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RESEARCH ARTICLE

Homogeneous Riemannian Structures in Thurston Geometries and Contact Riemannian Geometries

Jun-ichi Inoguchi

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ABSTRACT

We give explicit parametrizations for all the homogeneous Riemannian structures on model spaces of Thurston geometry. As an application, we give all the homogeneous contac Riemannian structures on 3-dimensional Sasakian space forms.

Keywords: *Homogeneous structures, Thurston geometry, Sasakian manifolds, Ambrose-Singer connections.*
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*** Corresponding author***

1. Introduction

According to Thurston's classification [220] of 3-dimensional geometries, there exist eight simply connected model spaces. The model spaces are the following homogeneous Riemannian spaces:

- space forms: Euclidean 3-space $\mathbb{E}^3$, 3-sphere $\mathbb{S}^3$, hyperbolic 3-space $\mathbb{H}^3$,
- reducible Riemannian symmetric spaces: $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$,
- the Heisenberg group Nil_3, the universal covering group $\widetilde{\mathrm{SL}}_2\mathbb{R} \cong \widetilde{\mathrm{SU}}(1,1)$.
- the Minkowski motion group Sol_3.

Other than space forms and product spaces, model spaces are *not* Riemannian symmetric spaces. As is well known, local symmetry of Riemannian manifolds is characterized by the parallelism of the Riemannian curvature due to E. Cartan. As a generalization of local symmetry, Ambrose and Singer [5] obtained an infinitesimal characterization of Riemannian homogeneity of Riemannian manifolds. They showed that local Riemannian homogeneity is equivalent to the existence of certain tensor field S. Such a tensor field S is referred as to a *homogeneous Riemannian structure*. The moduli space of homogeneous structures on a homogeneous Riemannian space (M, g) represents the all possible coset space representations of (M, g) up to isomorphisms. In other words, the moduli space is identified with the space of *canonical connections* (also called the *Ambrose-Singer connections*). Katsuda [137] obtained a pinching theorem for locally homogeneous Riemannian spaces by using homogeneous Riemannian structures. Recently Ni and Zheng studied Hermitian manifolds whose Chern connection is an Ambrose-Singer connection [166].

The moduli problem of 3-dimensional homogeneous Riemannian structures indicates us some interesting phenomena. For instance, Riemannian symmetric spaces are homogeneous Riemannian spaces with *trivial* homogeneous Riemannian structure $S = 0$. However the trivial homogeneous Riemannian structure may not determine uniquely the Riemannian symmetric spaces. For instance, the Euclidean plane $\mathbb{E}^2$ is represented by $\mathrm{SE}(2)/\mathrm{SO}(2)$ as a Riemannian symmetric space with $S = 0$. On the other hand, The homogeneous Riemannian space $\mathbb{E}^2 = \mathbb{E}^2/\{\mathbf{0}\}$ has trivial homogeneous Riemannian structure $S = 0$ (See [222, Introduction, p. II, Corollary 4.2]).

The 3-dimensional (homogeneous) geometry is rather special. Olmos and Reggiani [176, 177] proved the uniqueness of canonical connections for the hyperbolic space $\mathbb{H}^n$ for $n \neq 3$. More precisely represent $\mathbb{H}^n$ as $\mathbb{H}^n = \mathrm{SO}^+(1, n)/\mathrm{SO}(n)$ as a naturally reductive homogeneous space. Then the Levi-Civita connection is the only canonical connection associated to this representation when $n \neq 3$. But the hyperbolic 3-space $\mathbb{H}^3$ admits exactly a line of canonical connections. On the other hand, for the naturally reductive spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, the canonical connection is unique. But as we will see later the 3-sphere $\mathbb{S}^3$ has another naturally reductive space representation $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathrm{U}(1) = \mathrm{U}(2)/\mathrm{U}(1)$.

It is known that any oriented Riemannian 3-manifold $(M, g, \mathrm{d}v_g)$ admits an almost contact structure compatible to the metric and the orientation. Here an almost contact structure is a triplet (φ, ξ, η) of tensor fields consisting of an endomorphism field φ, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

An almost contact structure (φ, ξ, η) is said to be *compatible* to the metric g and the orientation if it satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y on M and

$$\mathrm{d}v_g = -3\eta \wedge \Phi,$$

where Φ is a 2-form defined by $\Phi(X, Y) = g(X, \varphi Y)$. An oriented Riemannian 3-manifold $(M, g, \mathrm{d}v_g)$ equipped an compatible almost contact structure is called an *almost contact Riemannian* 3*-manifold*.

An almost contact Riemannian 3-manifold M is said to be a *contact Riemannian* 3*-manifold* if $\Phi = \mathrm{d}\eta$. One can see that the 1-form η of a contact Riemannian 3-manifold M is a *contact form*, *i.e.*, $\mathrm{d}\eta \wedge \eta \neq 0$. Moreover ξ is the Reeb vector field of a contact 3-manifold (M, η).

On the other hand, an almost contact Riemannian 3-manifold M is said to be *normal* if the almost complex structure on the product manifold $M \times \mathbb{R}$ is integrable. An normal contact Riemannian 3-manifold is called a *Sasakian* 3*-manifold*.

From almost contact structure viewpoint, we emphasize that every seven model space other than Sol_3 admits normal almost contact structure compatible to to the metric. On the other hand, Sol_3 admits an almost contact structure compatible to the metric such that the resulting almost contact 3-manifold is non-normal contact Riemannian 3-manifold.

Among the eight model spaces, $\mathbb{S}^3$, Nil_3, $\widetilde{\mathrm{SL}}_2\mathbb{R}$ and Sol_3 admit homogeneous contact structure compatible to the metric. In particular, $\mathbb{S}^3$, Nil_3, $\widetilde{\mathrm{SL}}_2\mathbb{R}$ are homogeneous Sasakian 3-manifolds of constant holomorphic sectional curvature (*Sasakian space forms*). Note that 3-dimensional simply connected Sasakian space forms are exhausted by $\mathbb{S}^3$, Nil_3, $\widetilde{\mathrm{SL}}_2\mathbb{R}$ and the Berger 3-spheres. Moreover Sasakian space forms are *naturally reductive* homogeneous Riemannian spaces [21].

Model space	Homogeneity	Compatible almost contact structure
$\mathbb{E}^3$	Symmetric (space form)	CoKähler
$\mathbb{S}^3$	Symmetric (space form)	Sasakian
$\mathbb{H}^3$	Symmetric (space form)	Kenmotsu
$\mathbb{S}^2\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^1$	Symmetric	CoKähler
Nil_3, $\widetilde{\mathrm{SL}}_2\mathbb{R}$	Naturally reductive	Sasakian
Sol_3	Homogeneous (4-symmetric)	Contact

Table 1. The eight model spaces

This article has three purposes.

The first aim of this article is to give a survey on homogeneous Riemannian structures on model spaces of Thurston geometries.

The second purpose of the present paper is to describe all the homogeneous Riemannian structures of model spaces of Thurston geometries explicitly.

In contact Riemannian geometry or CR-geometry, certain kind of linear connections with non-vanishing torsion have been used. Tanaka [212] and Webster [228] introduced a linear connection on contact strongly pseudo-convex CR-manifolds. This connection is referred as to the *Tanaka-Webster connection*. Note that Sasakian manifolds are strongly pseudo-convex CR-manifolds.

Tanno [219] introduced the notion of generalized Tanaka-Webster connection on general contact Riemannian manifolds. The generalized Tanaka-Webster connection coincides with original Tanaka-Webster connection on contact strongly pseudo-convex CR-manifolds.

The third purpose of the present article is to study relations between these two kinds of connections, Ambrose-Singer connections and generalized Tanaka-Webster connections, derived from different geometric backgrounds. More precisely we shall study relations between generalized Tanaka-Webster connections and Ambrose-Singer connections on 3-dimensional Sasakian space forms.

Throughout this paper all manifolds are assumed to be connected.

Conventions. In this paper we use the following symbols and conventions for exterior differentiation of differential forms:

- Throughout this paper we denote the space of all smooth sections of a vector bundle E by $\Gamma(E)$. For instance the Lie algebra of all smooth vector fields on M is denoted by $\Gamma(\mathrm{T}M)$. Here $\mathrm{T}M$ is the tangent bundle of M. The space $\Gamma(\mathrm{T}M)$ forms an infinite dimensional Lie algebra with Lie bracket:
$$[X,Y]f = X(Y(f)) - Y(X(f)), \quad f \in C^\infty(M),$$
where $C^\infty(M)$ is the commutative ring of all smooth functions on M. The resulting Lie algebra is denoted by $\mathfrak{X}(M)$.
- The Lie differentiation by a vector field X is denoted by $\pounds_X$.
- The space of all real square matrices of degree n is denote by $\mathrm{M}_n\mathbb{R}$.
- The space of all complex square matrices of degree n is denote by $\mathrm{M}_n\mathbb{C}$.
- The skew field of quaternions is denoted by $\mathbf{H}$.
- The space of all quaternion square matrices of degree n is denote by $\mathrm{M}_n\mathbf{H}$.
- The unit element of a Lie group is denoted by e. In case $G=\mathrm{SU}(2)$, $\mathrm{SU}(1,1)$ and $\mathrm{SL}_2\mathbb{R}$, we denote the unit element also by $\mathbf{1}$.
- The matrix units are denoted by E_{ij}.

- The identity matrix of degree n is denoted by E_n.
- The zero matrix of type (m,n) are denoted by $O_{m,n}$ or simply O. In particular $O_{n,n}$ is denoted by O_n.
- The identity endomorphism field is denoted by $\mathrm{I} \in \Gamma(\mathrm{End}(\mathrm{T}M))$.
- Let M be a manifold and η a 1-form on M. Then the exterior derivative $\mathrm{d}\eta$ is defined by

$$2\mathrm{d}\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]), \quad X,Y \in \Gamma(\mathrm{T}M).$$

- The exterior derivative $\mathrm{d}\Phi$ of a 2-form Φ is defined by

$$\begin{aligned}\mathrm{d}\Phi =& X(\Phi(Y,Z)) + Y(\Phi(Z,X)) + Z(\Phi(X,Y)) \\ & - \Phi([X,Y],Z) - \Phi([Y,Z],X) - \Phi([Z,X],Y).\end{aligned}$$

- On an oriented Riemannian manifold (M,g), $\mathrm{d}\eta$ and $\mathrm{d}\Phi$ are rewritten as

$$\mathrm{d}\eta(X,Y) = \frac{1}{2}\left((\nabla_X\eta)Y - (\nabla_Y\eta)X\right), \quad \mathrm{d}\Phi(X,Y,Z) = \frac{1}{3}\underset{X,Y,Z}{\mathfrak{S}}(\nabla_X\Phi)(Y,Z)$$

  in terms of Levi-Civita connection ∇. The codifferential $\delta\eta$ and $\delta\Phi$ are given by

$$\delta\eta = -\mathrm{tr}(\nabla\eta), \quad (\delta\Phi)X = -\mathrm{tr}(\nabla.\Phi)(\cdot, X).$$

- On an oriented Riemannian n-manifold $(M, g, \mathrm{d}v_g)$, the volume element $\mathrm{d}v_g$ satisfies

$$\mathrm{d}v_g(e_1, e_2, \dots, e_n) = 1$$

  for any positively oriented local orthonormal frame field $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$. In terms of the coframe field $\{\vartheta^1, \vartheta^2, \dots, \vartheta^n\}$ metrically dual to $\mathcal{E}$, $\mathrm{d}v_g$ is expressed as

$$\mathrm{d}v_g = n!\,\vartheta^1 \wedge \vartheta^2 \wedge \cdots \wedge \vartheta^n.$$

- For a tensor field S of type $(1,2)$, its *covariant form* is denoted by $S_\flat$, *i.e.*,

$$S_\flat(X,Y,Z) = g(S(X)Y,Z), \quad X,Y,Z \in \Gamma(\mathrm{T}M).$$

2. Riemannian manifolds

2.1. *Linear connections*

Let M be a smooth manifold. A *linear connection*

$$D : \Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M); \quad (X,Y) \longmapsto D_X Y$$

is a differential operator on the tangent bundle $\mathrm{T}M$ satisfying that it

- is linear in the both first and second slots,
- is $C^\infty(M)$-linear in the first slot and
- satisfies the *Leipniz rule*:

$$D_X(fY) = X(f)Y + fD_XY, \quad f \in C^\infty(M),\ X,Y \in \Gamma(\mathrm{T}M).$$

The *torsion tensor field* (often called the torsion) $T = T^D$ is defined by

$$T(X,Y) = D_XY - D_YX - [X,Y].$$

A linear connection D is said to be *torsion free* if its torsion T vanishes. The curvature $R = R^D$ is defined by

$$R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}.$$

2.2. Curvatures

Let (M, g) be a Riemannian manifold. A linear connection D is said to be *metrical* with respect to g, if g is parallel with respect to D, *i.e.*,

$$(Dg)(Y, Z; X) = (D_X g)(Y, Z) = Xg(Y, Z) - g(D_X Y, Z) - g(Y, D_X Z) = 0$$

for all $X, Y, Z \in \Gamma(\mathrm{T}M)$. Such a linear connection D on (M, g) is called a *metric connection*.

On a Riemannian manifold (M, g), there exits a unique torsion free metrical connection ∇ called the *Levi-Civita connection*.

The Levi-Civita connection ∇ is determined by the *Koszul formula*:

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X).$$

The curvature $R = R^\nabla$ of the Levi-Civita connection is called the *Riemannian curvature*. The *Ricci tensor field* Ric of (M, g) is defined by

$$\mathrm{Ric}(X, Y) = \mathrm{tr}_g(Z \longmapsto R(Z, Y)X).$$

The self-adjoint endomorphism field Q metrically equivalent to Ric is called the *Ricci operator*. The scalar curvature s is defined by $\mathrm{s} = \mathrm{tr}_g \mathrm{Ric} = \mathrm{tr}_g Q$.

We define a curvature-like tensor field $(X \wedge Y)Z$ by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad X, Y, Z \in \Gamma(\mathrm{T}M).$$

One can see that (M, g) is of constant curvature c if and only if $R(X, Y)Z = c(X \wedge Y)Z$ for all vector fields $X, Y, Z \in \Gamma(\mathrm{T}M)$.

We denote by $\mathrm{Iso}(M) = \mathrm{Iso}(M, g)$ the full isometry group of (M, g). The identity component of $\mathrm{Iso}(M)$ is denote by $\mathrm{Iso}_\circ(M)$. The Lie algebra of $\mathrm{Iso}_\circ(M)$ is denote by $\mathfrak{iso}(M)$.

A vector field $X \in \Gamma(\mathrm{T}M)$ is said to be a *Killing vector field* if its (local) flows are isometric. The set $\mathfrak{i}(M)$ of all Killing vector fields is a subalgebra of $\mathfrak{X}(M)$. A vector field X is a Killing vector field if and only if $\pounds_X g = 0$, *i.e.*,

$$Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]) = 0, \quad Y, Z \in \Gamma(\mathrm{T}M).$$

The Killing property of X is equivalent to that ∇X is skew-adjoint with respect to g, *i.e.*,

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0, \quad Y, Z \in \Gamma(\mathrm{T}M).$$

2.3. Connection form

Let $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame field on a Riemannian n-manifold (M, g). Denote by $\Theta = (\vartheta^1, \vartheta^2, \ldots, \vartheta^n)$ the local orthonormal coframe field metrically dual to $\mathcal{E}$. We regard Θ as an $\mathbb{R}^n$-valued 1-form

$$\Theta = \begin{pmatrix} \vartheta^1 \\ \vartheta^2 \\ \vdots \\ \vartheta^n \end{pmatrix}$$

Since the Levi-Civita connection ∇ is torsion free, the following *first structure equation*:

$$\mathrm{d}\Theta + \omega \wedge \Theta = 0$$

holds. The $\mathfrak{so}(n)$-valued 1-form

$$\omega = \begin{pmatrix} 0 & \omega_2^1 & \cdots & \omega_n^1 \\ -\omega_2^1 & 0 & \cdots & \omega_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_n^1 & -\omega_n^2 & \cdots & 0 \end{pmatrix}$$

determined by the first structure equation is called the *connection form*. Here $\mathfrak{so}(n)$ is the Lie algebra of real skew-symmetric matrices of degree n (see Example 4.3). A component ω_j^i of ω is called a *connection 1-form*. The first structure equation is the differential system:

$$\mathrm{d}\vartheta^i + \sum_{j=1}^n \omega_j{}^i \wedge \vartheta^j = 0.$$

The connection coefficients $\{\Gamma_{jk}^{\ i}\}$ of the Levi-Civita connection ∇ is relative to $\mathcal{E}$ is defined by

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^{\ k} e_k.$$

Then the connection 1-forms are related to connection coefficients by

$$\omega_j^{\ k} = \sum_{\ell=1}^{n} \Gamma_{\ell j}^{\ k} \vartheta^\ell.$$

Hence we obtain

$$g(\nabla_X e_i, e_j) = \omega_i^{\ j}(X). \tag{2.1}$$

Thus

$$\omega_{ij} = \omega_i^{\ j} = -\omega_j^{\ i}.$$

Remark 2.1. Tricerri and Venhecke [222] used the convention:

$$g(\nabla_X e_i, e_j) = \omega_{ij}(X).$$

2.4. Curvature forms

Next, the $\mathfrak{so}(n)$-valued 2-form $\Omega = (\Omega_j^{\ i})$ defined by

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega$$

is called the *curvature form* relative to Θ. This formula is called the *second structure equation*. The components $\Omega_j^{\ i}$ are called *curvature 2-forms*. The second structure equation is the differential system:

$$\Omega_j^{\ i} = \mathrm{d}\omega_j^{\ i} + \sum_{k=1}^{n} \omega_k^{\ i} \wedge \omega_j^{\ k}.$$

One can see that

$$R(X,Y)e_i = 2\sum_{j=1}^{n} \Omega_i^{\ j}(X,Y)e_j.$$

If we express the Riemannian curvature R as

$$R(e_k, e_\ell)e_i = \sum_{j=1}^{n} R_{ik\ell}^{\ j}\, e_j,$$

and set

$$R_{ijk\ell} = g(R(e_k, e_\ell)e_i, e_j) = R_{ik\ell}^{\ j}, \quad \Omega_{ij} := \Omega_i^{\ j},$$

then we obtain

$$\Omega_{ij} = \frac{1}{2}\sum_{k,\ell=1}^{n} R_{ijk\ell}\, \theta^k \wedge \theta^\ell.$$

The components of Ricci tensor field are given by

$$R_{ij} = \mathrm{Ric}(e_i, e_j) = \sum_{k=1}^{n} R_{ikj}^{\ k}.$$

The scalar curvature is given by $\mathrm{s} = \sum_{i=1}^{n} R_{ii}$.

The sectional curvature K_{ij} of the tangent plane $e_i \wedge e_j$ spanned by e_i and e_j is given by

$$K_{ij} = K(e_i \wedge e_j) = g(R(e_i, e_j)e_j, e_i) = R_{jiij} = R_{ijji}.$$

In case $\dim M = 3$, we have

$$R_{12} = R_{1323}, \quad R_{23} = R_{1213}, \quad R_{31} = -R_{1223}.$$

The sectional curvature K_{ij} are related to Ric by

$$K_{12} = \frac{1}{2}(R_{11} + R_{22} - R_{33}), \quad K_{13} = \frac{1}{2}(R_{11} - R_{22} + R_{33}), \quad K_{23} = \frac{1}{2}(-R_{11} + R_{22} + R_{33}).$$

2.5. The vector product

Let $(M, g, \mathrm{d}v_g)$ be an oriented Riemannian 3-manifold. Then the volume element $\mathrm{d}v_g$ defines the *vector product operation* (also called the *cross product*) $\times$ on each tangent space T_pM by the rule

$$g(X \times Y, Z) = \mathrm{d}v_g(X, Y, Z), \quad X, Y, Z \in \mathrm{T}_pM.$$

The vector product operation satisfies:

$$(X \times Y) \times Z = g(Z, X)Y - g(Y, Z)X,$$

Comparing this with the curvature-like tensor field

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y,$$

we obtain the following formula:

$$(X \wedge Y)Z = -(X \times Y) \times Z = Z \times (X \times Y).$$

We denote by $\mathrm{d}V$ the tensor field of type $(1, 2)$ metrically equivalent to $\mathrm{d}v_g$;

$$\mathrm{d}v_g(X, Y, Z) = g(\mathrm{d}V(X)Y, Z). \tag{2.2}$$

In other words, $\mathrm{d}V_\flat = \mathrm{d}v_g$. Equivalently $\mathrm{d}V(X)Y = X \times Y$.

3. Homogeneous geometry

3.1. Homogeneous manifolds

Let $M = G/H$ be a homogeneous manifold with connected Lie group G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$, the Lie algebras of G and H, respectively. We denote by $\Pi_{\mathfrak{h}}$ the projection from $\mathfrak{g}$ onto $\mathfrak{h}$.

Let π be the natural projection of G onto M;

$$\pi : G \to M, \ \pi(a) = aH.$$

Denote by τ the natural left action of G on M.

$$\tau : G \times M \to M; \quad \tau(a, bH) = (ab)H, \quad a, b \in G.$$

The diffeomorphism τ_a on M defined by $\tau_a(bH) = \tau(a, bH)$ is called the *translation* on M by a. For $h \in H$, the differential $\rho(h) := \tau_{h*o}$ of τ_h at the origin $o := H$ defines a representation ρ of H over the tangent space T_oM at o. This representation ρ of H is called the *linear isotropy representation* of H. The group $\rho(H)$ is called the *linear isotropy group* of M.

