ *is symmetric i.e.,* $\tilde{\mathcal{T}}_U V = \tilde{\mathcal{T}}_V U$.

Proof. Using (4.1), (4.2) and the properties of $\tilde{\mathcal{T}}$, we get **(a)**. **(b)** follows from (2.10) and (4.1). □

In view of Lemma 4.2, we have that:

Corollary 4.1. *Let* $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a l.c.K. submersion. Then,* $\tilde{\mathcal{T}}$ *acts as a second fundamental form of* Ψ.

Now, we present the properties of $\tilde{\mathcal{A}}$.

Lemma 4.3. *Let* $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a l.c.K. submersion. For any* $X, Y \in \Gamma(ker\Psi_*)^\perp$, *we have*
(a) $\tilde{\mathcal{A}}_X(.)$, *reverse the vertical and horizontal distribution,*
(b) $\tilde{\mathcal{A}}$ *is neither symmetric nor skew-symmetric for horizontal vector fields, i.e.,* $\tilde{\mathcal{A}}_X Y \neq \tilde{\mathcal{A}}_Y X$,
(c) $\tilde{\mathcal{A}}_X Y = \frac{1}{2}\{g_1(X, Y) B^\nu - \frac{1}{2} g_1(X, J_1 Y) J_1 B^\nu\}$,
(d) $\tilde{\mathcal{A}}_X Y = \frac{1}{2} ||X||^2 B^\nu$,
(e) $\tilde{\mathcal{A}}_X J_1 X = \frac{1}{2} ||X||^2 J_1 B^\nu$,
(f) $\tilde{\mathcal{A}}_X J_1 Y = J_1 \tilde{\mathcal{A}}_X Y$, *i.e.,* $(\tilde{\mathcal{A}}_X Y) = 0$.

Proof. *(a)* comes from (4.3) and (4.4). By (4.4) and the skew-symmetricness of $\mathcal{A}$, we see that **(b)** is true. From Proposition 4.3 of [8], we have $\mathcal{A}_X Y = -\frac{1}{2} g_1(X, J_1 Y) J_1 B^\nu$ for $X, Y \in \Gamma(ker\Psi_*)^\perp$. Using this fact in (4.4), we immediately get **(c)**. **(d)**, **(e)** and **(f)** are simple consequences of **(c)**. □

From Proposition 3.34 of [5], we know that the mean curvature vector field of the fibers of a l.c.K. submersion is $-\frac{1}{2} B^\hbar$. Hence, if the fibers are totally umbilical with respect to $\mathcal{T}$, we have $\mathcal{T}_U V = -\frac{1}{2} g_1(U, V) B^\hbar$, for $U, V \in \Gamma(ker\Psi_*)$. Thus, by (4.1), we get the following result:

Theorem 4.1. *The fibers of a l.c.K. submersion $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ are totally umbilical with respect to $\mathcal{T}$ iff they are totally geodesic with respect to $\tilde{\mathcal{T}}$.*

Theorem 4.2. *The horizontal distribution of a l.c.K. submersion $\Psi : (N_1, g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ is integrable and totally geodesic iff $\tilde{\mathcal{A}} \equiv 0$. Namely, $\mathcal{A} \equiv 0 \Leftrightarrow \tilde{\mathcal{A}} \equiv 0$.*

Proof. If the horizontal distribution is integrable and totally geodesic, then, we have $\mathcal{A} \equiv 0$ and the Lee vector field B is horizontal from Proposition 4.3 [8]. Thus, by (4.4), we get $\tilde{\mathcal{A}} \equiv 0$. Conversely, if $\tilde{\mathcal{A}} \equiv 0$, we deduce that B is horizontal form Lemma 4.2 - **(d)**. Again, by Proposition 4.3 [8], we conclude that $\mathcal{A} \equiv 0$. □

5. Chen-Ricci Inequality

Chen [1] established an inequality between Ricci curvature and the squared mean curvature for any submanifold in a space form. The aforementioned inequality is now referred to as the Chen-Ricci inequality and has been inhanced by several authors (see references [1] and [2]). In this section, we shall give Chen-Ricci inequality for the fibers of an almost Hermitian submersion whose total manifold is a l.c.K. space form.

Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a locally conformal Kaehler space form submersion. We consider $ker\Psi_* = span\{U_1, ..., U_{2n}\}$ such that $J_1U_{2i-1} = U_{2i}$ for $1 \leq i \leq n$ and $(ker\Psi_*)^{\perp} = span\{U_{2n+1}, ..., U_{2m}\}$.
Then, we can give the following theorems.

Theorem 5.1. *Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a locally conformal Kaehler space form submersion. For any $U, V \in \Gamma(ker\Psi_*)$, we have*

$$\begin{aligned}
\widehat{Ric}(U,V) =& \frac{c}{2}(n+1)g_1(U,V) + \frac{3}{4}\Big[(1-2n)(\nabla_U\omega_1)V + (3-2n)\omega_1(U)\omega_1(V) \\
&+ \omega_1(J_1U)\omega_1(J_1V) + (2n-2)||B||^2 g_1(U,V) - ||B^{\nu}||^2 g_1(U,V) + (\nabla_{J_1U}\omega_1)J_1V \\
&+ \sum_{i=1}^{2n} g_1(U,U_i)(\nabla_{U_i}\omega_1)V - g_1(U,V)(\nabla_{U_i}\omega_1)U_i - g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V\Big] \\
&+ \frac{1}{4}\sum_{i=1}^{2n} g_1(J_1U,V)(\nabla_{J_1U_i}\omega_1)U_i + \sum_{i=1}^{2n} g_1(\mathcal{T}_{U_i}V, \mathcal{T}_U U_i) - 2n g_1(\mathcal{T}_U V, H),
\end{aligned} \tag{5.1}$$