We describe the tangent spaces of M. Let $\tau^\sharp : \mathfrak{g} \times M \to \mathrm{T}M$ be the *linearization* of the natural left action τ on M:

$$\tau^\sharp(X, p) = X^\sharp_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\exp(tX)}(p), \quad X \in \mathfrak{g}, \quad p \in M.$$

One can check that

$$[X, Y]^\sharp = -[X^\sharp, Y^\sharp].$$

Express p as $p = a \cdot o$. Then the kernel of $\tau^\sharp(\cdot, p)$ is $\mathfrak{h}_p = \mathrm{Ad}(a)\mathfrak{h}$. The kernel $\mathfrak{h}_p$ is the Lie algebra of the isotropy subgroup H_p at p.

Note that the isotropy subgroup H_p of G at $p = a \cdot o$ is aHa^{-1}. Thus we get the linear isomorphism

$$\mathfrak{g}/\mathrm{Ad}(a)\mathfrak{h} \cong \mathrm{T}_{a \cdot o}M; \quad X + \mathrm{Ad}(a)\mathfrak{h} \longmapsto X^\#_{a \cdot o}.$$

The *isotropy bundle* $\underline{\mathfrak{h}}$ defined by

$$\underline{\mathfrak{h}} = \bigcup_{p \in M} \mathrm{Ker}\, \tau^\#(\cdot, p)$$

is isomorphic to the vector bundle $G/H \times_H \mathfrak{h}$ associated to $G \to G/H$ with standard fiber $\mathfrak{h}$.

Note that $\tau_{a*o} X^{\#}_o$ is given by

$$\tau_{a*o} X^{\#}_o = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\exp(t\mathrm{Ad}(a)X)}(p), \ p = a \cdot o.$$

Here we define another map $\tau^{\natural} : M \times \mathfrak{g} \to \mathrm{T}M$ by

$$\tau^{\natural}(p, X) = X^{\natural}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\exp(t\mathrm{Ad}(a)X)}(p), \ p = a \cdot o.$$

Thus $\tau^{\natural}$ is a vector bundle morphism from the trivial bundle $M \times \mathfrak{g}$ to $\mathrm{T}M$. The kernel of each $\tau^{\natural}(p, \cdot)$ is $\mathfrak{h}$. Thus we get a linear isomorphism;

$$\mathfrak{g}/\mathfrak{h} \cong \mathrm{T}_{a\cdot o}M; \ \ X + \mathfrak{h} \longmapsto X^{\natural}_{a\cdot o}.$$

This linear isomorphism induces the isomorphism between the tangent bundle $\mathrm{T}M$ and the vector bundle $G \times_H \ \mathfrak{g}/\mathfrak{h}$ associated to the principal H-bundle G:

$$X^{\natural}_{a\cdot o} \longmapsto [(a, X + \mathfrak{h})].$$

3.2. *Reductive homogeneous spaces*

Hereafter we assume that $M = G/H$ is *reductive*. Namely there exists a linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ complementary to $\mathfrak{h}$ such that

$$\mathrm{ad}(\mathfrak{h})\mathfrak{m} \subset \mathfrak{m}.$$

The linear subspace $\mathfrak{m}$ is called the *Lie subspace* of $\mathfrak{g}$. We denote by $\Pi_{\mathfrak{m}}$ the projection from $\mathfrak{g}$ onto $\mathfrak{m}$.

Via the differential $\pi_{*\mathsf{e}} : \mathfrak{g} \to \mathrm{T}_oM$ of the natural projection at $\mathsf{e} \in G$, the tangent space T_oM is identified with the Lie subspace $\mathfrak{m}$. Under this identification the linear isotropy group $\rho(H)$ is identified with $\mathrm{Ad}(H)$. Moreover the tangent spaces of M are given by

$$\mathrm{T}_{a\cdot o}M \cong \mathrm{Ad}(a)\mathfrak{m}; \quad X^{\#}_{a\cdot o} \longmapsto \mathrm{Ad}(a)X.$$

Thus we obtain a linear isomorphism:

$$\tau^{\sharp}_{a\cdot o} : \mathfrak{m} \to \mathrm{T}_{a\cdot o}M; \quad \tau^{\sharp}_{a\cdot o}(X) = X^{\sharp}_{a\cdot o}.$$

Let us denote by $\beta_p : \mathrm{T}_pM \to \mathfrak{m}$ the inverse mapping of $\tau^{\sharp}_{a\cdot o}$ for $p = a \cdot o$. Then β_p is naturally extended to an $\mathfrak{m}$-valued 1-form on M.

Proposition 3.1 ([34])**.** *For $X \in \mathfrak{g}$ and $p = a \cdot o$, we have*

$$\beta_p(X^{\sharp}_p) = \mathrm{Ad}(a)\Pi_{\mathfrak{m}}(\mathrm{Ad}(a^{-1})X).$$

Note that

$$v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\exp(t\beta_p(v))}p$$

holds for any $v \in T_pM$.

Corollary 3.1 ([34])**.** *For any $X \in \mathfrak{m}$ and $p = a \cdot o \in M$:*

$$\beta_p(X^{\natural}_p) = \mathrm{Ad}(a)X, \quad X \in \mathfrak{m}.$$

The $\mathfrak{m}$-valued 1-form β is called the *Maurer-Cartan form* of the reductive homogeneous space $M = G/H$ (see [34]).

For a reductive homogeneous space $M = G/H$, the tangent bundle $\mathrm{T}M$ is identified with $\underline{\mathfrak{m}} := G \times_H \mathfrak{m}$ via the bundle isomorphism β.

The Maurer-Cartan form β is characterized as the unique bundle homomorphism from $\mathrm{T}M$ to $G/H \times \mathfrak{g}$ satisfying

$$\beta \circ \tau^{\#} = \Pi_{\underline{\mathfrak{m}}}, \quad \tau^{\#} \circ \beta = \mathrm{I}_{\mathrm{T}M}.$$

3.3. The canonical connection

The vector bundle $\underline{\mathfrak{g}} = G \times_H \mathfrak{g}$ is decomposed as $\underline{\mathfrak{g}} = \underline{\mathfrak{h}} \oplus \underline{\mathfrak{m}}$ according to the reductive splitting of $\mathfrak{g}$. Denote by $\Pi_{\underline{\mathfrak{h}}}$, the projection $\underline{\mathfrak{g}} \to \underline{\mathfrak{h}}$. The Maurer-Cartan form ϑ of G is the unique left invariant $\mathfrak{g}$-valued 1-form which satisfies

$$\vartheta_{\mathsf{e}}(X_{\mathsf{e}}) = X_{\mathsf{e}}, \quad X_{\mathsf{e}} \in \mathrm{T}_{\mathsf{e}}G.$$

The Maurer-Cartan form ϑ can be defined by the following formula:

$$\vartheta_a(X_a) = L_{a*}^{-1} X_a \in \mathrm{T}_{\mathsf{e}}G.$$

Here L_a is the left translation of G by $a \in G$. The 1-form $\vartheta_{\underline{\mathfrak{h}}} := \Pi_{\underline{\mathfrak{h}}} \circ \vartheta$ is a connection form of the principal H-bundle $\pi : G \to G/H$. The horizontal distribution of this connection is

$$\mathfrak{Q}_a = L_{a*}\mathfrak{m}.$$

This connection induces a linear connection ∇^{c} on M. The linear connection ∇^{c} is called the *canonical connection* of the reductive homogeneous space $M = G/H$. Note that ∇^{c} is the canonical connection of *second kind* in the sense of Nomizu [170].

Proposition 3.2 ([34]).

$$\beta(\nabla^{\mathrm{c}}_X Y) = X\beta(Y) - [\beta(X), \beta(Y)], \ X, Y \in \Gamma(\mathrm{T}M).$$

The torsion T^{c} of ∇^{c} is described by the following *structure equations*:

$$\mathrm{d}\beta = (\mathrm{I} - \frac{1}{2}\Pi_{\underline{\mathfrak{m}}})[\beta \wedge \beta], \quad \beta \circ T^{\mathrm{c}} = -\frac{1}{2}\Pi_{\underline{\mathfrak{m}}}[\beta \wedge \beta].$$

Take a representation of H on a finite dimensional linear space V. Then the vector bundle $\underline{V} = G \times_H V$ is identified with the trivial bundle $G/H \times V$ via

$$\underline{V} \ni [(g, v)] \longmapsto (\pi(g), g \cdot v).$$

Proposition 3.3 ([34]). *Let s be a section of $\underline{V}$, then* $\mathrm{d}s = \nabla^{\mathrm{c}} s + \beta \cdot s$.

3.4. Invariant connections

There is a bijective correspondence between the set of all G-invariant linear connections on a reductive homogeneous space G/H with Lie subspace $\mathfrak{m}$ and the set of all linear maps $\mu : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$ satisfying

$$\mu(\mathrm{Ad}(h)X) = \mathrm{ad}(\rho(h))\mu(X)$$

for any $X \in \mathfrak{m}$ and $h \in H$. Here ρ is the linear isotropy representation as before. The linear connection ∇^{μ} corresponding to μ is described as (see [145, 203]):

$$(\nabla^{\mu}_{X^\sharp} Y^\sharp)_o = -[X, Y]_{\mathfrak{m}} + \mu(X)Y. \tag{3.1}$$

The torsion T^{μ} and curvature R^{μ} of ∇^{μ} are given by

$$T^{\mu}(X^\sharp, Y^\sharp)|_o = \mu(X)Y - \mu(Y)X - [X, Y]_{\mathfrak{m}}, \quad R^{\mu}(X^\sharp, Y^\sharp)|_o = [\mu(X), \mu(Y)] - \mu([X, Y]_{\mathfrak{m}}) - \rho([X, Y]_{\mathfrak{h}}).$$

Here we used the notation

$$[X, Y]_{\mathfrak{h}} = \Pi_{\mathfrak{h}}([X, Y]), \quad [X, Y]_{\mathfrak{m}} = \Pi_{\mathfrak{m}}([X, Y]).$$

Obviously the canonical connection ∇^{c} corresponds to $\mu = 0$. Thus the torsion T^{c} and curvature R^{c} of ∇^{c} are given by ([145, Theorem 2.6]):

$$T^{\mathrm{c}}(X^\sharp, Y^\sharp)|_o = -[X, Y]_{\mathfrak{m}}, \quad R^{\mathrm{c}}(X^\sharp, Y^\sharp)Z^\sharp|_o = -[\,[X, Y]_{\mathfrak{h}}, Z], \quad X, Y, Z \in \mathfrak{m}.$$

In particular $\nabla^{\mathrm{c}} T^{\mathrm{c}} = 0$ and $\nabla^{\mathrm{c}} R^{\mathrm{c}} = 0$ holds.

3.5. *Canonical connections on Lie groups*

Since the model spaces of Thurston geometry (other than $\mathbb{S}^2 \times \mathbb{E}^1$) are realized as Lie groups equipped with specific left invariant metrics, here we discuss canonical connections on Lie groups.

We can regard a Lie group G as a homogeneous space in two ways: $G = G/\{\mathsf{e}\}$ and $G = (G \times G)/\Delta G$. The Lie algebra $\mathfrak{g}$ is defined as the tangent space $\mathrm{T}_{\mathsf{e}}G$ of G at the unit element and we often regard $\mathfrak{g}$ as the space of all smooth *left invariant* vector fields on M. It should be remarked that for $X \in \mathfrak{g} = \mathrm{T}_{\mathsf{e}}G$, $X^\sharp$ is a *right invariant* vector field on G. However it is desirable to use only left-invariant vector fields to describe everything on G. On this reason, here, we give left invariant formulation for canonical connections.

In the first representation, the isotropy algebra is $\{0\}$ and $\mathfrak{m} = \mathfrak{g}$. Obviously the splitting $\mathfrak{g} = \{0\} + \mathfrak{g}$ is reductive. The natural projection $\pi : G \to G/\{\mathsf{e}\}$ is the identity map. By definition, the canonical connection ∇^{c} is determined by

$$(\nabla^{\mathrm{c}}_{X^\sharp} Y^\sharp)_{\mathsf{e}} = -[X, Y].$$

The canonical connection of $G/\{\mathsf{e}\}$ is characterized as

$$\nabla^{\mathrm{c}}_X Y = 0, \quad X, Y \in \mathfrak{g}.$$

The torsion T^{c} of ∇^{c} is given by $T^{\mathrm{c}}(X, Y) = -[X, Y]$. The canonical connection ∇^{c} is also called the *Cartan-Schouten's* $(-)$*-connection* and denoted by $\nabla^{(-)}$.

Next, let us take the product Lie group $G \times G$. The Lie algebra of $G \times G$ is

$$\mathfrak{g} = \{(X, Y) \mid X, Y \in \mathfrak{g}\}$$

with Lie bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

The product Lie group $G \times G$ acts on G by the action:

$$(G \times G) \times G \to G; \quad (a, b)x = axb^{-1}. \tag{3.2}$$

The isotropy subgroup at the identity e is the *diagonal subgroup*

$$\Delta G = \{(a, a) \mid a \in G\}$$

with Lie algebra $\Delta\mathfrak{g} = \{(X, X) \mid X \in \mathfrak{g}\}$. We can consider the following three Lie subspaces;

$$\mathfrak{m}^+ = \{(0, X) \mid X \in \mathfrak{g}\}, \quad \mathfrak{m}^- = \{(X, 0) \mid X \in \mathfrak{g}\}, \quad \mathfrak{m}^0 = \{(X, -X) \mid X \in \mathfrak{g}\}.$$

Then $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}^+$, $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}^-$ and $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}^0$ are reductive.

The corresponding splittings are given explicitly by

$$\begin{aligned}
(X, Y) &= (X, X) + (0, -X + Y)) \in \Delta\mathfrak{g} \oplus \mathfrak{m}^+ \\
(X, Y) &= (Y, Y) + (X - Y, 0) \in \Delta\mathfrak{g} \oplus \mathfrak{m}^- \\
(X, Y) &= \left(\frac{X + Y}{2}, \frac{X + Y}{2}\right) + \left(\frac{X - Y}{2}, -\frac{X - Y}{2}\right) \in \Delta\mathfrak{g} \oplus \mathfrak{m}^0.
\end{aligned}$$

Let us identify the tangent space $\mathrm{T}_{\mathsf{e}}G$ of G at e with these Lie subspaces. Then the canonical connection with respect to the reductive decomposition $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}^+$ is denoted by $\nabla^{(+)}$ and given by

$$(\nabla^{(+)}_{X^\sharp} Y^\sharp)_{\mathsf{e}} = 0, \quad X, Y \in \mathfrak{g}.$$

In left invariant way, we have

$$\nabla^{(+)}_X Y = [X, Y], \quad X, Y \in \mathfrak{g}.$$

The torsion $T^{(+)}$ of $\nabla^{(+)}$ is given by $T^{(+)}(X, Y) = [X, Y]$. The canonical connection $\nabla^{(+)}$ is called the *Cartan-Schouten's* $(+)$*-connection* or *anti canonical connection* [60].

Next, the canonical connection with respect to the reductive decomposition $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}^-$ is $\nabla^{(-)}$. Finally the canonical connection with respect to the reductive decomposition $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathfrak{m}^0$ is denoted by $\nabla^{(0)}$ and given by

$$(\nabla^{(0)}_{X^\sharp} Y^\sharp)_{\mathsf{e}} = -\frac{1}{2}[X, Y], \quad X, Y \in \mathfrak{g},$$

equivalently,

$$\nabla^{(0)}_X Y = \frac{1}{2}[X,Y], \quad X, Y \in \mathfrak{g}.$$

The connection $\nabla^{(0)}$ is torsion free and called the *Cartan-Schouten's* (0)*-connection*, the *natural torsion free connection* [170] or *neutral connection* [60].

More generally, there exists a bijective correspondence between the set of all left invariant linear connections on G and the set $\{\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \mid \mu \text{ is bilinear}\}$. The linear connection ∇^{μ} corresponding to μ is described as

$$\nabla^{\mu}_X Y = \mu(X)Y, \quad X, Y \in \mathfrak{g}.$$

4. Homogeneous Riemannian spaces

4.1. Riemannian homogeneity

Definition 4.1. A Riemannian manifold (M, g) is said to be a *homogeneous Riemannian space* if there exits a Lie group G of isometries which acts transitively on M.

More generally, M is said to be *locally homogeneous Riemannian space* if for each p, $q \in M$, there exists a local isometry which sends p to q.

Without loss of generality, we may assume that a homogeneous Riemannian space (M, g) is reductive. In fact, the following result is known (see *e.g.*, [151]):

Proposition 4.1. *Any homogeneous Riemannian space* (M, g) *is a reductive homogeneous space.*

Moreover it is known that isotropy subgroup H of a homogeneous Riemannian space G/H is compact.

Since G acts isometrically on (M, g), for any $X \in \mathfrak{g}$, the vector field $X^\sharp$ is a Killing vector field. In particular, the correspondence $X \longmapsto X^\sharp$ is an *anti isomorphism*, *i.e.*,

$$[X^\sharp, Y^\sharp] = -[X, Y]^\sharp.$$

Let $M = G/H$ be a reductive homogeneous Riemannian space with Lie subspace $\mathfrak{m}$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathfrak{m}$ induced from the Riemannian metric g of M.

Let us introduce a bilinear map $\mathsf{U}_{\mathfrak{m}} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by

$$2\langle \mathsf{U}_{\mathfrak{m}}(X, Y), Z\rangle = \langle [Z, X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle. \tag{4.1}$$

The Levi-Civita connection ∇ at the origin o is given by:

$$(\nabla_{X^\sharp} Y^\sharp)_o = \mathsf{U}_{\mathfrak{m}}(X, Y) - \frac{1}{2}[X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m}.$$

Example 4.1. Let us consider a Lie group G equipped with a left invariant metric. We regard G as a homogeneous space $G/\{\mathsf{e}\}$ and denote the bilinear map (4.1) by U, that is,

$$2\langle \mathsf{U}(X, Y), Z\rangle = \langle [Z, X], Y\rangle + \langle Y, [Z, X]\rangle, \quad X, Y, Z \in \mathfrak{g}. \tag{4.2}$$

Then the Levi-Civita connection ∇ of the metric is given by

$$\nabla_X Y = \mathsf{U}(X, Y) + \frac{1}{2}[X, Y], \quad X, Y \in \mathfrak{g}.$$

Proposition 4.2. *The following properties for a left invariant Riemannian metric on a Lie group G are mutually equivalent*:

1. *The metric is bi-invariant.*
2. $\mathsf{U} = 0$.
3. $\nabla = \nabla^{(0)}$.

When the metric is bi-invariant, G is represented by $G = (G \times G)/\Delta G$ as a Riemannian symmetric space with Lie subspace $\mathfrak{m}^0$ (*see Example* 4.2).

4.2. Riemannian symmetric spaces

Let (M, g) be a Riemannian manifold. At a point $p \in M$, take a linear isometry L of T_pM and a normal neighborhood $\mathcal{U}$ of p such that $\exp_p$ is defined on $L(\exp_p^{-1}(\mathcal{U}))$. Then the *polar map* Ψ_L is defined by ([182, p. 221]):

$$\Psi_L = \exp_p \circ L \circ \exp_p^{-1} : \mathcal{U} \to M. \tag{4.3}$$

The polar map ζ_p of $L = -\mathrm{I}_p$ is called the local *geodesic symmetry* (or local geodesic reflection) of M at p.

A Riemannian manifold (M, g) is said to be *locally symmetric* if its local geodesic symmetries are isometric.

Proposition 4.3. *A Riemannian manifold (M, g) is locally symmetric if and only if its Riemannian curvature R is parallel with respect to Levi-Civita connection.*

A Riemannian manifold (M, g) is said to be a *Riemannian symmetric space* if at each point $p \in M$, there is a unique isometry ζ_p with differential map $(\mathrm{d}\zeta_p)_p = -\mathrm{I}_p$ and fixes p. The isometry ζ_p is called the *global symmetry* at p and is the unique extension of local geodesic symmetry. One can see that Riemannian symmetric spaces are complete and locally symmetric.

Proposition 4.4. *A complete, simply connected locally symmetric Riemannian manifold is a Riemannian symmetric space.*

Moreover Riemannian symmetric spaces are homogeneous Riemannian spaces.

Let M be a Riemannian symmetric space with global symmetries $\{\zeta_p\}_{p\in M}$. Since M is a homogeneous Riemannian space, $G = \mathrm{Iso}_\circ(M)$ acts isometrically and transitively on M. Then we obtain an involutive Lie group automorphism σ of G by

$$\sigma(a) = \zeta_o a \zeta_o, \quad a \in G,$$

where ζ_o is the global symmetry at the origin o. The isotropy subgroup H at o satisfies $G_\sigma^\circ \subset H \subset G_\sigma$, where

$$G_\sigma = \{a \in G \mid \sigma(a) = a\}$$

and its identity component is denoted by G_σ°. The isotropy algebra $\mathfrak{h}$ is characterized by the differential map $\mathrm{d}\sigma$ as

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \mathrm{d}\sigma(X) = X\}.$$

Since $\mathrm{d}\sigma$ has eigenvalues ± 1, we get the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = \{X \in \mathfrak{g} \mid \mathrm{d}\sigma(X) = -X\}.$$

The linear subspace $\mathfrak{m}$ is identified with T_oM and the eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reductive. Thus G/H is a reductive homogeneous Riemannian space. Moreover $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ holds. This condition implies that G/H is naturally reductive with respect to $\mathfrak{m}$.

The Riemannian curvature R of a Riemannian symmetric space $M = G/K$ is described as [145, Theorem 2.2]:

$$R(X, Y)Z = -[[X, Y], Z], \quad X, Y, Z \in \mathfrak{m}.$$

We recall the following notion from O'Neill's textbook [182, p. 317]:

Definition 4.2. A triplet $(G/H, \sigma, \langle\cdot,\cdot\rangle)$ is called a *symmetric data* if

1. H is a closed subgroup of a connected Lie group G.
2. σ is an involutive Lie group automorphism of G satisfying $G_\sigma^\circ \subset H \subset G_\sigma$.
3. $\langle\cdot,\cdot\rangle$ is an $\mathrm{Ad}(H)$-invariant inner product on the (-1)-eigenspace $\mathfrak{m}$ of $\mathrm{d}\sigma : \mathfrak{g} \to \mathfrak{g}$.

We obtain a symmetric data from a Riemannian symmetric space $M = G/H$. Note that the isotropy subgroup H is compact.

Conversely from a given symmetric date, we obtain a Riemannian symmetric space $M = G/H$. On the other hand, we recall the following definition form Helgason's textbook [92, p. 209]:

Definition 4.3. A pair (G, H) is said to be a *Riemannian symmetric pair* if

1. H is a closed subgroup of a connected Lie group G.

2. σ is an involutive Lie group automorphism of G satisfying $G_\sigma^\circ \subset H \subset G_\sigma$.
3. The group $\mathrm{Ad}(H)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$.

The third condition of Definition 4.3 implies the existence of inner product satisfying the third condition of Definition 4.2. Note that the notion of Riemannian symmetric pair in the sense of Takeuchi [129, part II] is the symmetric data in the sense of O'Neill (Definition 4.2). On the other hand, Definition 4.2 is valid also for indefinite semi-Riemannian symmetric spaces.

For more information on Riemannian symmetric spaces, we refer to [92] and [145].

Example 4.2 (Lie groups)**.** Let G be a connected Lie group equipped with a *bi-invariant* metric. Then the product Lie group $G \times G$ acts transitively and isometrically on G via the action (3.2). One can see that $(G \times G)/\Delta G$ is a Riemannian symmetric space with Lie subspace $\mathfrak{m}^0$. The Levi-Civita connection coincides with $\nabla^{(0)}$ (see Proposition 4.2).

The notion of invariant connection as well as that of canonical connection are generalized to any principal bundles over homogeneous spaces G/H. One can see that canonical connections are Yang-Mills [130].

Remark 4.1. Some generalizations of local symmetry have been proposed. Here we mention two examples. A Riemannian manifold (M, g) is said to be

- *semi-symmetric* if $R \cdot R = 0$. Here $R \cdot R$ is the derivative of R by R itself;
$$(R \cdot R)(U, V, W; Y, X) = (R(X, Y)R)(U, V)W,$$
$$(R(X, Y)R)(U, V)W = R(X, Y)R(U, V)W - R(R(X, Y)U, V)W - R(U, R(X, Y)V)W - R(U, V)R(X, Y)W.$$
Obviously local symmetry implies the semi-symmetry. The tensorial equation $R \cdot R = 0$ has a clear differential geometric meaning. At a point $p \in M$ of a Riemannian manifold M, denote by $\mathfrak{ph}_p$ the linear subspace of $\mathfrak{so}(\mathrm{T}_pM)$ spanned by the set $\{R_p(X, Y) \mid X, Y \in \mathrm{T}_pM\}$. Then the semi-symmetry condition $R \cdot R = 0$ is equivalent to that $\mathfrak{ph}_p$ is a Lie subalgebra of $\mathfrak{so}(\mathrm{T}_pM)$. The connected Lie group pH_p with Lie algebra $\mathfrak{ph}_p$ is called the *primitive holonomy group* at p (in the sense of Z. I. Szabó).
- *pseudo-symmetric* (in the sense of R. Deszcz) if there exists a smooth function L such that
$$R(X, Y) \cdot R = L(X \wedge Y) \cdot R$$
for all vector fields on M.

The semi-symmetry is equivalent to the pseudo-symmetry with $L = 0$. It should be remarked that pseudo-symmetry as well as semi-symmetry do not implies the (local) homogeneity. Indeed there exist non-homogeneous examples. Pseudo-symmetric almost contact manifolds are studied in [48, 49, 51]. See also [98].

Here we would like to point out that all the eight model spaces of Thurston geometry are pseudo-symmetric. In particular, Nil_3 and $\widetilde{\mathrm{SL}}_2\mathbb{R}$ and Sol_3 are pseudo-symmetric, but not semi-symmetric.

4.3. Riemannian symmetric space representations of space forms

In this subsection we exhibit Riemannian symmetric space representations of space forms.

Example 4.3 (The Euclidean space)**.** Let $\mathbb{E}^n$ the Euclidean n-space with natural coordinates $(x_1, x_2, \ldots, x_n)$ and Euclidean inner product
$$(\boldsymbol{x}|\boldsymbol{y}) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$
The full isometry group $\mathrm{E}(n)$ is the semi-direct product group $\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$, where
$$\mathrm{O}(n) = \{A \in \mathrm{M}_n\mathbb{R} \mid {}^tAA = E_n\}$$
is a compact Lie group called the *orthogonal group*. The orthogonal group has two connected components:
$$\mathrm{O}^+(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}, \quad \mathrm{O}^-(n) = \{A \in \mathrm{O}(n) \mid \det A = -1\}.$$
The identity component of $\mathrm{O}(n)$ is
$$\mathrm{SO}(n) = \mathrm{SL}_n\mathbb{R} \cap \mathrm{O}(n) = \mathrm{O}^+(n)$$

and called the *rotation group*. Note that the Lie algebra $\mathfrak{o}(n)$ of $\mathrm{O}(n)$ coincides with the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$:

$$\mathfrak{so}(n) = \{X \in \mathrm{M}_n\mathbb{R} \mid {}^tX = -X\}.$$

The Lie group $\mathrm{E}(n)$ is called the *Euclidean group*. The Euclidean group acts isometrically and transitively on $\mathbb{E}^n$ by the action:

$$(A, \boldsymbol{b})\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{b}, \quad (A, \boldsymbol{b}) \in \mathrm{O}(n) \ltimes \mathbb{R}^n, \ \ \boldsymbol{x} \in \mathbb{E}^n.$$

The identity component $\mathrm{SE}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$ of $\mathrm{E}(n)$ is called the *Euclidean motion group*. Here we identify $\mathrm{E}(n)$ with the following linear Lie group:

$$\left\{ \left(\begin{array}{cc} A & \boldsymbol{b} \\ {}^t\mathbf{0} & 1 \end{array} \right) \middle| \ A \in \mathrm{O}(n), \ \boldsymbol{b} \in \mathbb{R}^n \right\} \subset \mathrm{GL}_{n+1}\mathbb{R}.$$

Then the isotropy subgroup of $\mathrm{E}(n)$ at $o = \mathbf{0}$ is

$$\left\{ \left(\begin{array}{cc} A & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{array} \right) \middle| \ A \in \mathrm{O}(n) \right\} \cong \mathrm{O}(n).$$

Thus we obtain the homogeneous space representation

$$\mathbb{E}^n = \mathrm{E}(n)/\mathrm{O}(n) = \mathrm{SE}(n)/\mathrm{SO}(n).$$

The Lie algebra of $\mathrm{SE}(n)$ is

$$\mathfrak{se}(n) = \left\{ \left(\begin{array}{cc} X & \boldsymbol{x} \\ {}^t\mathbf{0} & 0 \end{array} \right) \middle| \ X \in \mathfrak{so}(n), \ \boldsymbol{x} \in \mathbb{R}^n \right\}$$

The isotropy algebra is given by

$$\left\{ \left(\begin{array}{cc} X & \mathbf{0} \\ {}^t\mathbf{0} & 0 \end{array} \right) \middle| \ X \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n).$$

Let us take a linear subspace

$$\mathfrak{m} = \left\{ \left(\begin{array}{cc} O_n & \boldsymbol{x} \\ {}^t\mathbf{0} & 0 \end{array} \right) \middle| \ \boldsymbol{x} \in \mathbb{R}^n \right\}.$$

Then the decomposition $\mathfrak{se}(n) = \mathfrak{so}(n) \oplus \mathfrak{m}$ is reductive and satisfies $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{so}(n)$. Thus $\mathbb{E}^n = \mathrm{SE}(n)/\mathrm{SO}(n)$ is a Riemannian symmetric space representation of $\mathbb{E}^n$.

Example 4.4 (The sphere)**.** The orthogonal group $\mathrm{O}(n+1)$ acts isometrically and transitively on the n-sphere

$$\mathbb{S}^n(c^2) = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{E}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1/c^2\}$$

of curvature $c^2 > 0$ $(c > 0)$ via the usual matrix multiplication. The isotropy subgroup of $\mathrm{O}(n+1)$ at $o = (0, \ldots, 0, c)$ is

$$\left\{ A = \left(\begin{array}{cc} A^\circ & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{array} \right) \middle| \ A^\circ \in \mathrm{O}(n) \right\} \cong \mathrm{O}(n).$$

Moreover the isotropy subgroup of $\mathrm{SO}(n+1)$ at $o = (0, 0, \ldots, c)$ is

$$\left\{ A = \left(\begin{array}{cc} A^\circ & \mathbf{0} \\ {}^t\mathbf{0} & 1 \end{array} \right) \middle| \ A^\circ \in \mathrm{SO}(n) \right\} \cong \mathrm{SO}(n).$$

Hence we obtain homogeneous space representations

$$\mathbb{S}^n(c^2) = \mathrm{O}(n+1)/\mathrm{O}(n) = \mathrm{SO}(n+1)/\mathrm{SO}(n).$$

The Lie algebra $\mathfrak{so}(n+1)$ is parametrized as

$$\mathfrak{so}(n+1) = \left\{ X = \left(\begin{array}{cc} X^\circ & \boldsymbol{x} \\ -{}^t\boldsymbol{x} & 0 \end{array} \right) \middle| \ X^\circ \in \mathfrak{so}(n), \ \boldsymbol{x} \in \mathbb{R}^n \right\}.$$

The isotropy algebra is given by

$$\left\{ X = \left(\begin{array}{cc} X^\circ & \mathbf{0} \\ \mathbf{0} & 0 \end{array} \right) \middle| \ X^\circ \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n).$$

The Killing form $\mathsf{B}_{\mathfrak{so}(n+1)}$ is given by

$$\mathsf{B}_{\mathfrak{o}(n+1)}(X,Y)=(n-1)\mathrm{tr}(XY)$$

and negative definite on $\mathfrak{so}(n+1)$.

Let us introduce an inner product

$$\langle X,Y\rangle=-\frac{1}{2c}\mathrm{tr}(XY)=\frac{1}{c}\left\{-\frac{1}{2}\mathrm{tr}(X^{\circ}Y^{\circ})+(\boldsymbol{x}|\boldsymbol{y})\right\}$$

on $\mathfrak{so}(n+1)$. The orthogonal complement $\overline{\mathfrak{m}}$ of the isotropy algebra in $\mathfrak{so}(n+1)$ is given by

$$\overline{\mathfrak{m}}=\left\{X=\begin{pmatrix}0 & \boldsymbol{x}\\ -{}^t\boldsymbol{x} & O\end{pmatrix}\ \middle|\ \boldsymbol{x}\in\mathbb{R}^n\right\}.$$

The tangent space $\mathrm{T}_o\mathbb{S}^n(c^2)$ is expressed as

$$\mathrm{T}_o\mathbb{S}^n(c^2)=\{(\boldsymbol{x},0)\mid \boldsymbol{x}\in\mathbb{R}^n\}\subset \mathrm{T}_o\mathbb{E}^{n+1}=\mathbb{E}^{n+1}.$$

We identify the tangent space $\mathrm{T}_o\mathbb{S}^n(c^2)$ with $\overline{\mathfrak{m}}$ via the correspondence:

$$(\boldsymbol{x},0)\longmapsto\begin{pmatrix}0 & -{}^t\boldsymbol{x}\\ \boldsymbol{x} & O\end{pmatrix}.$$

The orthogonal splitting $\mathfrak{so}(n+1)=\mathfrak{so}(n)\oplus\overline{\mathfrak{m}}$ is reductive and sasisfies $[\overline{\mathfrak{m}},\overline{\mathfrak{m}}]\subset\mathfrak{so}(n)$. Thus $\mathbb{S}^n(c^2)=\mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a Riemannian symmetric space. The involution σ corresponding to $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ is

$$\sigma(A)=\mathrm{Ad}(E_{n,1})A,\quad E_{n,1}=\mathrm{diag}(1,\dots,1,-1).$$

The action of the isotropy group on $\mathfrak{m}$ is given explicitly by

$$\mathrm{Ad}\begin{pmatrix}A^{\circ} & \mathbf{0}\\ {}^t\mathbf{0} & 0\end{pmatrix}X=\begin{pmatrix}O & A^{\circ}\boldsymbol{x}\\ -{}^t(A^{\circ}\boldsymbol{x}) & 0\end{pmatrix}.$$

For any vectors $X,Y\in\overline{\mathfrak{m}}$, we have

$$[X,Y]=\begin{pmatrix}-\boldsymbol{x}^t\boldsymbol{y}+\boldsymbol{y}^t\boldsymbol{x} & A\mathbf{0}\\ {}^t\mathbf{0} & 0\end{pmatrix}.$$

Hence the Riemannian curvature is

$$R(X,Y)Z=\begin{pmatrix}O & (\boldsymbol{x}\wedge\boldsymbol{y})\boldsymbol{z}\\ -{}^t\{(\boldsymbol{x}\wedge\boldsymbol{y})\boldsymbol{z}\} & 0\end{pmatrix}.$$

This shows that

$$R(X,Y)Z=c^2(X\wedge Y)Z,\quad X,Y,Z\in\mathrm{T}_o\mathbb{S}^n(c^2).$$

Example 4.5 (The unit tangent sphere bundle of $\mathbb{S}^n$)**.** The unit tangent sphere bundle $\mathrm{U}\mathbb{S}^n$ of the unit n-sphere $\mathbb{S}^n$ is described as

$$\mathrm{U}\mathbb{S}^n=\{(\boldsymbol{x},\boldsymbol{v})\in\mathbb{E}^{n+1}\times\mathbb{E}^{n+1}\mid(\boldsymbol{x}|\boldsymbol{x})=(\boldsymbol{v}|\boldsymbol{v})=1,\ (\boldsymbol{x}|\boldsymbol{v})=0\}.$$

The rotation group acts transitively on $\mathrm{U}\mathbb{S}^n$ via the action:

$$A(\boldsymbol{x},\boldsymbol{v})=(A\boldsymbol{x},A\boldsymbol{v}).$$

The isotropy group at $(o,(0,0,\dots,0,1,0))$ is

$$\left\{A=\begin{pmatrix}A^{\circ\circ} & O_{2,n-1}\\ O_{n-1,2} & E_2\end{pmatrix}\ \middle|\ A^{\circ\circ}\in\mathrm{SO}(n-1)\right\}\cong\mathrm{SO}(n-1)$$

with Lie algebra

$$\left\{X=\begin{pmatrix}X^{\circ\circ} & O_{2,n-1}\\ O_{n-1,2} & O_2\end{pmatrix}\ \middle|\ X^{\circ\circ}\in\mathfrak{so}(n-1)\right\}\cong\mathfrak{so}(n-1).$$

With respect to the inner product on $\mathfrak{so}(n+1)$, the orthogonal complement $\mathfrak{p} = \mathfrak{so}(n-1)^{\perp}$ is given by

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} O_{n-2} & Y \\ -{}^tY & s\mathsf{J} \end{pmatrix} \middle| \ s \in \mathbb{R},\ Y \in \mathrm{M}_{2,n-1}\mathbb{R} \right\}, \quad \mathsf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2).$$

One can see that $\mathrm{U}\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n-1)$ is naturally reductive with respect to $\mathfrak{p}$. In particular, we know that

$$\mathrm{U}\mathbb{S}^2 = \mathrm{SO}(3) = \mathbb{R}P^2.$$

Namely $\mathrm{U}\mathbb{S}^2$ is identified with the rotation group $\mathrm{SO}(3)$ equipped with a bi-invariant metric of curvature $1/4$ and hence it is isometric to the real projective 3-space $\mathbb{R}P^3$ of curvature $1/4$ (see Klingenberg-Sasaki [141] and [115]).

Next, let $\mathrm{Geo}(\mathbb{S}^n)$ be the space of all oriented geodesics in $\mathbb{S}^n$. Then as is well known, $\mathrm{Geo}(\mathbb{S}^n)$ is identified with the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbb{E}^{n+1})$ of oriented 2-planes in $\mathbb{E}^{n+1}$ in the following manner:

$$\widetilde{\mathrm{Gr}}_2(\mathbb{E}^{n+1}) \ni W \longleftrightarrow W \cap \mathbb{S}^n \in \mathrm{Geo}(\mathbb{S}^n).$$

The rotation group acts transitively on $\widetilde{\mathrm{Gr}}_2(\mathbb{E}^{n+1})$ via the action:

$$A(\boldsymbol{x} \wedge \boldsymbol{v}) = (A\boldsymbol{x}) \wedge (A\boldsymbol{v}).$$

The isotropy group at the plane spanned by $o = (0, 0, \dots, 0, 1)$ and $(0, 0, \dots, 0, 1, 0)$ is

$$\left\{ A = \begin{pmatrix} A^{\circ\circ} & O_{2,n-1} \\ O_{n-1,2} & R(\theta) \end{pmatrix} \middle| \ A^{\circ\circ} \in \mathrm{SO}(n-1),\ R(\theta) \in \mathrm{SO}(2) \right\} \cong \mathrm{SO}(n-1) \times \mathrm{SO}(2).$$

Thus the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbb{E}^{n+1})$ is represented by $\widetilde{\mathrm{Gr}}_2(\mathbb{E}^{n+1}) = \mathrm{SO}(n+1)/\mathrm{SO}(n-1) \times \mathrm{SO}(2)$ as a Hermitian symmetric space. The natural projection $\mathrm{U}\mathbb{S}^n \to \mathrm{Geo}(\mathbb{S}^n) = \widetilde{\mathrm{Gr}}_2(\mathbb{E}^{n+1})$ defines a principal circle bundle

$$\mathrm{SO}(n+1)/\mathrm{SO}(n-1) \to \mathrm{SO}(n+1)/\mathrm{SO}(n-1) \times \mathrm{SO}(2).$$

Moreover, with respect to the canonical contact form of $\mathrm{U}\mathbb{S}^n$, this fibering is the Boothby-Wang fibering. By performing a suitable normalization of the almost contact Riemannian structure $\mathrm{U}\mathbb{S}^n$ becomes a homogeneous Sasakian manifold, especially it is a Sasakian φ-symmetric space (see Section 9). With respect to the normalized $\mathrm{SO}(3)$-invariant metric, $\mathrm{U}\mathbb{S}^2$ is identified with the real projective plane $\mathbb{R}P^2$ equipped with a Riemannian metric of constant curvature 1.

Example 4.6 (The hyperbolic space)**.** Here we explain the hyperboloid model of the hyperbolic space.

Let us denote by $\mathbb{E}^{1,n}$ the Minkowski $(n+1)$-space which is realized as the Cartesian $(n+1)$-space $\mathbb{R}^{n+1}$ equipped with the scalar product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

The full isometry group $\mathrm{E}(1,n)$ of $\mathbb{E}^{1,n}$ is the semi-direct product $\mathrm{O}(1,n) \ltimes \mathbb{R}^{n+1}$, where

$$\mathrm{O}(1,n) = \{A \in \mathrm{M}_{n+1}\mathbb{R} \mid {}^tA E_{1,n} A = E_{1,n}\}, \quad E_{1,n} = \mathrm{diag}(-1, 1, \dots, 1)$$

is called the *Lorentz group*. For a matrix $A = (a_{ij})_{0 \leq i,j \leq n} \in \mathrm{O}(1,n)$, we set

$$A_{\mathtt{Time}} = a_{00}, \quad A_{\mathtt{Space}} = (a_{ij})_{1 \leq i,j \leq n}.$$

The Lorentz group $\mathrm{O}(1,n)$ has four connected components:

$$\begin{aligned}
\mathrm{O}^{++}(1,n) &= \{A \in \mathrm{O}(1,n) \mid A_{\mathtt{Time}} > 0,\ \det A_{\mathtt{Space}} > 0\}, \\
\mathrm{O}^{+-}(1,n) &= \{A \in \mathrm{O}(1,n) \mid A_{\mathtt{Time}} > 0,\ \det A_{\mathtt{Space}} < 0\}, \\
\mathrm{O}^{-+}(1,n) &= \{A \in \mathrm{O}(1,n) \mid A_{\mathtt{Time}} < 0,\ \det A_{\mathtt{Space}} > 0\}, \\
\mathrm{O}^{--}(1,n) &= \{A \in \mathrm{O}(1,n) \mid A_{\mathtt{Time}} < 0,\ \det A_{\mathtt{Space}} < 0\}.
\end{aligned}$$

One can see that the identity component is $\mathrm{O}^{++}(1,n)$. Next we set $\mathrm{SO}(1,n) = \mathrm{SL}_{n+1}\mathbb{R} \cap \mathrm{O}(1,n)$. Then we get ([182, p. 239]):

$$\mathrm{SO}(1,n) = \mathrm{O}^{++}(1,n) \cup \mathrm{O}^{--}(1,n).$$

Moreover $\mathrm{O}^{++}(1,n) \cup \mathrm{O}^{+-}(1,n)$ and $\mathrm{O}^{++}(1,n) \cup \mathrm{O}^{-+}(1,n)$ are Lie subgroups of $\mathrm{O}(1,n)$.