where

$$\widehat{Ric}(U,V) = \sum_{i=1}^{2n} \hat{R}(U,U_i,U_i,V).$$

Proof. Let $U, V \in \Gamma(ker\Psi_*)$. Then from (2.3), we have

$$\begin{aligned}
\sum_{i=1}^{2n} R(U,U_i,U_i,V) = &\sum_{i=1}^{2n}\Big[\frac{c}{4}\Big\{g_1(U,V)g_1(U_i,U_i) - g_1(U,U_i)g_1(U_i,V)\\
&+ g_1(J_1U,V)g_1(J_1U_i,U_i) - g_1(J_1U,U_i)g_1(J_1U_i,V)\\
&- 2g_1(J_1U,U_i)g_1(J_1U_i,V)\Big\}\\
&+ \frac{3}{4}\Big\{\Omega(U,V)g_1(U_i,U_i) - \Omega(U,U_i)g_1(U_i,V)\Big\}\\
&+ g_1(U,V)\Omega(U_i,U_i) - g_1(U,U_i)\Omega(U_i,V)\\
&+ \frac{1}{4}\Big\{-\tilde{\Omega}(U,V)g_1(J_1U_i,U_i) + \tilde{\Omega}(U,U_i)g_1(J_1U_i,V)\\
&- g_1(J_1U,V)\tilde{\Omega}(U_i,U_i) + g_1(J_1U,U_i)\tilde{\Omega}(U_i,V)\Big\}\\
&+ \frac{1}{2}\Big\{\tilde{\Omega}(U,U_i)g_1(J_1U_i,V) + \tilde{\Omega}(U_i,V)g_1(J_1U,U_i)\Big\}\Big]\\
= &\frac{c}{4}\big\{2n\big(g_1(U,V) - g_1(U,V) + 3g_1(U,V)\big)\big\}\\
&+ \sum_{i=1}^{2n}\Big[\frac{3}{4}\big\{2n\big[-(\nabla_U\omega_1)V - \omega_1(U)\omega_1(V) + \frac{1}{2}||B||^2g_1(U,V)\big]\\
&- g_1(U_i,V)\big[-(\nabla_{U_i}\omega_1)U_i - \omega_1(U)\omega_1(U_i) + \frac{1}{2}||B||^2g_1(U,U_i)\big]\\
&+ g_1(U,V)\big[-(\nabla_{U_i}\omega_1)U_i - \omega_1(U_i)\omega_1(U_i) + \frac{1}{2}||B||^2g_1(U_i,U_i)\big]\\
&- g_1(U_i,U)\big[-(\nabla_{U_i}\omega_1)V - \omega_1(U_i)\omega_1(V) + \frac{1}{2}||B||^2g_1(U_i,V)\big]\big\}\\
&+ \frac{1}{4}\big\{g_1(J_1U_i,V)\big[-(\nabla_{J_1U}\omega_1)U_i - \omega_1(U_i)\omega_1(J_1U) + \frac{1}{2}||B||^2g_1(J_1U,U_i)\big]\\
&- g_1(J_1U,V)\big[-(\nabla_{J_1U_i}\omega_1)U_i - \omega_1(J_1U_i)\omega_1(U_i) + \frac{1}{2}||B||^2g_1(J_1U_i,U_i)\big]\\
&+ g_1(J_1U,U_i)\big[-(\nabla_{J_1U_i}\omega_1)V - \omega_1(J_1U_i)\omega_1(V) + \frac{1}{2}||B||^2g_1(J_1U_i,V)\big]\big\}\\
&+ \frac{1}{2}\big\{g_1(J_1U_i,V)\big[-(\nabla_{J_1U}\omega_1)U_i - \omega_1(U_i)\omega_1(J_1U) + \frac{1}{2}||B||^2g_1(J_1U,U_i)\big]\\
&+ g_1(J_1U,U_i)\big[-(\nabla_{J_1U_i}\omega_1)V - \omega_1(J_1U_i)\omega_1(V) + \frac{1}{2}||B||^2g_1(J_1U_i,V)\big]\big\}\\
= &\frac{c}{4}(2n+2)g_1(U,V) + \frac{3}{4}\Big\{2n(-(\nabla_U\omega_1)V - \omega_1(U)\omega_1(V) + \frac{1}{2}||B||^2g_1(U,V))\\
&+ \sum_{i=1}^{2n}\Big[\big(\nabla_U\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2g_1(U,V)\\
&- g_1(U,V)(\nabla_{U_i}\omega_1)U_i - ||B||^2g_1(U,V) + n||B||^2g_1(U,V)\\
&+ g_1(U,U_i)(\nabla_{U_i}\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2g_1(U,V)\Big]\Big\}\\
&+ \frac{1}{4}\Big\{(\nabla_{J_1U}\omega_1)J_1V + \omega_1(J_1U)\omega_1(J_1V) - \frac{1}{2}||B||^2g_1(U,V)\\
&+ \sum_{i=1}^{2n}\Big[g_1(J_1U,V)(\nabla_{J_1U_i}\omega_1)U_i + g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V + \omega_1(U)\omega_1(V)\\
&- \frac{1}{2}||B||^2g_1(U,V)\Big]\Big\} + \frac{1}{2}\Big\{(\nabla_{J_1U}\omega_1)J_1V + \omega_1(J_1U)\omega_1(J_1V) - \frac{1}{2}||B||^2g_1(U,V)\\
&+ \sum_{i=1}^{2n}\Big[-g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2g_1(U,V)\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c}{2}(n+1)g_1(U,V) + \frac{3}{4}\big[(1-2n)(\nabla_U\omega_1)V + (1-2n)\omega_1(U)\omega_1(V) \\
&+ (n-\frac{1}{2})||B||^2 g_1(U,V)\big] + \sum_{i=1}^{2n}\Big[\frac{3}{4}\big\{-g_1(U,V)(\nabla_{U_i}\omega_1)U_i - ||B^v||^2 g_1(U,V) \\
&+ n||B||^2 g_1(U,V) + g_1(U,U_i)(\nabla_{U_i}\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2 g_1(U,V)\big\} \\
&+ \frac{1}{4}\big\{(\nabla_{J_1U}\omega_1)J_1V + \omega_1(J_1U)\omega_1(J_1V) - \frac{1}{2}||B||^2 g_1(U,V) + g_1(J_1U,V)(\nabla_{J_1U_i}\omega_1)U_i \\
&- g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2 g_1(U,V)\big\} \\
&+ \frac{2}{4}\big\{(\nabla_{J_1U}\omega_1)J_1V + \omega_1(J_1U)\omega_1(J_1V) - \frac{1}{2}||B||^2 g_1(U,V) \\
&- g(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2 g_1(U,V)\big\}\Big] \\
&= \frac{c}{2}(n+1)g_1(U,V) + \frac{3}{4}\big[(1-2n)(\nabla_U\omega_1)V + (1-2n)\omega_1(U)\omega_1(V) \\
&+ (n-\frac{1}{2})||B||^2 g_1(U,V)\big] + \sum_{i=1}^{2n}\Big[\frac{3}{4}\big\{-g_1(U,V)(\nabla_{U_i}\omega_1)U_i - ||B^\nu||^2 g_1(U,V) \\
&+ n||B||^2 g_1(U,V) + g_1(U,U_i)(\nabla_{U_i}\omega_1)V + \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2 g_1(U,V)\big\} \\
&+ \frac{3}{4}\big\{(\nabla_{J_1U}\omega_1)J_1V + \omega_1(J_1U)\omega_1(J_1V) - \frac{1}{2}||B||^2 g_1(U,V) - g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V \\
&+ \omega_1(U)\omega_1(V) - \frac{1}{2}||B||^2 g_1(U,V)\big\} + \frac{1}{4}g_1(J_1U,V)(\nabla_{J_1U_i}\omega_1)U_i\Big] \\
&= \frac{c}{2}(n+1)g_1(U,V) + \frac{3}{4}\Big[(1-2n)(\nabla_U\omega_1)V + (3-2n)\omega_1(U)\omega_1(V) \\
&+ \omega_1(J_1U)\omega_1(J_1V) + (2n-2)||B||^2 g_1(U,V) - ||B^\nu||^2 g_1(U,V) + (\nabla_{J_1U}\omega_1)J_1V \\
&+ \sum_{i=1}^{2n} g_1(U,U_i)(\nabla_{U_i}\omega_1)V - g_1(U,V)(\nabla_{U_i}\omega_1)U_i - g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V\Big] \\
&+ \frac{1}{4}\sum_{i=1}^{2n} g_1(J_1U,V)(\nabla_{J_1U_i}\omega_1)U_i.
\end{aligned}
$$