The hyperbolic n-space $\mathbb{H}^n(-c^2)$ of curvature $-c^2 < 0 (c > 0)$ is realized as the connected component

$$\{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1/c^2,\ x_0 > 0\}$$

of the hyperboloid of two sheets. The full isometry group $\mathrm{Iso}(\mathbb{H}^{n+1})$ is given by ([182, Corollary 9]):

$$\mathrm{Iso}(\mathbb{H}^n) = \mathrm{O}^{++}(1,n) \cup \mathrm{O}^{+-}(1,n).$$

Thus the identity component is $\mathrm{Iso}_\circ(\mathbb{H}^n) = \mathrm{O}^{++}(1,n)$. The identity component $\mathrm{SO}^+(1,n) := \mathrm{O}^{++}(1,n)$ acts isometrically and transitively on $\mathbb{H}^n(-c^2)$. The isotropy subgroup at $o = (c, 0, \ldots, 0)$ is

$$\left\{ A = \begin{pmatrix} 1 & {}^t\mathbf{0} \\ \mathbf{0} & A^\circ \end{pmatrix} \,\middle|\, A^\circ \in \mathrm{SO}(n) \right\}$$

which is isomorphic to the rotation group $\mathrm{SO}(n)$. Thus we have a homogeneous space representation:

$$\mathbb{H}^n(-c^2) = \mathrm{SO}^+(1,n)/\mathrm{SO}(n).$$

Let us perform the Iwasawa decomposition of $\mathrm{SO}^+(1,n)$ as $\mathrm{SO}^+(1,n) = \mathrm{SO}(n)AN$, where A and N are abelian part and nilpotent part of $\mathrm{SO}^+(1,n)$, respectively. The Lie subgroup $\mathcal{S} = AN$ is solvable and acts simple transitively on $\mathbb{H}^n(-c^2)$. Thus $\mathbb{H}^n(-c^2)$ is identified with the solvable Lie group $\mathcal{S} = AN$ equipped with a left invariant metric. Note that $\dim A = 1$ and $\dim N = n - 1$.

The Lie algebra $\mathfrak{so}(1,n)$ is expressed as

$$\mathfrak{so}(1,n) = \left\{ X = \begin{pmatrix} 0 & {}^t\boldsymbol{x} \\ \boldsymbol{x} & X^\circ \end{pmatrix} \,\middle|\, X^\circ \in \mathfrak{so}(n),\ \boldsymbol{x} \in \mathbb{R}^n \right\}.$$

We introduce a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{so}(1,n)$ by

$$\langle X, Y \rangle = \frac{1}{2c}\mathrm{tr}(XY) = \frac{1}{2c}\{2\,{}^t\boldsymbol{x}\boldsymbol{y} + \mathrm{tr}(X^\circ Y^\circ)\}.$$

The isotropy algebra is

$$\left\{ X = \begin{pmatrix} 0 & {}^t\mathbf{0} \\ \mathbf{0} & X^\circ \end{pmatrix} \,\middle|\, X^\circ \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n)$$

is a non-degenerate linear subspace of $\mathfrak{so}(1,n)$. Thus we can take

$$\mathfrak{m} = \mathfrak{so}(n)^\perp = \{X \in \mathfrak{so}(1,n) \mid \langle X, Y \rangle = 0,\ \forall Y \in \mathfrak{so}(n)\} = \left\{ \begin{pmatrix} 0 & {}^t\boldsymbol{x} \\ \boldsymbol{x} & O \end{pmatrix} \,\middle|\, \boldsymbol{x} \in \mathbb{R}^n \right\}.$$

On can see that $\mathfrak{m}$ is spacelike, *i.e.*, the restriction of the scalar product on $\mathfrak{m}$ is positive definite. The orthogonal decomposition $\mathfrak{so}(1,n) = \mathfrak{so}(n) \oplus \mathfrak{m}$ is reductive. Under the identification $\mathfrak{m} = \mathrm{T}_o\mathbb{H}^n(-c^2)$, the induced inner product on $\mathfrak{m}$ coincides with the one induced from the Riemannian metric of $\mathbb{H}^n(-c^2)$. Moreover $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{so}(n)$ holds. Thus $\mathbb{H}^n(-c^2) = \mathrm{SO}(1,n)/\mathrm{O}(n) = \mathrm{SO}^+(1,n)/\mathrm{SO}(n)$ with $\mathfrak{m} = \mathfrak{so}(n)^\perp$ is a Riemannian symmetric space.

Olmos and Reggiani [176, 177] proved the uniqueness of canonical connections for hyperbolic space $\mathbb{H}^n(-c^2)$ for $n \neq 3$. More precisely represent $\mathbb{H}^n(-c^2)$ as $\mathbb{H}^n(-c^2) = \mathrm{SO}^+(1,n)/\mathrm{SO}(n)$ as a naturally reductive homogeneous space. Then the Levi-Civita connection is the only canonical connection associated to this representation. In other words, the Levi-Civita connection is the only Ambrose-Singer connection associated to the naturally reductive $\mathbb{H}^n(-c^2)$ for $n \neq 3$. The hyperbolic 3-space $\mathbb{H}^3(-c^2)$ admits exactly a line of canonical connections. Compare [176, Remark 6.1] and [177, Remark 2.5] with Abe's classification (Proposition 16.2). Homogeneous Riemannian structures on $\mathbb{H}^n$ with $n > 3$ are investigated in [40, 41, 186].

Example 4.7 (The unit tangent sphere bundle of $\mathbb{H}^n$)**.** The unit tangent sphere bundle $\mathrm{U}\mathbb{H}^n$ of the hyperbolic n-space of constant curvature -1 is described as

$$\mathrm{U}\mathbb{H}^n = \{(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{E}^{1,n} \times \mathbb{E}^{1,n} \mid -\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1,\ \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0,\ x_0 > 0\}.$$

The Lorentz group acts transitively on $\mathrm{U}\mathbb{H}^n$ via the action:

$$A(\boldsymbol{x}, \boldsymbol{v}) = (A\boldsymbol{x}, A\boldsymbol{v}).$$

The isotropy group at $(o,(0,1,\dots,0))$ is

$$\left\{ A = \begin{pmatrix} E_2 & O_{n-1,2} \\ O_{2,n-1} & A^{\circ\circ} \end{pmatrix} \,\middle|\, A^{\circ\circ} \in \mathrm{SO}(n-1) \right\} \cong \mathrm{SO}(n-1)$$

with Lie algebra

$$\left\{ X = \begin{pmatrix} O_2 & O_{n-1,2} \\ O_{2,n-1} & X^{\circ\circ} \end{pmatrix} \,\middle|\, X^{\circ\circ} \in \mathfrak{so}(n-1) \right\} \cong \mathfrak{so}(n-1).$$

With respect to the scalar product on $\mathfrak{so}(1,n)$, the orthogonal complement $\mathfrak{p} = \mathfrak{so}(n-1)^{\perp}$ is given by

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} 0 & t & {}^t\boldsymbol{x} \\ t & 0 & {}^t\boldsymbol{y} \\ \boldsymbol{x} & -\boldsymbol{y} & \boldsymbol{0} \end{pmatrix} \,\middle|\, t \in \mathbb{R},\ \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^{n-1} \right\}.$$

The scalar product on $\mathfrak{so}(1,n)$ induces an $\mathrm{SO}^+(1,n)$-invariant semi-Riemannian metric of signature $(-,-,-,+,+)$ on $\mathrm{U}\mathbb{H}^n$. One can see that $\mathrm{U}\mathbb{H}^n = \mathrm{SO}^+(1,n)/\mathrm{SO}(n-1)$ is naturally reductive with respect to $\mathfrak{p}$ (see [59]).

Next, let $\mathrm{Geo}(\mathbb{H}^n)$ be the space of all oriented geodesics in $\mathbb{H}^n$. Then as is well known, $\mathrm{Geo}(\mathbb{H}^n)$ is identified with the Grassmannian manifold $\widetilde{\mathrm{Gr}}_{1,1}(\mathbb{E}^{1,n})$ of oriented timelike planes in $\mathbb{E}^{1,n}$ in the following manner:

$$\widetilde{\mathrm{Gr}}_{1,1}(\mathbb{E}^{1,n}) \ni W \longleftrightarrow W \cap \mathbb{H}^n \in \mathrm{Geo}(\mathbb{H}^n).$$

The Lorentz group acts transitively on $\widetilde{\mathrm{Gr}}_{1,1}(\mathbb{E}^{1,n})$ via the action:

$$A(\boldsymbol{x} \wedge \boldsymbol{v}) = (A\boldsymbol{x}) \wedge (A\boldsymbol{v}).$$

The isotropy group at the plane spanned by $o = (1,0,\dots,0)$ and $(0,1,0,\dots,0)$ is

$$\left\{ A = \begin{pmatrix} A^{\circ} & O_{2,n-1} \\ O_{n-1,2} & A^{\circ\circ} \end{pmatrix} \,\middle|\, A^{\circ} \in \mathrm{SO}(1,1),\ A^{\circ\circ} \in \mathrm{SO}(n-1) \right\} \cong \mathrm{SO}(1,1) \times \mathrm{SO}(n-1).$$

Thus the Grassmannian manifold $\widetilde{\mathrm{Gr}}_{1,1}(\mathbb{E}^{1,n})$ is represented by $\widetilde{\mathrm{Gr}}_{1,1}(\mathbb{E}^{1,n}) = \mathrm{SO}^+(1,n)/\mathrm{SO}(1,1)\times\mathrm{SO}(n-1)$ as para-Kähler symmetric space of neutral signature. The natural projection $\mathrm{U}\mathbb{H}^n \to \mathrm{Geo}(\mathbb{H}^n) = \widetilde{\mathrm{Gr}}_{1,1}(\mathbb{E}^{1,n})$ defines a principal line bundle

$$\mathrm{SO}^+(1,n)/\mathrm{SO}(n-1) \to \mathrm{SO}^+(1,n)/\mathrm{SO}(1,1)\times\mathrm{SO}(n-1).$$

In contact Riemannian geometry, another Riemannian metric (Sasaki-lift metric) is introduced on $\mathrm{U}\mathbb{H}^n$. See Section 5.8 and Section 13.2.

Remark 4.2 ($\mathrm{SE}(1,1)$). In the Minkowski plane $\mathbb{E}^{1,1} = (\mathbb{R}^2(x_0,x_1), -\mathrm{d}x_0^2 + \mathrm{d}x_1^2)$, we may take a null coordinate system (u,v) defined by

$$u := x_0 + x_1, \quad v := -x_0 + x_1.$$

Then the scalar product is rewritten as $\mathrm{d}u \odot \mathrm{d}v = (\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u)/2$. With respect to the null coordinate system, the identity component of the isometry group is expressed as

$$\left\{ \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \,\middle|\, x,y,t \in \mathbb{R} \right\} \cong \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x,y,t \in \mathbb{R} \right\}. \tag{4.4}$$

4.4. *Ambrose-Singer connections*

Ambrose and Singer [5] gave an *infinitesimal characterization* of local homogeneity of Riemannian manifolds. To explain their characterization we recall the following notion:

Definition 4.4. A *homogeneous Riemannian structure* S on (M,g) is a tensor field of type $(1,2)$ which satisfies

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0. \tag{4.5}$$

Here $\tilde{\nabla}$ is a linear connection on M defined by $\tilde{\nabla} = \nabla + S$. The linear connection $\tilde{\nabla}$ is called the *Ambrose-Singer connection*.

Note that $\tilde{\nabla} R = 0$ is equivalent to the condition $\tilde{\nabla}\tilde{R} = 0$, where $\tilde{R}$ is the curvature of $\tilde{\nabla}$.

Let $(M, g) = G/H$ be a homogeneous Riemannian space. Here G is a connected Lie group acting transitively on M as a group of isometries. Without loss of generality we can assume that G acts *effectively* on M. The subgroup H is the isotropy subgroup of G at a point $o \in M$ which will be called the *origin* of M. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Then there exists a linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ which is $\mathrm{Ad}(H)$-invariant. If H is connected, then $\mathrm{Ad}(H)$-invariant property of $\mathfrak{m}$ is equivalent to the condition $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, *i.e.*, G/H is reductive. Then as we saw in the preceding section, $\mathrm{T}_p M$ is identified with $\mathfrak{m}$ via the isomorphism $\tau^\natural(p, \cdot)$:

$$\tau^\natural(p, \cdot) : \mathfrak{m} \ni X \longmapsto X^\natural_p \in \mathrm{T}_p M.$$

Then the canonical connection $\tilde{\nabla} = \nabla^{\mathrm{c}}$ is given by

$$(\tilde{\nabla}_{X^\natural} Y^\natural)_o = -([X, Y]_{\mathfrak{m}})^\natural_o, \quad X, Y \in \mathfrak{m}.$$

One can see the difference tensor field $S = \tilde{\nabla} - \nabla$ is a homogeneous Riemannian structure. Thus every homogeneous Riemannian space admits at least one homogeneous Riemannian structure.

Conversely, let (M, S) be a *simply connected* and *complete* Riemannian manifold equipped with a homogeneous Riemannian structure. Fix a point $o \in M$ and put $\mathfrak{m} = \mathrm{T}_o M$. Denote by $\tilde{R}$ the curvature of the Ambrose-Singer connection $\tilde{\nabla}$. Then the holonomy algebra $\mathfrak{h}$ of $\tilde{\nabla}$ the Lie subalgebra of the Lie algebra $\mathfrak{so}(\mathfrak{m}, g_o)$ generated by the curvature operators $\tilde{R}(X, Y)$ with $X, Y \in \mathfrak{m}$, since $\tilde{\nabla}\tilde{R} = 0$ and $\tilde{\nabla}\tilde{T} = 0$ (see [149, p. 178, B9]). Here $\tilde{T}$ is the torsion tensor field of $\tilde{\nabla}$. Now we define a Lie algebra structure on the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ by

$$\begin{aligned}
[U, V] &= UV - VU, \\
[U, X] &= U(X), \\
[X, Y] &= -\tilde{R}(X, Y) - S(X)Y + S(Y)X
\end{aligned}$$

for all $X, Y \in \mathfrak{m}$ and $U, V \in \mathfrak{h}$. The Lie algebra $\mathfrak{g}$ is the *transvection algebra* of $(M, \tilde{\nabla})$ (see [222, pp. 31–32] and [**?**]).

Now let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then M is a coset manifold $\tilde{G}/\tilde{H}$, where $\tilde{H}$ is a Lie subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{h}$. Let Γ be the set of all elements in G which act trivially on M. Then Γ is a discrete normal subgroup of $\tilde{G}$ and $G = \tilde{G}/\Gamma$ acts transitively and effectively on M as an isometry group. The isotropy subgroup H of G at o is $H = \tilde{H}/\Gamma$. Hence (M, g) is a homogeneous Riemannian space with coset space representation $M = G/H$.

Theorem 4.1 ([5])**.** *A Riemannian manifold (M, g) with a homogeneous Riemannian structure S is locally homogeneous.*

Definition 4.5. Let (M, g, S) and (M', g', S') be homogeneous Riemannian spaces with homogeneous Riemannian structures. Then (M, g, S) and (M', g', S') are said to be *isomorphic* if there exits an isometry $f : M \to M'$ satisfying $f^* S' = S$, that is

$$\mathrm{d}f(S(X)Y) = S'(\mathrm{d}f(X))\mathrm{d}f(Y)$$

for all $X, Y \in \Gamma(\mathrm{T}M)$.

Theorem 4.2 ([222])**.** *Let (M, g) be a homogenous Riemannian space and G, G' connected Lie subgroups of the identity component $\mathrm{Iso}_\circ(M, g)$ of the full isometry group acting transitively on M. Assume that the Lie algebras $\mathfrak{g}$ of G and $\mathfrak{g}'$ of G' has reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$, respectively. Then the homogeneous Riemannian structures S determined by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and S' determined by $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ are isomorphic if and only if there exits a Lie algebra isomorphism $F : \mathfrak{g} \to \mathfrak{g}'$ such that*

$$F(\mathfrak{h}) = \mathfrak{h}', \quad F(\mathfrak{m}) = \mathfrak{m}'$$

and $F|_{\mathfrak{m}}$ is a linear isometry.

Note that there exist locally homogeneous Riemannian manifolds which are *not* locally isometric to any homogeneous Riemannian spaces. Indeed Kowalski gave explicit examples in [150] (see also [189]). Bazdar and Teleman gave a reformulation of homogeneous Riemannian structure [9].

4.5. The eight classes of homogeneous Riemannian structures

Let (M, g) be a Riemannian manifold and S a homogeneous Riemannian structure on M. Then the metrical condition $\tilde{\nabla} g = 0$ for the Ambrose-Singer connection $\tilde{\nabla} = \nabla + S$ is rewritten as

$$S_\flat(X, Y, Z) + S_\flat(X, Z, Y) = 0 \tag{4.6}$$

for all vector fields X, Y and Z. Here $S_\flat$ is the covariant form of S.

Tricerri and Vanhecke [222] obtained the following decompositions of all possible types of homogeneous Riemannian structures into eight classes (Table 2):

Classes	Defining conditions
Symmetric	$S = 0$
$\mathcal{T}_1$	$S_\flat(X,Y,Z) = g(X,Y)\omega(Z) - g(Z,X)\omega(Y)$ for some 1-form ω
$\mathcal{T}_2$	$\underset{X,Y,Z}{\mathfrak{S}} S_\flat(X,Y,Z) = 0$ and $c_{12}(S_\flat) = 0$
$\mathcal{T}_3$	$S_\flat(X,Y,Z) + S_\flat(Y,X,Z) = 0$
$\mathcal{T}_1 \oplus \mathcal{T}_2$	$\underset{X,Y,Z}{\mathfrak{S}} S_\flat(X,Y,Z) = 0$
$\mathcal{T}_1 \oplus \mathcal{T}_3$	$S_\flat(X,Y,Z) + S_\flat(Y,X,Z) = 2g(X,Y)\omega(Z) - g(Z,X)\omega(Y) - g(Y,Z)\omega(X)$ for some 1-form ω
$\mathcal{T}_2 \oplus \mathcal{T}_3$	$c_{12}(S_\flat) = 0$
$\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$	no conditions

Table 2. The eight classes

Here $\underset{X,Y,Z}{\mathfrak{S}} S_\flat$ denotes the cyclic sum of $S_\flat$, *i.e.*,

$$\underset{X,Y,Z}{\mathfrak{S}} S_\flat(X,Y,Z) = S_\flat(X,Y,Z) + S_\flat(Y,Z,X) + S_\flat(Z,X,Y).$$

Next c_{12} denotes the contraction operator in $(1,2)$-entries;

$$c_{12}(S_\flat)(Z) = \sum_{i=1}^{n} S_\flat(e_i, e_i, Z),$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame field.

A homogeneous Riemannian structure S is said to be of *linear type* if it is of type $\mathcal{T}_1$ [36].

A reductive homogeneous Riemannian manifold $M = G/H$ is said to be *cyclic* ([77]) if there exists a Lie subspace $\mathfrak{m}$ satisfying

$$\underset{X,Y,Z}{\mathfrak{S}} \langle [X,Y]_{\mathfrak{m}}, Z\rangle = 0, \quad X, Y, Z \in \mathfrak{m}.$$

If in addition G is unimodular, then M is said to be *traceless cyclic*. On can see that $M = G/H$ is cyclic [resp. traceless cyclic] with respect to $\mathfrak{m}$ if and only if corresponding homogeneous Riemannian structure is of type $\mathcal{T}_1 \oplus \mathcal{T}_2$. [type $\mathcal{T}_2$]. Kowalski and Tricerri classified traceless cyclic homogeneous Riemannian spaces of dimension less than 5. Their classification is extended to cyclic homogeneous Riemannian spaces of dimension less than 5 in [76, 77].

In addition, Gadea, González-Dávila, and Oubiña proved the following fact.

Theorem 4.3 ([78]). *A homogeneous spin Riemannian manifold has a Dirac operator like that on a Riemannian symmetric spin space if and only if it is traceless cyclic.*

Early studies on homogeneous Riemannian structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$, $\mathcal{T}_3$ and $\mathcal{T}_1 \oplus \mathcal{T}_2$, we refer to [183], [184, 185] and [187], respectively.

4.6. Homogeneous Riemannian structures on Riemannian 2-manifolds

In this subsection we collect results on homogeneous Riemannian structures on homogeneous Riemannian 2-manifolds for later use.

First we recall the notion of homogeneous almost Hermitian structures.

Definition 4.6. A Hermitian manifold (N, h, J) is said be a *locally homogeneous Hermitian manifold* if for each $p, q \in N$, there exits a local holomorphic isometry which sends p to q.

Definition 4.7. A Hermitian manifold (N, h, J) is said be a *homogeneous Hermitian manifold* if there exits a Lie group G of holomorphic isometries acts transitively on M.

The following fundamental fact is due to Sekigawa and generalized by Kiričhenko for more general geometric structures (For a detailed proof of the so-called Sekigawa-Kiričhenko theorem, see [36, §2.3]).

Theorem 4.4 ([207, 142])**.** *Let $N = (G/H, h, J)$ be a homogeneous almost Hermitian manifold with Levi-Civita connection ∇ and the canonical connection $\tilde{\nabla}$. Then $S = \tilde{\nabla} - \nabla$ is a homogeneous Riemannian structure satisfying $\tilde{\nabla} J = 0$.*

Conversely, an almost Hermitian manifold (N, h, J) with a homogeneous Riemannian structure S satisfying $\tilde{\nabla} J = 0$ for $\tilde{\nabla} = \nabla + S$, then N is locally homogeneous almost Hermitian manifold.

A homogeneous Riemannian structure S satisfying $\tilde{\nabla} J = 0$ for $\tilde{\nabla} = \nabla + S$ is called a *homogeneous almost Hermitian structure*. In case N is Hermitian [resp. Kähler], a homogeneous almost Hermitian structure S is called a *homogeneous Hermitian structure* [resp. *homogeneous Kähler structure*].

Now let us turn our attention to homogeneous Riemannian 2-manifolds. The assumption *dimension 2* implies that possible types of homogeneous Riemannian structures are *trivial* or type $\mathcal{T}_1$.

Homogeneous Riemannian 2-manifolds with non-trivial homogenous Riemannian structure are classified as follows:

Theorem 4.5 ([222])**.** *Let S be a non-trivial homogeneous Riemannian structure on a Riemannian 2-manifold, then M is of constant negative curvature.*

Theorem 4.6 ([222])**.** *The only homogeneous Riemannian structures on the 2-sphere $\mathbb{S}^2(c^2)$ of curvature $c^2 > 0$ and Euclidean plane $\mathbb{E}^2$ are trivial one.*

Let us exhibit explicit expression of homogeneous Riemannian structures on 2-dimensional space forms.

Example 4.8 (The 2-sphere)**.** In Example 4.4, we exhibit the Riemannian symmetric space representation of $\mathbb{S}^n(c^2)$. Here we investigate homogeneous Riemannian structures on

$$\mathbb{S}^2(c^2) = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{E}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1/c^2\}.$$

We take a basis

$$\bar{E}_1 = E_{32} - E_{23}, \quad \bar{E}_2 = E_{13} - E_{31}, \quad \bar{E}_3 = E_{21} - E_{12}$$

of $\mathfrak{so}(3)$. Then the commutation realtions are

$$[\bar{E}_1, \bar{E}_2] = \bar{E}_3, \quad [\bar{E}_2, \bar{E}_3] = \bar{E}_1, \quad [\bar{E}_3, \bar{E}_1] = \bar{E}_2.$$

The Lie algebra $\mathfrak{so}(3)$ is parametrized as

$$\left\{ \left(\begin{array}{rrr} 0 & -y_3 & y_2 \\ x_3 & 0 & -y_1 \\ -y_2 & y_2 & 0 \end{array} \right) \middle| \; y_1, y_2, y_3 \in \mathbb{R} \right\}.$$

The isotropy subgroup $\overline{H}$ of $\mathrm{SO}(3)$ at $\bar{o} = (0, 0, c)$ is

$$\overline{H} = \left\{ \left(\begin{array}{rrr} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array} \right) \middle| \; 0 \leq \theta < 2\pi \right\} \cong \mathrm{SO}(2).$$

The isotropy algebra is

$$\bar{\mathfrak{h}} = \left\{ \left(\begin{array}{rrr} 0 & -y_3 & 0 \\ y_3 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| \; y_3 \in \mathbb{R} \right\} \cong \mathfrak{so}(2).$$

Let us look for Lie subspace. Since $\overline{\mathfrak{h}}$ is spanned by $\bar{E}_3$, the only possible Lie subspace $\overline{\mathfrak{m}}$ is

$$\overline{\mathfrak{m}} = \left\{ \left(\begin{array}{ccc} 0 & 0 & y_2 \\ 0 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{array} \right) \middle| \; y_1, y_2 \in \mathbb{R} \right\}.$$

One can see that $[\overline{\mathfrak{m}}, \overline{\mathfrak{m}}] \subset \overline{\mathfrak{h}}$. We identify the tangent space

$$\mathrm{T}_o\mathbb{S}^2(c^2) = \{(y_1, y_2, 0) \mid y_1, y_2 \in \mathbb{R}\}$$

via the correspondence

$$(y_1, y_2, 0) \longmapsto \left(\begin{array}{ccc} 0 & 0 & y_2 \\ 0 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{array} \right).$$

Thus $\mathbb{S}^2(c^2) = \mathrm{SO}(3)/\mathrm{SO}(2)$ is a Riemannian symmetric space. This fact confirms that the only homogeneous Riemannian structure of $\mathbb{S}^2(c^2)$ is the trivial one $S = 0$.

On the tangent space

$$\mathrm{T}_o\mathbb{S}^2(c^2) \cong \{y_1\bar{E}_1 + y_2\bar{E}_2 \mid y_1, y_2 \in \mathbb{R}\}$$

Next we define a linear endomorphism J_o on $\overline{\mathfrak{m}}$ by

$$J\bar{E}_1 = \bar{E}_2, \quad J\bar{E}_2 = -\bar{E}_2.$$

Then J gives an orthogonal complex structure on $\mathbb{S}^2(c^2)$. The resulting Kähler manifold $(\mathbb{S}^2, g, J)$ is Hermitian symmetric. If we regard $\mathrm{T}_o\mathbb{S}^2(c^2)$ as a linear subspace of $\mathbb{E}^3$, then the complex structure J is rewritten as

$$J_o(\boldsymbol{y}, 0) = (0, 0, 1) \times (\boldsymbol{y}, 0).$$

Here $\times$ is the vector product of $\mathbb{E}^3$ with respect to the orientation determined by $\mathrm{d}v = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Remark 4.3. The Cartesian 3-space $\mathbb{R}^3$ is isomorphic to the Lie algebra $\mathfrak{so}(3)$ via the isomorphism $\iota : \mathbb{R}^3 \to \mathfrak{so}(3)$;

$$\iota(\boldsymbol{x}) = \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \left(\begin{array}{ccc} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_2 & 0 \end{array} \right) =: X. \tag{4.7}$$

Then we have

$$\iota(\boldsymbol{x} \times \boldsymbol{y})\boldsymbol{z} = (\boldsymbol{x} \wedge \boldsymbol{y})\boldsymbol{z} = \boldsymbol{z} \times (\boldsymbol{x} \times \boldsymbol{y}),$$
$$\iota(\boldsymbol{x} \times \boldsymbol{y}) = [\iota(\boldsymbol{x}), \iota(\boldsymbol{y})].$$

Hence we have

$$[X, Y]\boldsymbol{z} = (\boldsymbol{x} \wedge \boldsymbol{y})\boldsymbol{z}, \quad X = \iota(\boldsymbol{x}),\ Y = \iota(\boldsymbol{y}).$$

The Euclidean inner product $(\cdot|\cdot)$ corresponds to the inner product

$$(X|Y) = -\frac{1}{2}\mathrm{tr}(XY), \quad X, Y \in \mathfrak{so}(3).$$

Under the identification $\mathbb{E}^3 = \mathfrak{so}(3)$, $\mathbb{S}^2(c^2)$ is identified with

$$\{X \in \mathfrak{so}(3) \mid \mathrm{tr}(X^2) = -2c^2\}.$$

The action of $\mathrm{SO}(3)$ on $\mathbb{S}^2(c^2)$ is rewritten as

$$\mathrm{Ad} : \mathrm{SO}(3) \times \mathbb{S}^3(c^2) \to \mathbb{S}^3(c^2); \quad (A, X) \longmapsto \mathrm{Ad}(A)X = AXA^{-1}.$$

Example 4.9 (Euclidean plane)**.** On the Euclidean plane $\mathbb{E}^2$, the Euclidean motion group $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ acts isometrically and transitively. The isotropy subgroup $\overline{H}$ at $\bar{o} = (0, 0)$ is $\mathrm{SO}(2)$. The Lie algebra $\mathfrak{se}(2)$ and the isotropy algebra $\overline{\mathfrak{h}}$ are given by

$$\mathfrak{se}(2) = \left\{ \left(\begin{array}{ccc} 0 & u_3 & u_1 \\ -u_3 & 0 & u_2 \\ 0 & 0 & 0 \end{array} \right) \middle| \; u_1, u_2, u_3 \in \mathbb{R} \right\}, \; \overline{\mathfrak{h}} = \left\{ \left(\begin{array}{ccc} 0 & u_3 & 0 \\ -u_3 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| \; u_3 \in \mathbb{R} \right\}.$$

The only possible Lie subspace $\overline{\mathfrak{m}}$ is

$$\overline{\mathfrak{m}} = \left\{ \left(\begin{array}{ccc} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{array} \right) \middle| \ u_1, u_2, u_3 \in \mathbb{R} \right\}.$$

Thus $\mathrm{SE}(2)/\mathrm{SO}(2)$ is a Riemannian symmetric space corresponding to the trivial homogeneous Riemannian structure. The Lie subspace $\overline{\mathfrak{m}}$ is a Lie subalgebra of $\mathfrak{se}(2)$. The corresponding simply connected Lie group is

$$\left\{ \left(\begin{array}{ccc} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \ x, y, z \in \mathbb{R} \right\}$$

which is isomorphic and isometric to the abelian Lie group $(\mathbb{E}^2(x,y), +)$. Thus $\mathbb{E}^2$ is interpreted also as $\mathbb{E}^2/\{\mathbf{0}\}$. The homogeneous Riemannian structure of $\mathbb{E}^2/\{\mathbf{0}\}$ is the trivial one $S = 0$. The transvection algebra determined by $S = 0$ is $\overline{\mathfrak{m}}$. Thus $S = 0$ corresponds to both $\mathrm{SE}(2)/\mathrm{SO}(2)$ and $\mathbb{E}^2/\{\mathbf{0}\}$.

Remark 4.4. Let G be a Lie group equipped with a bi-invariant Riemannian metric. Then G is a homogeneous space $(G \times G)/\Delta G$ under the action (3.2). However, in case G is abelian, we need to pay attention to the homogeneous space representation of G. Indeed if $G = (\mathbb{E}^2, +)$, then we the action (3.2) is described as

$$(\boldsymbol{a}, \boldsymbol{b}) \cdot \boldsymbol{x} = \boldsymbol{a} + \boldsymbol{x} - \boldsymbol{b}.$$

However we know that $\dim \mathrm{Iso}_{\circ}(\mathbb{E}^2) = 3$. On the other hand, $\dim(\mathbb{E}^2 \times \mathbb{E}^2) = 4$. Thus $(\mathbb{E}^2 \times \mathbb{E}^2)/\Delta\mathbb{E}^2$ is not the correct homogeneous space representation for $(\mathbb{E}^2, +)$. In case if we regard $\mathbb{E}^2$ as an abelian Lie group, the correct homogeneous space representation is $\mathbb{E}^2 = \mathbb{E}^2/\{\mathbf{0}\}$.

Here we exhibit homogeneous Riemannian structures on the hyperbolic plane $\mathbb{H}^2(-c^2)$ of curvature $-c^2 < 0$. We realize $\mathbb{H}^2(-c^2)$ as the upper half plane:

$$\mathbb{H}^2(-c^2) = (\{(x,y) \in \mathbb{R}^2 \mid y > 0\}, \bar{g}) = (\{x + \sqrt{-1}y \in \mathbb{C} \mid y > 0\}, \bar{g})$$

equipped with the Poincaré metric

$$\bar{g} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{c^2 y^2}.$$

We can take a globally defined orthonormal frame field

$$\bar{e}_1 = (cy)\frac{\partial}{\partial x}, \quad \bar{e}_2 = (cy)\frac{\partial}{\partial y}.$$

Th dual coframe field $\bar{\Theta} = \{\bar{\vartheta}^1, \bar{\vartheta}^2\}$ is given by

$$\bar{\vartheta}^1 = \frac{\mathrm{d}x}{cy}, \quad \bar{\vartheta}^2 = \frac{\mathrm{d}y}{cy}.$$

Since

$$\mathrm{d}\bar{\vartheta}^1 = c\,\bar{\vartheta}^1 \wedge \bar{\vartheta}^2, \quad \mathrm{d}\bar{\vartheta}^2 = 0,$$

the connection form ω and curvature form Ω relative to $\bar{\Theta}$ are given by

$$\omega = \left(\begin{array}{cc} 0 & -c\,\bar{\vartheta}^1 \\ c\,\bar{\vartheta}^1 & 0 \end{array} \right), \quad \Omega = \left(\begin{array}{cc} 0 & -c^2\,\bar{\vartheta}^1 \wedge \bar{\vartheta}^2 \\ c^2\,\bar{\vartheta}^1 \wedge \bar{\vartheta}^2 & 0 \end{array} \right).$$

The Levi-Civita connection $\overline{\nabla}$ of $\bar{g}$ is computed as

$$\overline{\nabla}_{\bar{e}_1}\bar{e}_1 = c\,\bar{e}_2, \quad \overline{\nabla}_{\bar{e}_1}\bar{e}_2 = -c\,\bar{e}_1, \quad \overline{\nabla}_{\bar{e}_2}\bar{e}_1 = \overline{\nabla}_{\bar{e}_2}\bar{e}_2 = 0.$$

The complex structure J is determined by

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

Then the Kähler form $\bar{\Omega} = \bar{g}(\cdot, J)$ is given by

$$\bar{\Omega} = -2\,\bar{\vartheta}^1 \wedge \bar{\vartheta}^2 = -\frac{2\mathrm{d}x \wedge \mathrm{d}y}{c^2y^2}.$$

Note that the area element is

$$\mathrm{d}v_g = 2\,\bar{\vartheta}^1 \wedge \bar{\vartheta}^2 = -\bar{\Omega}.$$

The Kähler form is an exact 2-form with potential

$$\bar{\omega} = -\frac{2}{c}\vartheta^1 = -\frac{2\mathrm{d}x}{c^2y}.$$

The resulting almost Hermitian manifold $(\mathbb{H}^2(-c^2), \bar{g}, J)$ is Kähler.

The special linear group

$$\mathrm{SL}_2\mathbb{R} = \left\{ \mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle|\, a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R},\ a_{11}a_{22} - a_{12}a_{21} = 1 \right\}$$

acts isometrically, holomorphically and transitively on $\mathbb{H}^2(-c^2)$ as the linear fractional transformation group. The *linear fractional action* $\mathrm{T} : \mathrm{SL}_2\mathbb{R} \times \mathbb{H}^2(-c^2) \to \mathbb{H}^2(-c^2)$ is described as

$$\mathrm{T}_{\mathsf{A}}(x + \sqrt{-1}y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot (x + \sqrt{-1}y) = \frac{a_{11}(x + \sqrt{-1}y) + a_{12}}{a_{21}(x + \sqrt{-1}y) + a_{22}}.$$

The isotropy subgroup at $\sqrt{-1} = (0, 1)$ is the rotation group

$$K = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle|\, 0 \leq \theta < 2\pi \right\}.$$

The Lie algebra $\mathfrak{sl}_2\mathbb{R}$ is given by

$$\mathfrak{sl}_2\mathbb{R} = \left\{ X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathrm{M}_2\mathbb{R} \,\middle|\, \mathrm{tr}\, X = x_{11} + x_{22} = 0 \right\}$$

and spanned by the *split-quaternionic basis* ([94, 96]):

$$\boldsymbol{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{j}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{k}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

The isotropy algebra is $\mathfrak{k} = \mathbb{R}\boldsymbol{i}$. The tangent space $\mathrm{T}_{\bar{o}}\mathbb{H}^2(-c^2)$ at the origin $\bar{o} = (0, 1)$ is identified with

$$\overline{\mathfrak{m}}_0 = \mathbb{R}\boldsymbol{j}' \oplus \mathbb{R}\boldsymbol{k}'.$$

One can see that $[\overline{\mathfrak{m}}_0, \overline{\mathfrak{m}}_0] \subset \mathfrak{k}$. Hence $\mathbb{H}^2(-c^2) = \mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2)$ equipped with $\overline{\mathfrak{m}}_0$ is a Riemannian symmetric space. Moreover it is Hermitian symmetric. The corresponding homogeneous Riemannian structure is $\bar{S} = 0$.

The hyperbolic plane $\mathbb{H}^2(-c^2)$ admits a non-trivial homogeneous Riemannian structure. To describe it, here we recall the Iwasawa decomposition of $\mathrm{SL}_2\mathbb{R}$.

The *Iwasawa decomposition* of the special linear group $\mathrm{SL}_2\mathbb{R}$ is given explicitly by $\mathrm{SL}_2\mathbb{R} = NAK$, where

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \middle|\, y > 0 \right\}$$

and $K = \mathrm{SO}(2)$ as mentioned before. Next the *polar decomposition* of $\mathrm{SL}_2\mathbb{R}$ is the Lie group decomposition

$$\mathrm{SL}_2\mathbb{R} = \overline{\mathcal{S}} \cdot \mathrm{SO}(2), \quad \overline{\mathcal{S}} = NA = \left\{ \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \middle|\, x \in \mathbb{R},\ y > 0 \right\}.$$

On the solvable part $\overline{\mathcal{S}} = NA$ of $\mathrm{SL}_2\mathbb{R}$, the Poincaré metric is left invariant on $\overline{\mathcal{S}}$. Hence $\overline{\mathcal{S}}$ is identified with $\mathbb{H}^2(-c^2)$. Let us describe the action of $\mathrm{SL}_2\mathbb{R}$ on $\overline{\mathcal{S}}$.

Proposition 4.5 (Dressing action)**.** The special linear group $\mathrm{SL}_2\mathbb{R}$ acts on $\overline{\mathcal{S}}$ by the action:

$$\mathrm{SL}_2\mathbb{R}\times\overline{\mathcal{S}}\to\overline{\mathcal{S}};\quad a\cdot \mathsf{z}:=(a\mathsf{z})_{\mathcal{S}},\quad a\in\mathrm{SL}_2\mathbb{R},\ \mathsf{z}\in\overline{\mathcal{S}}.$$

Here we decompose $a\mathsf{z}$ as $a\mathsf{z}=(a\mathsf{z})_{\mathrm{SO}(2)}(a\mathsf{z})_{\overline{\mathcal{S}}}$ along the polar decomposition $\mathrm{SL}_2\mathbb{R}=\mathrm{SO}(2)\cdot\overline{\mathcal{S}}$.

Here we describe the dressing action in detail. By performing the polar decomposition, we obtain

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} \sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}=\begin{pmatrix} a_{11}\sqrt{y} & (a_{11}x+b)/\sqrt{y}\\ a_{21}\sqrt{y} & (a_{21}x+d)/\sqrt{y}\end{pmatrix}=\begin{pmatrix} \sqrt{\tilde{y}} & \tilde{x}/\sqrt{\tilde{y}}\\ 0 & 1/\sqrt{\tilde{y}}\end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix},$$

where

$$\tilde{x}=\frac{(a_{11}x+a_{12})(a_{21}x+a_{22})+a_{11}a_{21}y^2}{(a_{21}x+a_{22})^2+(a_{21}y)^2},\quad \tilde{y}=\frac{y}{(a_{21}x+a_{22})^2+(a_{21}y)^2},\quad e^{i\phi}=\frac{a_{21}x+a_{22}-\sqrt{-1}a_{21}y}{\sqrt{(a_{21}x+a_{22})^2+(a_{21}y)^2}\sqrt{y}}.$$

One can see that the complex coordinate $x+\sqrt{-1}y$ and $\tilde{x}+\sqrt{-1}\tilde{y}$ are related by the linear fractional transformation:

$$\tilde{x}+\sqrt{-1}\tilde{y}=\frac{a_{11}(x+\sqrt{-1}y)+a_{12}}{a_{21}(x+\sqrt{-1}y)+a_{22}}.$$

The tangent space $\mathrm{T}_{\bar{o}}\mathbb{H}^2(-c^2)$ is identified with the Lie algebra $\bar{\mathfrak{s}}$ of $\overline{\mathcal{S}}$;

$$\bar{\mathfrak{s}}=\left\{\begin{pmatrix} u_2 & u_1\\ 0 & -u_2\end{pmatrix}\right\}$$

Thus we have a decomposition

$$\begin{pmatrix} v_1 & v_2\\ v_3 & -v_1\end{pmatrix}=\begin{pmatrix} 0 & -v_3\\ -(-v_3) & 0\end{pmatrix}+\begin{pmatrix} v_1 & v_2+v_3\\ 0 & -v_1\end{pmatrix}$$

along $\mathfrak{sl}_2\mathbb{R}=\mathfrak{k}+\bar{\mathfrak{s}}$. Note that

$$\bar{e}_1=cy\frac{\partial}{\partial x}\Big|_{\bar{o}}=\begin{pmatrix} 0 & c\\ 0 & 0\end{pmatrix},\quad \bar{e}_2=cy\frac{\partial}{\partial y}\Big|_{\bar{o}}=\begin{pmatrix} c/2 & 0\\ 0 & -c/2\end{pmatrix}.$$

The solvable Lie part $\overline{\mathcal{S}}$ is isomorphic to the following solvable Lie group

$$\overline{\mathcal{M}}=\left\{\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\ \middle|\ x,y\in\mathbb{R},\ y>0\right\}\tag{4.9}$$

via the Lie group isomorphism

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\longmapsto\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}.$$

The Lie algebra of $\overline{\mathcal{M}}$ is

$$\overline{\mathfrak{m}}=\left\{\begin{pmatrix} v & u\\ 0 & 0\end{pmatrix}\ \middle|\ u,v\in\mathbb{R}\right\}.$$

The Poincaré metric $\bar{g}$ is a left invariant metric on $\overline{\mathcal{M}}$. Moreover $\{\bar{e}_1,\bar{e}_2\}$ is a global left invariant orthonormal frame field on $\overline{\mathcal{M}}$. At the origin, we have

$$\bar{e}_1\big|_o=\begin{pmatrix} 0 & c\\ 0 & 0\end{pmatrix},\quad \bar{e}_2\big|_o=\begin{pmatrix} c & 0\\ 0 & 0\end{pmatrix}.$$

The homogeneous Riemannian structures on $\mathbb{H}^2(-c^2)$ are classified as follows (see also [80, Corollary 3.3]):

Proposition 4.6 ([222])**.** *Up to isomorphisms, there are two homogeneous Riemannian structures on* $\mathbb{H}^2(-c^2)$*:*

- $\bar{S}=0$*: The corresponding coset space representation is*

$$\mathbb{H}^2=\mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2)=\mathrm{SO}^+(1,2)/\mathrm{SO}(2).$$

- *The nontrivial homogeneous structure $\bar{S}$ of type $\mathcal{T}_1$ given by*

$$\bar{S}(X)Y = -c\{\bar{g}(X,Y)\bar{e}_2 - \bar{g}(\bar{e}_2,Y)X\} = -c\bar{g}(X,\bar{e}_1)JY.$$

The corresponding coset space representation is the solvable Lie group model $\mathcal{S} = \mathcal{S}/\{E_2\}$ of $\mathbb{H}^2(-c^2)$. The canonical connection $\nabla + \bar{S}$ coincides with Cartan-Schouten's $(-)$-connection.