Thus we get,

$$
\begin{aligned}
Ric(U,V) &= \frac{c}{2}(n+1)g_1(U,V) + \frac{3}{4}\Big[(1-2n)(\nabla_U\omega_1)V + (3-2n)\omega_1(U)\omega_1(V) \\
&+ \omega_1(J_1U)\omega_1(J_1V) + (2n-2)||B||^2 g_1(U,V) - ||B^\nu||^2 g_1(U,V) + (\nabla_{J_1U}\omega_1)J_1V \\
&+ \sum_{i=1}^{2n}\big[g_1(U,U_i)(\nabla_{U_i}\omega_1)V - g_1(U,V)(\nabla_{U_i}\omega_1)U_i - g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)V\Big] \\
&+ \frac{1}{4}\sum_{i=1}^{2n} g_1(J_1U,V)(\nabla_{J_1U_i}\omega_1)U_i
\end{aligned}
$$

From (2.14), we obtain (5.1). □

Corollary 5.1. *Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a locally conformal Kaehler space form submersion. For any unit vector field $U \in \Gamma(ker\Psi_*)$, we have*

$$
\begin{aligned}
\widehat{Ric}(U) \quad &\geq (n+1)\frac{c}{2} + \frac{3}{4}\Big\{(3-2n)(\omega_1(U))^2 + (2n-2)||B||^2 + (\omega_1(J_1U))^2 - ||B^\nu||^2 \\
&+(1-2n)(\nabla_U\omega_1)U + (\nabla_{J_1U}\omega_1)J_1U + \sum_{i=1}^{2n}\Big(g_1(U_i,U)(\nabla_{U_i}\omega_1)U - (\nabla_{U_i}\omega_1)U_i \\
&-g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega)U\Big)\Big\} - 2ng_1(\mathcal{T}_U U, H),
\end{aligned} \tag{5.2}
$$

where

$$\widehat{Ric}(U) = \sum_{i=1}^{2n} \widehat{R}(U, U_i, U_i, U). \tag{5.3}$$

The equality case of (5.2) holds when each fiber is totally geodesic.

Proof. If we take $U = V$ unit vector field in the equation (5.1), then we have

$$\begin{aligned}\widehat{Ric}(U) = (n+1)\frac{c}{2} + \frac{3}{4}\Big\{&(3-2n)(\omega_1(U))^2 + (2n-2)||B||^2 + (\omega_1(J_1U))^2 - ||B^\nu||^2 \\ &+ (1-2n)(\nabla_U\omega_1)U + (\nabla_{J_1U}\omega_1)J_1U + \sum_{i=1}^{2n}\Big(g_1(U_i,U)(\nabla_{U_i}\omega_1)U - (\nabla_{U_i}\omega_1)U_i \\ &- g_1(J_1U,U_i)(\nabla_{J_1U_i}\omega_1)U\Big)\Big\} - 2ng_1(\mathcal{T}_UU,H) + \sum_{i=1}^{2n}||T_UU_i||^2\end{aligned}$$

Thus, from the above equation we get the inequality (5.2). The equality occurs iff $\mathcal{T} \equiv 0$, in which case each fiber is totally geodesic. □

Corollary 5.2. *Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a Hopf space form submersion. For any unit vector field $U \in \Gamma(ker\Psi_*)$, we have*

$$\widehat{Ric}(U) \geq (n+1)\frac{c}{2} + \frac{3}{4}\Big\{(3-2n)(\omega_1(U))^2 + (2n-2)||B||^2 + (\omega_1(J_1U))^2 \tag{5.4}$$

$$- ||B^v||^2\} - 2ng_1(\mathcal{T}_UU, H). \tag{5.5}$$

Proof. Since $(N_1^{2m}(c), g_1, J_1, \omega_1)$ is a Hopf space form, then we get $\nabla\omega_1 = 0$. Hence, we obtain (5.4) from (5.2). □

Corollary 5.3. *Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a locally conformal Kaehler space form submersion. Then, we have*

$$\begin{aligned}\widehat{\tau} \;&\geq cn(n+1) + \frac{3}{4}\Big[4n(n-1)||B||^2 + 4(1-n)||B^\nu||^2 \\ &+ \sum_{j=1}^{2n}\Big((1-2n)(\nabla_{J_1U_j}\omega_1)U_j + (\nabla_{J_1U_j}\omega_1)J_1U_j\Big) \\ &- \sum_{i,j=1}^{2n} g_1(J_1U_j, U_i)(\nabla_{J_1U_i}\omega_1)J_1U_j\Big] - 4n^2||H||^2.\end{aligned} \tag{5.6}$$

Proof. If we take $U = V = U_j$ in (5.1), we obtain

$$
\begin{aligned}
\widehat{\tau} = & \sum_{j=1}^{2n} \widehat{Ric}(U_j, U_j) = \sum_{j=1}^{2n} \Bigg[\frac{c}{2}(n+1) g_1(U_j, U_j) + \frac{3}{4}\Big[(1-2n)(\nabla_{U_j}\omega_1)U_j + (3-2n)\omega_1(U_j)\omega_1(U_j) \\
& + \omega_1(J_1U_j)\omega_1(J_1U_j) + (2n-2)||B||^2 g_1(U_j,U_j) - ||B^\nu||^2 g_1(U_j,U_j) + (\nabla_{J_1U_j}\omega_1)J_1U_j \\
& + \sum_{i=1}^{2n} \Big[g_1(U_j,U_i)(\nabla_{U_i}\omega_1)U_j - g_1(U_j,U_j)(\nabla_{U_i}\omega_1)U_i - g_1(J_1U_j,U_i)(\nabla_{J_1U_i}\omega_1)U_j \Big] \\
& + \sum_{i=1}^{2n} g_1(J_1U_j,U_j)(\nabla_{J_1U_i}\omega_1)U_i \Bigg] + \sum_{i,j=1}^{2n} ||\mathcal{T}_{U_i}U_j||^2 - 4r^2||H||^2 \\
& = cn(n+1) + \frac{3}{4}\Big[\sum_{j=1}^{2n} (1-2n)(\nabla_{U_j}\omega_1)U_j + (3-2n)||B^\nu||^2 + ||B^\nu||^2 \\
& + 2n(2n-2)||B||^2 - 2n||B^\nu||^2 + \sum_{j=1}^{2n}(\nabla_{J_1U_j}\omega_1)J_1U_j \\
& + \sum_{i=1}^{2n} \big(2n(\nabla_{U_i}\omega_1)U_i - 2n(\nabla_{U_i}\omega_1)U_i - \sum_{j=1}^{2n} g_1(J_1U_j,U_i)(\nabla_{J_1U_j}\omega_1)J_1U_j \big) \Big] \\
& + \sum_{i,j=1}^{2n} ||\mathcal{T}_{U_i}U_i||^2 - 4r||H||^2 \\
& = cn(n+1) + \frac{3}{4}\Big[4(1-n)||B^\nu||^2 + 4n(n-1)||B||^2 \\
& + \sum_{j=1}^{2n} \big((1-2n)(\nabla_{U_j}\omega_1)U_j + (\nabla_{J_1U_j}\omega_1)J_1U_j \big) - \sum_{i,j=1}^{2n} g_1(J_1U_j,U_i)(\nabla_{J_1U_j}\omega_1)U_j \Big] \\
& + \sum_{i,j=1}^{2n} ||\mathcal{T}_{U_i}U_i||^2 - 4r^2||H||^2
\end{aligned}
$$