One can check that the homogeneous Riemannian structure $\bar{S}$ of type $\mathcal{T}_1$ satisfies $\bar{S}(X)JY + J\bar{S}(X)Y = 0$. Thus we get the following result (see [80, Corollary 3.3]):

Corollary 4.1. *All the homogeneous Riemannian structures of $\mathbb{H}^2(-c^2)$ are homogeneous Kähler structures.*

Since the area form $\mathrm{d}v_{\bar{g}}$ of $\mathbb{H}^2(-c^2)$ is given by $-\bar{\Omega}$, the covariant form of the homogeneous Riemannian structure $\bar{S}$ of type $\mathcal{T}_1$ is rewritten as

$$\bar{S}_\flat = -2c\,\bar{\vartheta}^1 \otimes (\bar{\vartheta}^1 \wedge \bar{\vartheta}^2) = c\bar{\vartheta}^1 \otimes \bar{\Omega} = -c\,\bar{\vartheta}^1 \otimes \mathrm{d}v_{\bar{g}} = \frac{c^2}{2}\,\bar{\omega} \otimes \mathrm{d}v_{\bar{g}}. \tag{4.10}$$

From the proof of [222, Theorem 4.1], we deduce the following result.

Proposition 4.7. *Let M be a Riemannian 2-manifold which posses a non-trivial homogeneous Riemannian structure $\bar{S}$, then $\bar{S}$ is represented as $\bar{S}_\flat = -2c\,\bar{\vartheta}^1 \otimes (\bar{\vartheta}^1 \wedge \bar{\vartheta}^2)$ with respect to a suitable local orthonormal coframe field $\Theta = (\vartheta^1, \vartheta^2)$ and the Gaussian curvature is $K = -c^2 < 0$.*

4.7. Homogeneous Riemannian structures of type $\mathcal{T}_3$ on 3-dimensional space forms

Let (M,g) be a Riemannian 3-manifold. Take a $S_\flat$ be a tensor field of type $(0,3)$ satisfying

$$S_\flat(X,Y,Z) + S_\flat(X,Z,Y) = 0. \tag{4.11}$$

Then $S_\flat$ is expressed as

$$S_\flat(X,Y,Z) = 2\{S_\flat(X,e_1,e_2)(\vartheta^1 \wedge \vartheta^2) + S_\flat(X,e_2,e_3)(\vartheta^2 \wedge \vartheta^3) + S_\flat(X,e_3,e_1)(\vartheta^3 \wedge \vartheta^1)\}(Y,Z). \tag{4.12}$$

We define a tensor field S by

$$S_\flat(X,Y,Z) = g(S(X)Y,Z). \tag{4.13}$$

By using S, we define a linear connection $\tilde{\nabla}$ by $\tilde{\nabla} = \nabla + S$. Then we have the following useful lemma.

Lemma 4.1. *Let $S_\flat$ be a tensor field of type $(0,3)$ on a Riemannian 3-manifold. Define a tensor field S of type $(1,2)$ by (4.13) and set $\tilde{\nabla} = \nabla + S$, then $\tilde{\nabla}$ satisfies $\tilde{\nabla}R = 0$ if and only if*

$$(\nabla_X \mathrm{Ric})(Y,Z) = \mathrm{Ric}(S(X)Y,Z) + \mathrm{Ric}(Y,S(X)Z) \tag{4.14}$$

holds.

Tricerri and Vanhecke classified homogeneous Riemannian structures of type $\mathcal{T}_3$ on 3-dimensional space forms [222, Theorem 6.3].

Theorem 4.7. *Let $M^3(\varepsilon c^2) = (M^3, g, \mathrm{d}v_g)$ be the one of the following 3-dimensional space forms of constant curvature εc^2 ($c > 0$, $\varepsilon = 0, \pm 1$):*

- $\varepsilon = 1:\quad \mathbb{S}^3(c^2)$.
- $\varepsilon = 0:\quad \mathbb{E}^3$.
- $\varepsilon = -1:\quad \mathbb{H}^3(-c^2)$.

Then all the non-vanishing homogeneous Riemannian structures of type $\mathcal{T}_3$ on $M^3(\varepsilon c^2)$ are given by the following 1-parameter family:

$$S_\flat^\lambda = -\lambda\,\mathrm{d}v_g, \quad \lambda \in \mathbb{R}^\times.$$

Two homogeneous Riemannian structures $S_\flat^{\lambda_1}$ and $S_\flat^{\lambda_2}$ are isomorphic each other if and only if $\lambda_1 = \pm\lambda_2$.

4.8. Naturally reductive homogeneous spaces

A reductive homogeneous Riemannian space $M = G/H$ with Lie subspace $\mathfrak{m}$ is said to be *naturally reductive* with respect to $\mathfrak{m}$ if the geodesic through $o \in M$ and tangent to $X \in \mathfrak{m} = \mathrm{T}_oM$ is the orbit $\{\exp(tX) \cdot o\}_{t\in\mathbb{R}}$ of the one-parameter subgroup $\{\exp(tX)\}_{t\in\mathbb{R}}$ for all X. Note that every Riemannian symmetric space $M = G/H$ satisfies this property.

The following infinitesimal reformulation of natural reducibility is useful [145].

Proposition 4.8. *A reductive homogeneous Riemannian space $M = G/H$ with Lie subspace $\mathfrak{m}$ is naturally reductive with respect to $\mathfrak{m}$ if and only if the bilinear map $\mathsf{U}_{\mathfrak{m}}$ defined by (4.1) vanishes.*

Definition 4.8. A reductive homogeneous Riemannian space $M = G/H$ is said to be *naturally reductive* if $M = G/H$ is reductive and has a Lie subspace $\mathfrak{m}$ with respect to which G/H is naturally reductive.

Naturally reductive spaces may be regarded as generalizations of Riemannian symmetric spaces. For instance, as we mentioned above, every geodesic through $o \in M$ and tangent to $X \in \mathfrak{m} = \mathrm{T}_oM$ is the orbit $\{\exp(tX) \cdot o\}_{t\in\mathbb{R}}$ of the one-parameter subgroup $\{\exp(tX)\}_{t\in\mathbb{R}}$ for all X. Next, every (local) geodesic symmetry is volume preserving up to sign. For more information on naturally reductive homogeneous spaces, we refer to [85, 55].

Remark 4.5. There are several attempts to generalize the class of naturally reductive homogeneous spaces. One of the generalization is the class of Riemannian g. o. space [154]. On the other hand, studies on Riemannian manifolds with volume preserving local geodesic symmetries (up to sign) was initiated by D'Atri and Nickerson [53, 54]. Those spaces are called *D'Atri spaces*. See [194]. Another attempt was proposed by Kowalski and Vanhecke [153].

Tricerri and Vanhecke obtained the following characterizations.

Theorem 4.8 ([222], [223])**.** *Let $M = G/H$ be a naturally reductive homogeneous space with canonical connection $\tilde{\nabla}$. Then $S = \tilde{\nabla} - \nabla$ is a homogeneous Riemannian structure which satisfies*

$$S(X)X = 0, \quad X \in \Gamma(\mathrm{T}M). \tag{4.15}$$

Namely S is of type $\mathcal{T}_3$. Conversely, a simply connected and complete Riemannian manifold (M, g, S) together with a homogeneous structure S is naturally reductive if and only if S satisfies (4.15)*.*

Theorem 4.9 ([222])**.** *Let (M, g) be a complete and simply connected Riemannian manifold. Then M is a naturally reductive homogeneous space if and only if there exists a homogeneous Riemannian structure S so that $\tilde{\nabla} = \nabla + S$ is projectively equivalent to the Levi-Civita connection ∇.*

Three-dimensional simply connected and connected naturally reductive homogeneous spaces are classified by Tricerri and Vanhecke [222]. See also Theorem 15.1 of the present article.

4.9. Canonical connections on general Riemannian manifolds

Olmos and Sánchez [178] introduced the notion of "canonical connection" in the following manner:

Definition 4.9. Let (M, g) be a Riemannian manifold with Levi-Civita connection. A metric connection D is said to be a *canonical connection* in the sense of Olmos-Sánchez if it satisfies $DS = 0$, where $S = D - \nabla$.

Obviously, any Ambrose-Singer connections are canonical connections in the sense of Olmos-Sánchez. Under this definition, they proved the following result.

Theorem 4.10 ([178])**.** *Let M be linearly full immersed submanifold of Euclidean space $\mathbb{E}^n$ with vector valued second fundamental form α. Then the following properties are mutually equivalent*:

1. *M admits a canonical connection in the sense of Olmos-Sánchez satisfying $D\alpha = 0$.*
2. *M is an (extrinsically) homogeneous submanifold with constant principal curvatures.*
3. *M is an orbit of an s-representation. Namely M is a standardly embedded symmetric R-spaces.*

4.10. Cartan-Riemannian manifolds

In [62], a Riemannian manifold (M, g, D) equipped with a metric linear connection D is called a *Cartan-Riemannian manifold*. Note that the authors of [84] use the terminology Riemann-Cartan manifold. Obviously, a Riemannian manifold $(M, g, \nabla + S)$ together with a homogeneous Riemannian structure S is a Cartan-Riemannian manifold.

Proposition 4.9. *Let (M, g, D) be a Cartan-Riemannian manifold. The linear connection has the same geodesics as the Levi-Civita connection if and only if the difference tensor field $S := D - \nabla$ is symmetric.*

Note that Theorem 4.9 is a special case of Proposition 4.9.

Let us denote by T^D the torsion tensor field of D and set $S = D - \nabla$, then by the definition of torsion, we have

$$T^D(X, Y) = D_X Y - D_Y X - [X, Y] = S(X)Y - S(Y)X.$$

The covariant tensor field $T^D_\flat$ is called the *torsion form* of (M, g, D).

Definition 4.10. A Cartan-Riemannian manifold (M, g, D) is said to be a *Cartan-Riemannian manifold of totally skew-symmetric torsion* if $T^D_\flat$ is a 3-form on M. If in addition, D satisfies $DT^D = 0$, then M is said to be a *Cartan-Riemannian manifold of parallel totally skew-symmetric torsion*.

If a Cartan-Riemannian manifold (M, g, D) has totally skew-symmetric torsion and D is a canonical connection in the sense of Olmos-Sánchez, then (M, g, D) has parallel totally skew-symmetric torsion.

Differential geometry of Cartan-Riemannian manifold of totally skew-symmetric torsion is motivated by string theory and special geometry. For more information, we refer to [2, 52, 71, 73, 74].

4.11. Sekigawa's theorem

Sekigawa proved the following fundamental fact for homogeneous Riemannian 3-spaces.

Theorem 4.11 ([204])**.** *Let M be a homogeneous Riemannian 3-space, then M is locally symmetric or M is a Lie group equipped with a left invariant metric.*

On the other hand Lie groups equipped with a left invariant metric are classified by Milnor [163]. See also [86, 147].

Sekigawa gave a list of all homogeneous Riemannian 3-spaces in [205, 206].

4.12. Three dimensional unimodular Lie groups

Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. A Lie group G is said to be *unimodular* if its left invariant Haar measure is right invariant. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups [163]. We recall it briefly here.

Let $\mathfrak{g}$ be a 3-dimensional *oriented* Lie algebra with the inner product $\langle \cdot, \cdot \rangle$. Denote by $\times$ the *vector product operation* of the oriented inner product space $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The vector product operation is a skew-symmetric bilinear map $\times : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is uniquely determined by the following conditions (*cf.* Section 2.5):

(i) $\langle X, X \times Y \rangle = \langle Y, X \times Y \rangle = 0$,
(ii) $\langle X \times Y, X \times Y \rangle = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2$,
(iii) if X and Y are linearly independent, then $\det(X, Y, X \times Y) > 0$,

for all $X, Y \in \mathfrak{g}$. On the other hand, the Lie-bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $L_{\mathfrak{g}}$ which is uniquely determined by the formula

$$[X, Y] = L_{\mathfrak{g}}(X \times Y), \quad X, Y \in \mathfrak{g}.$$

Now let G be an oriented 3-dimensional Lie group equipped with a left invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G, the endomorphism field $L_{\mathfrak{g}}$ is uniquely determined. The unimodularity of G is characterized as follows.

Proposition 4.10 ([163])**.** *Let G be an oriented 3-dimensional Lie group with a left invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $L_{\mathfrak{g}}$ is self-adjoint with respect to the metric.*

Let G be a 3-dimensional unimodular Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ (called a *unimodular basis*) of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \qquad c_1, c_2, c_3 \in \mathbb{R}. \tag{4.16}$$

Three-dimensional unimodular Lie groups are classified by Milnor as in Table 3 (up to numeration of c_1, c_2 and c_3). Note that without loss of generality we may assume that $c_3 \geq 0$.

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+,+,+)$	$\mathrm{SU}(2)$	compact and simple
$(-,-,+)$ or $(+,-,+)$	$\widetilde{\mathrm{SL}}_2\mathbb{R}$	non-compact and simple
$(0,+,+)$	$\widetilde{\mathrm{SE}}(2)$	solvable
$(0,-,+)$	$\mathrm{SE}(1,1)$	solvable
$(0,0,+)$	Heisenberg group	nilpotent
$(0,0,0)$	$(\mathbb{R}^3, +)$	Abelian

Table 3. Three dimensional unimodular Lie groups

Here $\widetilde{\mathrm{SE}}(2)$ is the universal covering group of the *Euclidean motion group* $\mathrm{SE}(2)$ (see, Example 4.9). The solvable Lie group $\mathrm{SE}(1,1)$ is the identity component of the isometry group (*Minkowski motion group*) of the Minkowski plane $\mathbb{E}^{1,1}$ (see, Remark 4.2) and isomorphic to

$$\left\{ \left(\begin{array}{ccc} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbb{R} \right\}.$$

To describe the Levi-Civita connection ∇ of G, we introduce the following constants:

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i, \quad i = 1, 2, 3. \tag{4.17}$$

Proposition 4.11. *The Levi-Civita connection ∇ is given by*

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \mu_1 e_3, & \nabla_{e_1} e_3 = -\mu_1 e_2 \\ \nabla_{e_2} e_1 = -\mu_2 e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \mu_2 e_1 \\ \nabla_{e_3} e_1 = \mu_3 e_2, & \nabla_{e_3} e_2 = -\mu_3 e_1 & \nabla_{e_3} e_3 = 0. \end{array}$$

The Riemannian curvature R is given by

$$R(e_1, e_2)e_1 = (\mu_1\mu_2 - c_3\mu_3)e_2, \;\; R(e_1, e_2)e_2 = -(\mu_1\mu_2 - c_3\mu_3)e_1,$$

$$R(e_2, e_3)e_2 = (\mu_2\mu_3 - c_1\mu_1)e_3, \;\; R(e_2, e_3)e_3 = -(\mu_2\mu_3 - c_1\mu_1)e_2,$$

$$R(e_1, e_3)e_1 = (\mu_3\mu_1 - c_2\mu_2)e_3, \;\; R(e_1, e_3)e_3 = -(\mu_3\mu_1 - c_2\mu_2)e_1.$$

The unimodular basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci tensor field. The components of the Ricci tensor field are given by

$$R_{11} = 2\mu_2\mu_3, \quad R_{22} = 2\mu_1\mu_3, \quad R_{33} = 2\mu_1\mu_2.$$

The bilinear map U defined by (4.2) is given by

$$\mathsf{U}(e_1, e_2) = \frac{1}{2}(-c_1 + c_2)e_3, \quad \mathsf{U}(e_1, e_3) = \frac{1}{2}(c_1 - c_3)e_2, \quad \mathsf{U}(e_2, e_3) = \frac{1}{2}(-c_2 + c_3)e_1.$$

Here we compute the difference tensor fields of Cartan-Schouten connections. Set $S^{(+)} = \nabla^{(+)} - \nabla$, $S^{(-)} = \nabla^{(-)} - \nabla$ and $S^{(0)} = \nabla^{(+)} - \nabla$, then we have

$$S^{(+)}(X,Y) = -\mathsf{U}(X,Y) + \frac{1}{2}[X,Y], \quad S^{(-)}(X,Y) = -\mathsf{U}(X,Y) - \frac{1}{2}[X,Y], \quad S^{(0)}(X,Y) = -\mathsf{U}(X,Y).$$

5. Contact 3-manifolds

As is well known contact structures play important roles in 3-dimensional topology and geometry (see *e.g.* [63, 81]). In this section we collect fundamental facts on contact structures on 3-dimensional Riemannian geometry.

5.1. Contact manifolds

Let M be a $(2n+1)$-manifold. A one form η is called a *contact form* on M if $\eta \wedge (\mathrm{d}\eta)^n \neq 0$. A $(2n+1)$-manifold M together with a contact form is called a *contact manifold*. The distribution $\mathcal{D}$ defined by

$$\mathcal{D} = \{X \in \mathrm{T}M \mid \eta(X) = 0\}$$

is called the *contact structure* (or *contact distribution*) determined by η. We denote the vector subbundle of $\mathrm{T}M$ determined by $\mathcal{D}$ by the same letter $\mathcal{D}$:

$$\mathcal{D} := \bigcup_{p \in M} \mathcal{D}_p, \quad \mathcal{D}_p = \{X_p \in \mathrm{T}_pM \mid \eta_p(X_p) = 0\}$$

On a contact manifold (M, η), there exists a unique vector field ξ such that

$$\eta(\xi) = 1, \quad \mathrm{d}\eta(\xi, \cdot) = 0.$$

Namely ξ is transversal to the contact structure $\mathcal{D}$. This vector field ξ is called the *Reeb vector field* of (M, η). The flows of ξ are called *Reeb flows*. Martinet proved that any orientable 3-manifold admits a contact form.

Definition 5.1. A diffeomorphism f on a contact manifold (M, η) is said to be a *contact transformation* if f preserves the contact distribution $\mathcal{D}$. In particular, a diffeomorphism f satisfying $f^*\eta = \eta$ is called a *strict contact transformation*.

We denote by $\mathfrak{D}_{\mathcal{D}}(M)$ the group of all contact transformations. Next we set $\mathfrak{D}_\eta(M)$ the group of all strict contact transformations. Both are subgroups of the diffeomorphism group $\mathfrak{D}(M)$ of M. In case M is compact we can equip a infinite dimensional Lie group structures on $\mathfrak{D}_{\mathcal{D}}(M)$ and $\mathfrak{D}_\eta(M)$ (Omori [180, 181]). Obviously, any strict contact transformation f satisfies $\mathrm{d}f(\xi) = \xi$.

On a contact manifold (M, η), there exists an endomorphism field φ and a Riemannian metric g on a contact manifold (M, η) such that

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{5.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\xi, \cdot) = \eta, \tag{5.2}$$

$$\mathrm{d}\eta(X, Y) = g(X, \varphi Y) \tag{5.3}$$

for all vector fields X, Y on M. The pair (φ, g) (or quartet (η, ξ, φ, g)) is called the *associated almost contact Riemannian structure* of (M, η). A contact manifold (M, η) together with an associated contact Riemannian structure is called a *contact Riemannian manifold* (or *contact metric manifold*) and denote it by $M = (M, \eta, \xi, \varphi, g)$.

On a contact Riemannian manifold M, we define an endomorphism field h by $h = (\pounds_\xi \varphi)/2$.

$$hX = \frac{1}{2}\{[\xi, \varphi X] - \varphi([\xi, X])\}, \quad X \in \Gamma(\mathrm{T}M).$$

Definition 5.2 ([18])**.** A contact Riemannian manifold M is said to be a *contact (κ, μ)-space* if there exits constants κ and μ such that

$$R(X, Y)\xi = \eta(Y)(\kappa\mathrm{I} + \mu h)X - \eta(X)(\kappa\mathrm{I} + \mu h)Y$$

for all $X, Y \in \Gamma(\mathrm{T}M)$.

5.2. Almost contact manifolds

More generally, a $(2n+1)$-manifold M is said to be an *almost contact manifold* if it admits a triplet (η, ξ, φ) satisfying (5.1). A $(2n+1)$-manifold M is said to be an *almost contact Riemannian manifold* if it admits a quartet (η, ξ, φ, g) satisfying (5.1)–(5.2). The vector field ξ will be called the *characteristic vector field*. The (local) flows of ξ will be called the *characteristic flows*.

An almost contact Riemannian manifold M is said to be a *contact Riemannian manifold* if it satisfies (5.3). Every contact Riemannian manifold is orientable. Here we recall the following fundamental fact:

Proposition 5.1. *On a $(2n+1)$-dimensional contact Riemannian manifold (M,φ,ξ,η,g), the volume element $\mathrm{d}v_g$ induced from the associated metric g is related to the contact form η by*

$$\mathrm{d}v_g = \frac{(-1)^n(2n+1)!}{2^n n!}\,\eta\wedge(d\eta)^n. \tag{5.4}$$

Remark 5.1. In [17, Theorem 4.6], the volume element $\mathrm{d}v_g$ is expressed as

$$\mathrm{d}v_g = \frac{(-1)^n}{2^n n!}\eta\wedge(\mathrm{d}\eta)^n.$$

Because, the convention for the volume element used in [17] is $\mathrm{d}v_g = \theta^1\wedge\theta^2\wedge\cdots\wedge\theta^n$ for Riemannian n-manifolds. On the other hand, in the present article, we use the convention $\mathrm{d}v_g = n!\,\theta^1\wedge\theta^2\wedge\cdots\wedge\theta^n$ (see [222, p. 67]).

Even if M is non-contact, we orient M by the volume element

$$\mathrm{d}v_g = \frac{(-1)^n(2n+1)!}{2^n n!}\,\eta\wedge\Phi^n.$$

Definition 5.3. A contact Riemannian manifold M is said to be a K-*contact manifold* if its Reeb vector field is a Killing vector field.