So we obtain the inequality (5.6) from the above equation. □

Corollary 5.4. *Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a Hopf space form submersion. Then we get*

$$
\widehat{\tau} \geq cn(n+1) + \frac{3}{4}\Big[4n(n-1)||B||^2 + 4(1-n)||B^\nu||^2 - 4r^2||H||^2 \Big]. \tag{5.7}
$$

Theorem 5.2. *Let $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ be a locally conformal Kaehler space form submersion. For any $X, Y \in \Gamma(ker\Psi_*)^\perp$, we have*

$$
\begin{aligned}
Ric^*(X,Y) = & \frac{c}{2}(m-n+1)g_1(X,Y) + \frac{3}{4}\Big\{ (1-2(m-n))(\nabla_X\omega_1)Y + (3-2(m-n))\omega_1(X)\omega_1(Y) \\
& + (2(m-n)-2)||B||^2 g_1(X,Y) - ||B^h||^2 g_1(X,Y) + \omega_1(J_1X)\omega_1(J_1Y) + (\nabla_{JX}\omega_1)J_1Y \\
& + \sum_{i=2n+1}^{2m} \Big[g_1(X,U_i)(\nabla_{U_i}\omega_1)Y - g_1(X,Y)(\nabla_{U_i}\omega_1)U_i - g_1(J_1X,U_i)(\nabla_{J_1U_i}\omega_1)Y \Big] \Big\} \\
& + \frac{1}{4}\sum_{i=2n+1}^{2m} g_1(J_1X,Y)(\nabla_{J_1U_i}\omega_1)U_i + 3g(A_XU_i, A_{U_i}Y),
\end{aligned} \tag{5.8}
$$

where

$$
Ric^*(X,Y) = \sum_{i=2n+1}^{2m} R^*(X,U_i,U_i,Y). \tag{5.9}
$$

Proof. If we take $F = G = U_i$, $E = X$ and $Z = Y$ in the equation (2.3) and using (2.15), then we obtain the equation (5.8). □

Corollary 5.5. *Let* $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a locally conformal Kaehler space form submersion. For any unit vector field* $X \in \Gamma(ker\Psi_*)^{\perp}$*, we obtain*

$$\begin{aligned} Ric^*(X) \leq \frac{c}{2}(m-n+1) + \frac{3}{4}\Big\{&(1-2k)(\nabla_X \omega_1)X + (3-2(m-n))(\omega_1(X))^2 \\ &+ (2(m-n)-2)||B||^2 - ||B^h||^2 + (\omega_1(J_1X))^2 + (\nabla_{J_1X}\omega_1)J_1X \\ &+ \sum_{i=2n+1}^{2m} \Big[g_1(X,U_i)(\nabla_{U_i}\omega_1)X - (\nabla_{U_i}\omega_1)U_i - g_1(J_1X,U_i)(\nabla_{J_1U_i}\omega_1)X\Big]\Big\}, \end{aligned} \tag{5.10}$$

where

$$Ric^*(X) = \sum_{i=2n+1}^{2m} Ric^*(X, U_i, U_i, X). \tag{5.11}$$

Proof. If we take $X = Y$ in the equation (5.8), then we get this inequatlity directly. □

Corollary 5.6. *Let* $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a Hopf space form submersion. Then, we have*

$$\begin{aligned} Ric^*(X) \leq \frac{c}{2}(m-n+1) + \frac{3}{4}\Big\{&(3-2(m-n))(\omega_1(X))^2 + (2(m-n)-2)||B||^2 \\ &- ||B^h||^2 + (\omega_1(J_1X))^2\Big\} \end{aligned}$$

for any unit vector field $X \in \Gamma(ker\Psi_*)^{\perp}$.