On can see that a contact Riemannian manifold M is K-contact if and only if $h=0$.

5.3. Holomorphic maps

A smooth map $f:(M,\varphi,\xi,\eta)\to(\tilde{M},\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ between almost contact manifolds is said to be a *holomorphic map* if it satisfies

$$\mathrm{d}f\circ\varphi = \tilde{\varphi}\circ\mathrm{d}f.$$

We denote by the subgroup of $\mathfrak{D}(M)$ consisting of all holomorphic diffeomorphisms by $\mathfrak{D}_\varphi(M)$.

Definition 5.4. A diffeomorphism $f:(M,\varphi,\xi,\eta)\to(\tilde{M},\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ between almost contact Riemannian manifolds is called an *almost contact isomorphism* if it preserves the structure tensor fields, *i.e.*,

$$\mathrm{d}f\circ\varphi = \tilde{\varphi}\circ\mathrm{d}f,\quad \mathrm{d}f(\xi)=\tilde{\xi},\quad f^*\tilde{\eta}=\eta.$$

Two almost contact manifolds M and $\tilde{M}$ are said to be *isomorphic each other* if there exits an almost contact isomorphim between them.

Here we set

$$\mathfrak{D}_{\varphi,\xi,\eta}(M) = \mathfrak{D}_\varphi(M)\cap\mathfrak{D}_\eta(M).$$

Then $\mathfrak{D}_{\varphi,\xi,\eta}(M)$ is the group of all almost contact isomorphisms from M onto M itself. An element of $\mathfrak{D}_{\varphi,\xi,\eta}(M)$ is called an *almost contact automorphism*.

Definition 5.5. A diffeomorphism $f:(M,\varphi,\xi,\eta,g)\to(\tilde{M},\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ between almost contact Riemannian manifolds is called an *isomorphism* if it preserves the structure tensor fields, *i.e.*,

$$\mathrm{d}f\circ\varphi = \tilde{\varphi}\circ\mathrm{d}f,\quad \mathrm{d}f(\xi)=\tilde{\xi},\quad f^*\tilde{\eta}=\eta,\quad f^*\tilde{g}=g.$$

Two almost contact Riemannian manifolds M and $\tilde{M}$ are said to be *isomorphic each other* if there exits an isomorphim between them.

The set $\mathrm{Aut}(M)$ of isomorphims from an almost contact Riemannian manifold M onto M itself forms a finite dimensional Lie group. Indeed, it is a Lie subgroup of the isometry group $\mathrm{Iso}(M)$. An element of $\mathrm{Aut}(M)=\mathfrak{D}_{\varphi,\xi,\eta}(M)\cap\mathrm{Iso}(M)$ is called an *automorphism* of M. Tanno [217] showed that $\dim\mathrm{Aut}(M)\leq(n+1)^2$.
Morimoto [164] proved the following fundamental fact.

Lemma 5.1. *Let $M=(M,\varphi,\xi,\eta)$ and $\tilde{M}=(\tilde{M},\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ be almost contact manifolds. If a diffeomorphims $f:M\to\tilde{M}$ is holomorphic and satisfies $\mathrm{d}f(\xi)=\tilde{\xi}$, then $f^*\tilde{\eta}=\eta$.*

Contact Riemannian manifolds satisfy the following property.

Proposition 5.2 ([214]). *If a diffeomorphism $f : M \to \tilde{M}$ between contact Riemannian manifolds is holomorphic, then there exits a positive constant a such that*

$$f^*\tilde{\eta} = a\eta, \quad \mathrm{d}f(\xi) = a\tilde{\xi}, \quad f^*\tilde{g} = ag + a(a-1)\eta \otimes \eta.$$

Based on this result due to Tanno, we introduce the following notion.

Definition 5.6. A diffemomorphism f on a contact Riemannian manifold M is said to be a *transversally homothety* or *$\mathcal{D}$-homothety* if there exists a positive constant a such that

$$f^*\eta = a\eta, \quad \mathrm{d}f(\xi) = a\xi, \quad f^*g = ag + a(a-1)\eta \otimes \eta.$$

Moreover the following theorem holds ([29, Proposition 8.1.11]).

Theorem 5.1. *The automorphism group $\mathrm{Aut}(M)$ of a contact Riemannian manifold M satisfies*

$$\mathrm{Aut}(M) = \mathfrak{D}_{\varphi,\xi,\eta}(M) = \mathfrak{D}_{\eta}(M) \cap \mathrm{Iso}(M).$$

Motivated by Proposition 5.2, we introduce the following notion:

Definition 5.7. Let $(M, \varphi, \xi, \eta, g)$ be a contact Riemannian manifold and a a positive constant. Then the deformation

$$\tilde{\varphi} := \varphi, \quad \tilde{\xi} := \frac{1}{a}\xi, \quad \tilde{\eta} := a\eta, \quad \tilde{g} := ag + a(a-1)\eta \otimes \eta \tag{5.5}$$

gives a new contact Riemannian structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on M. The new structure is called the *transversally homotehtic deformation* or *$\mathcal{D}$-homothetic deformation* of the original structure.

5.4. *Holomorphic sectional curvature*

At a point p of an almost contact Riemannian manifold M. We can take a vector $v \in \mathrm{T}_pM$ orthogonal to ξ_p. Then the tangent plane $v \wedge \varphi_p v$ is called a *holomorphic plane* at p since it is invariant under φ_p. A holomorphic plane is also called φ-section [16, 17]. The sectional curvature of a holomorphic plane is called a *holomorphic sectional curvature* (or *φ-sectional curvature*). On the other hand, a plane section $v \wedge \xi_p$ is called a ξ-section. The sectional curvature of a ξ-section is called a *ξ-sectional curvature*.

5.5. *Normality*

On the direct product $M \times \mathbb{R}(t)$ of an almost contact manifold (M, φ, ξ, η) and the real line $\mathbb{R}(t)$, we can extend naturally the endomorphism field φ to an almost complex structure J on $M \times \mathbb{R}(t)$:

$$J\left(X, f\frac{\partial}{\partial t}\right) = \left(\varphi X - f\xi, \eta(X)\frac{\partial}{\partial t}\right), \; X \in \Gamma(\mathrm{T}M), \quad f \in C^\infty(M \times \mathbb{R}).$$

If the almost complex structure J on $M \times \mathbb{R}$ is integrable then (M, φ, ξ, η) is said to be *normal*. The normality is equivalent to the vanishing of the *Sasaki-Hatakeyama torsion* $\mathcal{N}$:

$$\mathcal{N}(X, Y) = [\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y).$$

Here $[\varphi, \varphi]$ is the *Nijenhuis torsion* of φ:

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \quad X, Y \in \Gamma(\mathrm{T}M).$$

Note that the notion of normality is defined without metric.

Tanno [217] showed that if an almost contact Riemannian manifold M satisfies $\dim \mathrm{Aut}(M) = (n+1)^2$, then M is normal.

A normal contact Riemannian manifold $(M, \varphi, \eta, \xi, g)$ is called a *Sasakian manifold* (or *Sasaki manifold*).

A Sasakian manifold is regarded as a contact (κ, μ)-space with $\kappa = 1$ and $h = 0$.

Remark 5.2. Let M be a contact Riemannian manifold, then the product manifold $M \times \mathbb{R}$ equipped with the above almost complex structure is almost Hermitian with respect to the product metric. Then M is Sasakian if and only if $M \times \mathbb{R}$ is Hermitian. More strongly, the Hermitian structure on $M \times \mathbb{R}$ is locally conformal Kähler.

Here is another characterization of normality. Let us consider the product manifold $C(M) = \mathbb{R}^+ \times M$ of the positive real line $\mathbb{R}^+$ with coordinate $r > 0$ and an almost contact manifold (M, φ, ξ, η). The vector field $r\partial_r$ is called the *Liouville vector field* (also called the *Euler vector field*). Then we can define an almost complex structure J on $C(M)$ by

$$JX = \eta(X)(r\partial_r) + \varphi X, \quad J(r\partial_r) = -\xi, \quad X \in \Gamma(\mathrm{T}M). \tag{5.6}$$

One can see that the normality of M is equivalent to the integrability of J on $C(M)$ (see [29]).

Now let us consider almost contact Riemannian manifolds.

Definition 5.8. Let (M, g) be a Riemannian manifold. The product manifold $C(M) = \mathbb{R}^+ \times M$ equip with a Riemannian metric $dr^2 + r^2 g$ is called the *Riemannian cone* of M.

The metric $dr^2 + r^2 g$ is called the *cone metric* of the Riemannian cone $C(M)$.

Let M be a contact Riemannian manifold. Then M is Sasakian if and only if its Riemannian cone $C(M)$ equipped with the almost complex structure (5.6) is a Kähler manifold [29].

Definition 5.9 ([131])**.** An almost contact Riemannian manifold $(M, \varphi, \eta, \xi, g)$ is said to be

1. *almost coKähler* (or *almost cosymplectic*) if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$.
2. *almost b-Kenmotsu* if $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2b\eta \wedge \Phi$, where b is a non-zero constant. In particular an almost 1-Kenmotsu manifold is simply called an *almost Kenmotsu manifold*.

A *coKähler manifold* [resp. *Kenmotsu manifold*] is a normal almost coKähler manifold [resp. normal almost Kenmotsu manifold].

Proposition 5.3. *An almost contact Riemannian manifold M is*

- *coKähler if and only if $\nabla\varphi = 0$. In this case $\nabla\xi = 0$.*
- *Sasakian if and only if $(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$. In this case $\nabla_X\xi = -\varphi X$.*
- *Kenmotsu if and only if $(\nabla_X\varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X$. In this case $\nabla_X\xi = X - \eta(X)\xi$.*

A complete Sasakian manifold of constant holomorphic sectional curvature is called a *Sasakian space form*. Analogously, a complete coKähler manifold of constant holomorphic sectional curvature is called a *coKähler space form*.

Example 5.1 (Real hypersurfaces)**.** Let (N, h, J) be an almost Hermitian manifold and M is an orientable real hypersurface of N with inclusion map $\iota : M \subset N$ and unit normal vector field ν. Then the almost Hermitian structure (h, J) induces an almost contact Riemannian structure on M. First, let $g = \iota^* h$ be the induced metric. Next, set $\xi = -J\nu$. For any vector field X on M, we decompose JX into its tangential and normal components:

$$JX = \varphi X + \eta(X)\nu.$$

Then one can see that (φ, ξ, η, g) is an almost contact Riemannian structure on M.

Now let $\mathbb{C}^{n+1}$ the complex Euclidean $(n+1)$-space. Then the Kähler structure of $\mathbb{C}^{n+1}$ induces an almost contact Riemannian structure on the unit $(2n+1)$-sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. The resulting almost contact Riemannian manifold $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$ is Sasakian. In particular $\mathbb{S}^{2n+1}$ is a Sasakian space form of constant holomorphic sectional curvature 1.

Example 5.2 (Product manifolds)**.** Let $(\overline{M}, \bar{g}, J)$ be an almost Hermitian manifold. Take a Riemannian product $M = \overline{M} \times \mathbb{R}$ or $M = \overline{M} \times \mathbb{S}^1$ equipped with the product metric $g = \bar{g} + dt^2$. Set $\xi = \partial/\partial_t$ and $\eta = g(\xi, \cdot)$. We can extend J to an endomorphism field φ on M by the rule

$$\varphi(\overline{X} + a\xi) = J\overline{X}, \quad \overline{X} \in \Gamma(\mathrm{T}\overline{M}).$$

Then we obtain an almost contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$. One can see that M is almost coKähler if and only if $\overline{M}$ is almost Kähler. In particular M is coKähler if and only if $\overline{M}$ is Kähler (see [102]). Now let take a complex space form $(\overline{M}, \bar{g}, J)$ of constant holomorphic sectional curvature k. Then M is a coKähler space form of constant holomorphic sectional curvature k. The 3-dimensional coKqhler space forms are given by (see also [217]):

$$\mathbb{S}^2(c^2) \times \mathbb{E}^1, \quad \mathbb{E}^3 = \mathbb{E}^2 \times \mathbb{E}^1, \quad \mathbb{H}^2(-c^2) \times \mathbb{E}^1, \quad \mathbb{S}^2(c^2) \times \mathbb{S}^1, \quad \mathbb{E}^2 \times \mathbb{S}^1, \quad \mathbb{H}^2(-c^2) \times \mathbb{S}^1.$$

Here $\mathbb{S}^2(c^2)$ and $\mathbb{H}^2(-c^2)$ are sphere of curvature c^2 and hyperbolic plane of curvature $-c^2$, respectively.

Example 5.3 (Bianchi-Cartan-Vrănceanu models)**.** Let κ and τ be real constants. Define a region $\mathcal{R}$ of $\mathbb{R}^3$ by

$$\mathcal{R} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; 1 + \frac{\kappa}{4}(x^2+y^2) > 0 \right\}.$$

On the region $\mathcal{R}$, we equip a Riemannian metric

$$g_{\kappa,\tau} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\{1+\frac{\kappa}{4}(x^2+y^2)\}^2} + \left(\mathrm{d}z + \frac{\tau(y\mathrm{d}x - x\mathrm{d}y)}{1+\frac{\kappa}{4}(x^2+y^2)}\right)^2. \tag{5.7}$$

This 2-parameter family $\{g_{\kappa,\tau}\}$ of Riemannian metrics is called the *Bianchi-Cartan-Vrănceanu metrics* [13, 38, 226]. The metrics as above are defined over the whole 3-space $\mathbb{R}^3$ for $\kappa \geq 0$ and over the region $x^2 + y^2 < -4/\kappa$ for $\kappa < 0$. The resulting Riemannian 3-manifold $E(\kappa,\tau) = (\mathcal{R}, g_{\kappa,\tau})$, the *Bianchi-Cartan-Vrănceanu spaces* (*BCV-spaces* in short [10]). This 2-parameter family includes all Riemannian metrics with 4 or 6-dimensional isometry groups except constant negative curvature ones.

Take an orthonormal frame field $\mathcal{U} = (u_1, u_2, u_3)$:

$$u_1 = \{1+\frac{\kappa}{4}(x^2+y^2)\}\frac{\partial}{\partial x} - \tau y\frac{\partial}{\partial z}, \quad u_2 = \{1+\frac{\kappa}{4}(x^2+y^2)\}\frac{\partial}{\partial y} + \tau x\frac{\partial}{\partial z}, \quad u_3 = \frac{\partial}{\partial z}.$$

The Levi-Civita connection ∇ of the BCV-space $E(\kappa,\tau)$ is described by the formulas:

$$\begin{gathered}
\nabla_{u_1}u_1 = \frac{\kappa y}{2}u_2, \quad \nabla_{u_1}u_2 = -\frac{\kappa y}{2}u_1 + \tau u_3, \quad \nabla_{u_1}u_3 = -\tau u_2, \\
\nabla_{u_2}u_1 = -\frac{\kappa x}{2}u_2 - \tau u_3, \quad \nabla_{u_2}u_2 = \frac{\kappa x}{2}e_1, \quad \nabla_{u_2}u_3 = \tau u_1, \\
\nabla_{u_3}u_1 = -\tau u_2, \quad \nabla_{u_3}u_2 = \tau u_1, \quad \nabla_{u_3}u_3 = 0.
\end{gathered} \tag{5.8}$$

$$[u_1,u_2] = -\frac{\kappa}{2}yu_1 + \frac{\kappa}{2}xu_2 + 2\tau u_3, \quad [u_2,u_3] = [u_3,u_1] = 0. \tag{5.9}$$

The Riemannian curvature R is described by the formula:

$$R_{1212} = \kappa - 3\tau^2, \quad R_{1313} = R_{2323} = \tau^2. \tag{5.10}$$

The BCV-space is isomorphic to the following model spaces:

Model space	base and bundle curvatures
$\mathbb{S}^3(\kappa/4) \setminus \{\infty\}$	$\kappa = 4\tau^2 \neq 0$
$(\mathbb{S}^2(\kappa) \setminus \{\infty\}) \times \mathbb{E}^1$	$\kappa > 0, \tau = 0$
$\mathbb{H}^2(\kappa) \times \mathbb{E}^1$	$\kappa < 0, \tau = 0$
Berger sphere	$\kappa > 0, \tau \neq 0$
Nil_3	$\kappa = 0, \tau \neq 0$
$\widetilde{\mathrm{SL}}_2\mathbb{R}$	$\kappa < 0, \tau \neq 0$

Table 4. The Bianchi-Cartan-Vrănceanu model spaces

Define an endomorphism field φ by $\varphi u_1 = u_2$, $\varphi u_2 = -u_1$, $\varphi u_3 = 0$ and set $\xi := u_3$. Then it is easy to check that $(\eta, \xi, \varphi, g_{\kappa,\tau})$ is an almost contact structure. In particular if $\tau \neq 0$, $(\varphi, \xi, g_{\kappa,\tau})$ is the associated almost contact structure of η up to a constant multiple. More precisely the exterior derivative $\mathrm{d}\eta$ is related to φ by

$$\mathrm{d}\eta(X,Y) = \tau g(X, \varphi Y). \tag{5.11}$$

This almost contact structure satisfies the following:

$$(\nabla_X \varphi)Y = \tau\{g(X,Y)\xi - \eta(Y)X\}, \quad \nabla_X \xi = -\tau\varphi X. \tag{5.12}$$

These formulas show that ξ is a Killing vector field on the BCV-space. Moreover the BCV-space is normal. In case $\tau = 0$, the BCV-space is a coKähler manifold of constant holomorphic sectional curvature.

Thus if we fix the bundle curvature $\tau = 1$. Then we obtain a Sasakian manifold $(\mathcal{R}, \eta, \xi, \varphi, g_{\kappa,1})$ of constant holomorphic sectional curvature $-3 + \kappa$.

In case $\tau \neq 0$, we perform the normalization

$$\tilde{\eta} := \tau\eta, \quad \tilde{\xi} := \frac{1}{\tau}\xi, \quad \tilde{\varphi} := \varphi, \quad \tilde{g} = \tau^2 g.$$

Then the resulting manifold $(\mathcal{R}, \tilde{\eta}, \tilde{\xi}, \tilde{\varphi}, \tilde{g})$ is a Sasakian manifold of constant holomorphic sectional curvature $\tilde{c} := -3 + \kappa$.

Example 5.4 (The hyperbolic Sasakian space form)**.** Let $\mathbb{H}^2(-c^2)$ be the upper half plane model of the hyperbolic plane of curvature $-c^2$ exhibited in Section 4.6. On the product manifold $\mathbb{H}^2(-c^2) \times \mathbb{R}(t)$, we equip a contact form

$$\eta = \mathrm{d}t - \frac{2\mathrm{d}x}{c^2 y}$$

with Reeb vector field $\xi = \partial_t$. By using the contact form η, we introduce a Riemannian metric

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{c^2 y^2} + \eta \otimes \eta.$$

We can take an orthonormal frame field

$$e_1 = (cy)\frac{\partial}{\partial x} + \frac{2}{c}\frac{\partial}{\partial t}, \quad e_2 = (cy)\frac{\partial}{\partial y}, \quad e_3 = \xi.$$

The endomorphism field φ is determined by the formula $g(X, \varphi Y) = \mathrm{d}\eta(X, Y)$. Since

$$\mathrm{d}\eta = -\frac{2}{c^2 y^2}\mathrm{d}x \wedge \mathrm{d}y,$$

we deduce that

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0.$$

The resulting contact Riemannian 3-manifold $(\mathbb{H}^2(-c^2) \times \mathbb{R}(t), \varphi, \xi, \eta, g)$ is a Sasakian space form of constant holomorphic sectional curvature $-c^2 - 3 < -3$ [218]. Note that this Sasakian manifold is isomorphic to the universal coverings $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of $\mathrm{SL}_2\mathbb{R}$ as well as the universal coverings $\widetilde{\mathrm{SU}}(1,1)$ of $\mathrm{SU}(1,1)$. See Section 14.

Example 5.5 (Warped products)**.** Let $(\overline{M}, \bar{g}, J)$ be an almost Kähler manifold. Consider a warped product:

$$M = \mathbb{R} \times_{ce^t} M, \quad g = \mathrm{d}t^2 + c^2 e^{2t}\bar{g},$$

where c is a non-zero constant. Then we can introduce an almost contact structure (φ, ξ, η) by

$$\varphi X = JX, \quad X \in \Gamma(\mathrm{T}\overline{M}), \quad \varphi\xi = 0, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = g(\xi, \cdot).$$

Then M is almost Kenmotsu. In particular M is Kenmotsu if and only if $\overline{M}$ is Kähler. Now let us take

$$\overline{M} = \mathbb{C}^n(z_1, z_2, \dots, z_n) \quad \bar{g} = \sum_{k=1}^n \mathrm{d}x_k^2 + \sum_{i=1}^n \mathrm{d}y_k^2, \quad z_k = x_k + y_k i$$

Then the warped product metric

$$g = \mathrm{d}t^2 + c^2 e^{2t}\left(\sum_{k=1}^n \mathrm{d}x_k^2 + \sum_{i=1}^n \mathrm{d}y_k^2\right)$$

is of constant curvature -1. Thus M is isometric to the hyperbolic $(2n+1)$-space $\mathbb{H}^{2n+1}$.

The notion of Kenmotsu space form is defined in much the same way to that of Sasakian space form. However the constancy of holomorphic sectional curvature is a strong restriction for Kenmotsu manifolds. Indeed, Kenmotsu [139] proved that Kenmotsu manifolds of constant holomorphic sectional curvature are of constant curvature -1. Thus the only Kenmotsu space form is the hyperbolic space $\mathbb{H}^{2n+1}$. The Kenmotsu structure of $\mathbb{H}^3$ will be discussed again in Section 16. For 3-dimensional Kenmtsu manfolds, we have the following fact:

Proposition 5.4 ([100, 101]). *A Kenmotsu 3-manifold has constant scalar curvature if and only if it is of constant curvature* -1.