Corollary 5.7. *Let* $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a locally conformal Kaehler space form submersion. Then, we have*

$$\begin{aligned} \tau^* \leq c(m-n)(m-n+1) + \frac{3}{4}\Big\{&4(m-n)(m-n-1)||B||^2 \\ &+ 4(1-(m-n))||B^h||^2 + \sum_{i=2n+1}^{2m}\Big[(1-2(m-n)(\nabla_{U_j}\omega_1)U_j \\ &+ (\nabla_{J_1U_j}\omega_1)J_1U_j\Big]\Big\} - \frac{1}{4}\sum_{i,j=2n+1}^{2m} g_1(J_1U_j,U_i)(\nabla_{J_1U_i}\omega_1)U_j. \end{aligned} \tag{5.12}$$

Proof. If we take $X = Y = U_j$ in the equation (5.8) and using the equation (2.15), then we get the inequality (5.12). □

Corollary 5.8. *Let* $\Psi : (N_1^{2m}(c), g_1, J_1, \omega_1) \to (N_2, g_2, J_2, \omega_2)$ *be a Hopf space form submersion. Then, we obtain*

$$\begin{aligned} \tau^* \leq c(m-n)(m-n+1) + \frac{3}{4}\Big\{&4(m-n)(m-n-1)||B||^2 \\ &+ 4(1-(m-n))||B^h||^2\Big\}. \end{aligned}$$

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

References

[1] Chen, B.Y.: *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*. Glasgow Math. J., **41**, 33-41 (1999).
[2] Deng, S.: *An improved Chen-Ricci inequality*. Int. Elec. J. Geom., **2**(2), 39-45 (2009).
[3] Dragomir, S.: *Generalized Hopf manifolds locally conformal Kahler structures and real hypersurfaces*. Kodai Math. J., **14**, 366-391 (1991).
[4] Dragomir, S., Ornea, L.: Locally conformal Kahler geometry. Boston, Basel, Berlin: Birkhauser (1998).
[5] Falcitelli, M., Lanus, S., Pastore, A.M.: Riemannian Submersion and Related Topics. Singapore: Worl Scientific Publishing Co. Pte. Ltd. (2004).
[6] Gray , A.:*Pseudo-Riemannian Almost Product Manifolds and Submersions*. Journal of Mathematics and Mechanics, **16** (7), 715-737 (1967).
[7] Eells, J., Sampson, J.H: *Harmonic Mapping of Riemannian Manifolds*, Amer. J. Math., 109-160 (1964).
[8] Marrero, J.C., Rocha, J.: *Locally conformal Kaehler submersions*. Geom. Dedicata., **52**(3), 271-289 (1994).
[9] Musso, E.: *Submersioni localmente conformemente Kahleriane*. Boll. Unione Mat. It., **7** 3-A, 171-176 (1989).
[10] O'Neill, B.: *The fundamental equations of a submersion*. Mich. Math. J., **13**, 458-469 (1966).
[11] Vaisman, I. *On locally conformal almost Kahler manifolds*, Israel Journal of Mathematics, **24** (3-4), 338-351. (1976).
[12] Vilms, J.: *Totally geodesic maps*. J. Differential Geom., **4**, 73-79 (1970).
[13] Watson, B.: *Almost Hermitian submersions*. J. Differ. Geom., **11**(1), 147-165 (1976).

Affiliations

ÇAĞRIHAN ÇİMEN
ADDRESS: Marmara University, Dept. of Mathematics, 34722, İstanbul-Türkiye.
E-MAIL: cagrihan.cimen@marmara.edu.tr
ORCID ID: 0009-0000-6331-9615

BERAN PİRİNÇÇİ
ADDRESS: İstanbul University, Dept. of Mathematics, 34134 İstanbul-Türkiye.
E-MAIL: beranp@istanbul.edu.tr
ORCID ID: 0000-0002-4692-9590

HAKAN METE TAŞTAN
ADDRESS: İstanbul University, Dept. of Mathematics, 34134, İstanbul-Türkiye.
E-MAIL: hakmete@istanbul.edu.tr
ORCID ID: 0000-0002-0773-9305

DENIZ ULUSOY
ADDRESS: Istinye University, Dept. of Management Information Systems, 34396, İstanbul-Türkiye.
E-MAIL: deniz.ulusoy@istinye.edu.tr
ORCID ID: 0000-0002-0742-4047