Tanno [217] proved that almost contact Riemannian manifolds with automorphism group of maximum dimension are Sasakian space forms, coKähler space forms or Kenmotsu manifolds of constant curvature -1 (Table 5).

Model space	Almost contact structure
$\mathbb{E}^3$	CoKähler space form
$\mathbb{S}^3$	Sasakian space form
$\mathbb{H}^3$	Kenmotsu space form
$\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$	CoKähler space form

Table 5. The model spaces of constant holomorphic sectional curvature

5.6. Regularity

Example 5.2 and Example 5.3 motivates us to study homogeneous Riemannian 3-manifolds fibered over homogeneous Riemannian 2-manifolds.

A non-vanishing vector field ξ on an m-manifold M is said to be *quasi regular* if if there exits some positive integer k and each point p has a cubical coordinate neighborhood $(U; x^1, x^2, \dots, x^m)$ such that

- each integral curve of the vector field ξ passes through U at most k times, and
- each component of the intersection of an integral curve with U has the form $x^1 = a^1$, $x^2 = a^2, \dots, x^{m-1} = a^m$ with $a^1, a^2, \dots, a^{m-1}$ are constant. In case $k = 1$, ξ is said to be *regular*. If in addition, all the integral curves are homeomorphic to each other, then ξ is said to be a *strictly regular vector field*.

Tanno [215] showed that the following three conditions are mutually equivalent for regular (and complete) vector field ξ:

1. The period function of ξ is constant (maybe infinite).
2. There exists a 1-form η satisfying $\eta(\xi) = 1$ and $\pounds_\xi \eta = 0$.
3. There exists a Riemannian metric g satisfying $g(\xi, \xi) = 1$ and $\pounds_\xi g = 0$.

In such a case, M is a principal bundle over the orbit space $\overline{M} = M/\mathcal{G}$, where $\mathcal{G}$ is the 1-parameter group (of isometries) $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$. The prescribed vector field ξ is a unit Killing vector field with respect to the Riemannian metric g. In addition, there exists a Riemannian metric $\bar{g}$ on $\overline{M}$ so that $\pi : M \to \overline{M}$ is a Riemannian submersion. Moreover the 1-form η is a connection form of the principal bundle $\pi : M \to \overline{M}$.

Definition 5.10 ([171]). An almost contact manifold (M, φ, ξ, η) is said to be *regular* [resp. *strictly regular*] if ξ is regular [resp. strictly regular].

Proposition 5.5 ([171]). *Let M be a strictly regular almost contact manifold. If φ and η are invariant under the 1-parameter group $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$, then M is a principal $\mathcal{G}$-bundle over $\overline{M}$ with connection form η.*

Corollary 5.1 ([171]). *Let M be a compact regular almost contact manifold. If φ and η are invariant under the 1-parameter group $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$, then $\mathcal{G} = \mathbb{S}^1$ and M is a principal circle bundle over $\overline{M}$ with connection form η.*

Lemma 5.2. *If the endomorphism field φ and the 1-form η of a regular M are invariant under the 1-parameter group $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$, then*

$$J_{\pi(p)}\overline{X}_{\pi(p)} = \pi_{*p}(\varphi_p \overline{X}^{\mathsf{h}}_p), \quad \overline{X} \in \Gamma(\mathrm{T}\overline{M}) \tag{5.13}$$

defines an almost complex structure J on the orbit space $\overline{M}$. Here the superscript h means the horizontal lift operation with respect to the connection form η.

Let us consider contact Riemannian manifold. Assume that $M = (M, \eta)$ is a regular contact manifold, then one can see that φ and η are invariant under the 1-parameter group $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$. When M is compact, Boothby and Wang proved the following prominent theorem (see also Kobayashi [143]).

Theorem 5.2 ([28]). *Let (M, η') be a compact regular contact manifold. Then there exists a contact form $\eta = \mu\eta'$ on M, where μ is a non-vanishing smooth function so that the Reeb vector field ξ generates a free effective $\mathbb{S}^1$-action on M. Moreover M is a principal circle bundle over $\overline{M} = M/\mathcal{G}$. The orbit space $\overline{M}$ inherits a symplectic form $\overline{\Omega}$ satisfying $\mathrm{d}\eta = \pi^*\overline{\Omega}$. The symplectic form determines an integral cocycle on $\overline{\Omega}$. The contact form is a connection form of $\pi : M \to \overline{M}$.*

The fibering $\pi : (M, \eta) \to (\overline{M}, \overline{\Omega})$ is called the *Boothby-Wang fibering*.

Proposition 5.6 ([90, 171]). *Let M be a regular K-contact manifold. Then φ and η are invariant under the 1-parameter group $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$ and M is a principal $\mathcal{G}$-bundle over $\overline{M} = M/\mathcal{G}$ with connection form η. The orbit space inherits an almost Kähler structure and the fundamental 2-form $\overline{\Omega}$ determines an integral cocycle on $\overline{\Omega}$. In particular, M is Sasakian if and only if $\overline{M}$ is a Kähler manifold.*

Corollary 5.2 ([90, 171]). *Let M be a regular Sasakian manifold. Then φ and η are invariant under the 1-parameter group $\mathcal{G} = \{\mathrm{Exp}(t\xi)\}_{t\in\mathbb{R}}$ and M is a principal $\mathcal{G}$-bundle over a Hodge manifold $\overline{M} = M/\mathcal{G}$ with connection form η.*

5.7. The Boeckx invariant

Let M be a non-Sasakian contact (κ, μ)-space. Take a positive constant a and perform the transversally homothetic deformation (5.5) to M. Then $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a contact $(\tilde{\kappa}, \tilde{\mu})$-space with

$$\tilde{\kappa} = \frac{\kappa + a^2 - 1}{a}, \quad \tilde{\mu} = \frac{\mu + 2a - 2}{a^2}.$$

The *Boeckx invariant* $\mathcal{I}$ of a non-Sasakian contact (κ, μ)-space is defined by

$$\mathcal{I} = \frac{1}{\sqrt{1-\kappa}}\left(1 - \frac{\mu}{2}\right).$$

Then one can check that two non-Sasakian contact (κ, μ)-spaces are related by pseudo-homothetic deformation if and only if their Boeckx invariants agree [24].

5.8. Unit tangent sphere bundles

Let M be an n-manifold, then its tangent bundle $\mathrm{T}M$ is a smooth $(2n)$-manifold. Denote by $\pi : \mathrm{T}M \to M$ the projection. Take a local coordinate sytem $(x_1, x_2, \dots, x_n)$ of M, then

$$(x_1 \circ \pi, x_2 \circ \pi, \dots, x_n \circ \pi, u_1, u_2, \dots, u_n), \quad u_i := \mathrm{d}x_i$$

gives a local coordinate system of $\mathrm{T}M$. One can confirm that

$$\boldsymbol{U} = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}$$

is globally defined on $\mathrm{T}M$ and called the *canonical vertical vector field*. The vertical distribution $\mathcal{V}$ of $\mathrm{T}M$ is defined as $\mathcal{V} = \mathrm{Ker}\,\pi_*$. For any vector field X on M there exits a unique vector field X^{v} on $\mathrm{T}M$ satisfying $X^{\mathrm{v}}_{(p;v)} \in \mathcal{V}_{(p;v)}$. The vector field X^{v} is called the *vertical lift* of X.

Let us equip a Riemannian metric g on M, then the Levi-Civita connection ∇ of g defines the horizontal distribution $\mathcal{H}$. Moreover for any vector field X on M, there exits a unique vector field X^{h} on $\mathrm{T}M$ satisfying $X^{\mathrm{h}}_{(p;v)} \in \mathcal{H}_{(p;v)}$. The vector field X^{h} is called the *horizontal lift* of X.

Sasaki introduced a Riemannian metric g^{s} on $\mathrm{T}M$ by

$$g^{\mathrm{s}}_{(p;v)}(X^{\mathrm{h}}, Y^{\mathrm{h}}) = g^{\mathrm{s}}_{(p;v)}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = g_p(X, Y), \quad g^{\mathrm{s}}_{(p;v)}(X^{\mathrm{h}}, Y^{\mathrm{v}}) = 0.$$

This Riemannian metic g^{s} is called the *Sasaki lift metric*. On the other hand, Dombrowski [58] and Hsu [93] introduced the following almost complex structure

$$JX^{\mathrm{h}} = X^{\mathrm{v}}, \quad JX^{\mathrm{v}} = -X^{\mathrm{h}}.$$

One can see that $(\mathrm{T}M, g^{\mathrm{s}}, J)$ is almost Kähler. If we identify $\mathrm{T}M$ with the cotangent bundle T^*M via the metric g, then the fundamental 2-form of $\mathrm{T}M$ is identified with the canonical symplectic form of T^*M (see [112, 161]).

The unit tangent sphere bundle $\mathrm{U}M$ is defined as the hypersurface

$$\mathrm{U}M = \{(p; v) \in \mathrm{T}M \mid g_p(v, v) = 1\}$$

of $\mathrm{T}M$ with unit normal vector field $\boldsymbol{U}$. Thus the almost Kähler structure (g^{s}, J) of $\mathrm{T}M$ induces an almost contact Riemannian structure $(\varphi^{\mathrm{s}}, \xi^{\mathrm{s}}, \eta^{\mathrm{s}}, g^{\mathrm{s}})$ on $\mathrm{U}M$ [200]. The 1-form η^{s} is a contact form on $\mathrm{U}M$ and called the *canonical contact form* of $\mathrm{U}M$. The Reeb vector field ξ^{s} is called the *geodesic flow vector field* ([199, 200]) or *geodesic spray* ([4]). The integral curves of ξ project to geodesics on M.

The almost contact Riemannian structure of $\mathrm{U}M$ satisfies

$$g^{\mathrm{s}}(E, \varphi F) = 2\mathrm{d}\eta(E, F), \quad \Gamma(\mathrm{T}(\mathrm{T}M)).$$

To adapt contact Riemanian condition, Blair considered the following change of the structure

$$\tilde{\varphi}^{\mathrm{s}} = \varphi^{\mathrm{s}}, \quad \tilde{\xi}^{\mathrm{s}} = 2\xi^{\mathrm{s}}, \quad \tilde{\eta}^{\mathrm{s}} = \frac{1}{2}\eta^{\mathrm{s}}, \quad \tilde{g}^{\mathrm{s}} = \frac{1}{4}g^{\mathrm{s}}. \tag{5.14}$$

The contact Riemannian structure $(\tilde{\varphi}^{\mathrm{s}}, \tilde{\xi}^{\mathrm{s}}, \tilde{\eta}^{\mathrm{s}}, \tilde{g}^{\mathrm{s}})$ is referred as to the *standard contact Riemannian structure* (standard contact metric structure) in [16, 17]. For curvature properties and homogeneity of unit tangent sphere bundles, see [50].

Remark 5.3. The idea of Sasaki lift metric can be traced back to Poincaré. See Sasaki's article [199, 201]. In many articles, g^{s} is called the *Sasaki metric*. To avoid the confusion with *Sasakian metrics* (metrics of Sasakian manifolds), in this article, we use the terminology "Sasaki-lift metric".

5.9. Some adapted connections

Tanno [219] introduced the following linear connection on a contact Riemannian manifolds:

$${}^{*}\nabla_X Y = \nabla_X Y + \{(\nabla_X \eta)Y\}\xi - \eta(Y)\nabla_X \xi + \eta(X)\varphi Y. \tag{5.15}$$

This connection is called the *generalized Tanaka-Webster connection*. By using the operator h, Tanno's generalized Tanaka-Webster connection is rewritten as

$${}^{*}\nabla_X Y = \nabla_X Y - g(\varphi(\mathrm{I} + h)X, Y)\xi + \eta(X)\varphi Y + \eta(Y)\varphi(\mathrm{I} + h)X.$$

The generalized Tanaka-Webster connection satisfies

$${}^{*}\nabla\xi = 0, \quad {}^{*}\nabla\eta = 0, \quad {}^{*}\nabla g = 0.$$

The covariant derivative ${}^{*}\nabla$ is given by

$$({}^{*}\nabla_X \varphi)Y = \mathcal{Q}(Y, X),$$

where $\mathcal{Q}$ is the *Tanno tensor field* defined by

$$\mathcal{Q}(Y, X) = (\nabla_X \varphi)Y - g((\mathrm{I} + h)X, Y)\xi + \eta(Y)(\mathrm{I} + h)X.$$

On the other hand, in our previous work [110], we introduced the following one-parameter family of linear connections on almost contact Riemannian manifolds:

$$\begin{aligned} \nabla^r_X Y =& \nabla_X Y + A^r_X Y, \\ A^r_X Y =& -\frac{1}{2}\varphi(\nabla_X \varphi)Y - \frac{1}{2}\eta(Y)\nabla_X \xi - r\eta(X)\varphi Y + \{(\nabla_X \eta)Y\}\xi, \quad r \in \mathbb{R}. \end{aligned} \tag{5.16}$$

One can verify that

$$\nabla^r \varphi = 0, \quad \nabla^r \xi = 0, \quad \nabla^r \eta = 0, \quad \nabla^r g = 0$$

The connections ∇^r are called the *almost contact connections*.

The linear connection $\nabla^r|_{r=0}$ coincides with the (φ, ξ, η)*-connection* in the sense of Sasaki-Hatakeyama [202]. Thus we may call ∇^r by the name *generalized Sasaki-Hatakeyama connection*. In [110], we call it *generalized Tanaka-Webster-Okumura connection*. Note that the linear connection $\nabla^r|_{r=1}$ was introduced by Cho [47]. As we will see later many Ambrose-Singer connections are given by almost connections.

For more details on contact Riemannian manifolds, we refer to [16, 17, 29] .

6. CR-manifolds

6.1. CR-structures

An *almost CR-structure* $\mathcal{S}$ of a smooth manifold M is a complex vector subbundle $\mathcal{S} \subset T^{\mathbb{C}}M$ of the complexified tangent bundle of M satisfying $\mathcal{S} \cap \overline{\mathcal{S}} = \{0\}$. A manifold M equipped with an almost CR-structure is called an *almost CR-manifold*.

An almost CR-structure $\mathcal{S}$ is said to be *integrable* if it satisfies the *integrability condition*:

$$[\Gamma(\mathcal{S}), \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S}).$$

In such a case, $(M, \mathcal{S})$ is called a *CR-manifold*.

Now let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. Then we define an almost CR-structure $\mathcal{S}$ of M by

$$\mathcal{S} = \{X - \sqrt{-1}\varphi X \mid X \in \Gamma(\mathcal{D})\}$$

with

$$\mathcal{D} = \{X \in TM \mid \eta(X) = 0\}.$$

We call $\mathcal{S}$ the *almost CR-structure* associated to (φ, ξ, η, g). It should be emphasized that the integrability of $\mathcal{S}$ is equivalent to the vanishing of the Sasaki-Hatakeyama torsion on $\mathcal{D}$. Thus the associated almost CR-structures of normal almost contact Riemannian manifolds are integrable.

Note that when $\dim M = 3$, the associated almost CR-structure $\mathcal{S}$ is automatically integrable.

Assume that M is a contact Riemannian manifold. Define a section L of $\Gamma(\mathcal{D}^* \otimes \mathcal{D}^*)$ by

$$L(X, Y) = -\mathrm{d}\eta(X, \varphi Y).$$

Then L is positive definite on $\mathcal{D} \otimes \mathcal{D}$ and called the *Levi-form* of M. When the associated almost CR-structure $\mathcal{S}$ is integrable, the resulting CR-manifold $(M, \mathcal{S})$ is called a *strongly pseudo-convex CR-manifold* or *strongly pseudo-convex pseudo-Hermitian manifold*.

Proposition 6.1. *Let M be a contact Riemannian manifold. Then its associated almost CR-structure is integrable if and only if its Tanno tensor field Q vanishes.*

On a strongly pseudo-convex CR-manifolds, the following formula holds:

$$(\nabla_X \varphi)Y = g((\mathrm{I} + h)X, Y)\xi - \eta(Y)(\mathrm{I} + h)X \tag{6.1}$$

for all vector fields X and Y. The formula (6.1) implies

$$\nabla_X \xi = -\varphi(\mathrm{I} + h)X, \quad X \in \Gamma(\mathrm{T}M)$$

According to Tanaka [213], a strongly pseudo-convex CR manifold is said to be *normal* if its Reeb vector field is *analytic*, that is, $[\xi, \Gamma(\mathcal{S})] \subset \Gamma(\mathcal{S})$. The Reeb vector field is analytic if and only if ξ is an infinitesimal contact transformation and $[X, \varphi Y] = \varphi[X, Y]$ for all X, $Y \in \Gamma(\mathcal{D})$. One can see that a strongly pseudo-convex CR-manifold is normal if and only if its underlying contact Riemannian structure is Sasakian.

6.2. Tanaka-Webster connections

In the study of strongly pseudo-convex CR-manifolds, the linear connection $\hat{\nabla}$ introduced by Tanaka and Webster is highly useful:

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \{(\nabla_X \eta)Y\}\xi - \eta(Y)\nabla_X \xi. \tag{6.2}$$

Here ∇ is the Levi-Civita connection of the associated metric. The linear connection $\hat{\nabla}$ is referred as to the *Tanaka-Webster connection* [213, 228]. The Tanaka-Webster connection is rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi(\mathrm{I} + h)X - g(\varphi(\mathrm{I} + h)X, Y)\xi.$$

It should be remarked that the Tanaka-Webster connection has *non-vanishing* torsion $\hat{T}$:

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi h X - \eta(X)\varphi h Y.$$

With respect to the Tanaka-Webster connection, all the structure tensor fields (φ,ξ,η,g) are parallel, *i.e.*,

$$\hat{\nabla}\varphi=0,\quad \hat{\nabla}\xi=0,\quad \hat{\nabla}\eta=0,\quad \hat{\nabla}g=0.$$

On a strongly pseudo-convex CR-manifold M, the generalized Tanaka-Webster connection coincides with the Tanaka-Webster connection.

On a strongly pseudo-convex CR-manifold M, the almost contact connection ∇^r has the form

$$\nabla^r_XY=\nabla_XY-g(\varphi(\mathrm{I}+h)X,Y)\xi-r\eta(X)\varphi Y+\eta(Y)\varphi(\mathrm{I}+h)X.$$

Hence $\nabla^r|_{r=-1}$ is the Tanaka-Webster connection. Thus, on a strongly pseudo-convex CR-manifold M, we have ${}^*\nabla=\hat{\nabla}=\nabla^r|_{r=-1}$.

7. Three dimensional almost contact geometry

7.1. The vector product of almost contact Riemannian 3-manifolds

Let (M,φ,ξ,η,g) be a 3-dimensional contact metric manifold. Then as we have seen before, the volume element dv_g derived from the associated metric g is related to the contact form η by

$$\mathrm{d}v_g=-3\eta\wedge\Phi. \tag{7.1}$$

Even if M is non-contact, M is orientable by the 3-form $-3\eta\wedge\Phi$ and the volume element $\mathrm{d}v_g$ coincides with this 3-form. Thus hereafter we orient 3-dimensional almost contact metric manifolds by $\mathrm{d}v_g=-3\eta\wedge\Phi$ given in (7.1). With respect to this orientation, the vector product $\times$ is computed as

$$X\times Y=-\Phi(X,Y)\xi+\eta(X)\varphi Y-\eta(Y)\varphi X. \tag{7.2}$$

Note that for a unit vector field X orthogonal to ξ, the local frame field $\{X,\varphi X,\xi\}$ is positively oriented and

$$\xi\times X=\varphi X.$$

Here we emphasize that the existence of almost contact structure has no topological restriction for orientable 3-manifolds. Indeed, let $(M,g,\mathrm{d}v_g)$ be an oriented Riemannian 3-manifold. Then there exists a non-vanishing vector field $Z\in\Gamma(\mathrm{T}M)$. Set $\xi=Z/\|Z\|$, $\eta=g(\xi,\cdot)$ and define an endomorphism field φ by $\varphi X:=\xi\times X$. One can check that (φ,ξ,η,g) is an almost contact Riemannian structure such that $\mathrm{d}v_g=-3\Phi\wedge\eta$.

Proposition 7.1 ([35]). *Let $(M,g,\mathrm{d}v_g)$ be an oriented Riemannian 3-manifold. Then there exits an almost contact structure (φ,ξ,η) compatible to g and $\mathrm{d}v_g$.*

7.2. Normal almost contact Riemannian manifolds

For an arbitrary almost contact Riemannian 3-manifold M, we have the following *Olszak's formula* [179]:

$$(\nabla_X\varphi)Y=g(\varphi\nabla_X\xi,Y)\xi-\eta(Y)\varphi\nabla_X\xi. \tag{7.3}$$

Moreover, we have

$$\mathrm{d}\eta=\eta\wedge\nabla_\xi\eta+\alpha\Phi,\quad \mathrm{d}\Phi=2\beta\eta\wedge\Phi,$$

where α and β are the functions defined by

$$\alpha=\frac{1}{2}\mathrm{tr}\,(\varphi\nabla\xi),\quad \beta=\frac{1}{2}\mathrm{tr}\,(\nabla\xi)=\frac{1}{2}\mathrm{div}\,\xi. \tag{7.4}$$

The functions α and β are related by

$$\mathrm{d}\alpha(\xi)+2\alpha\beta=0.$$

On an almost contact Riemannian 3-manifold M, the almost contact connection has the form:

$$\nabla^r_XY=\nabla_XY+A^r_XY,\quad A^r_XY=-\eta(Y)\nabla_X\xi+g(\nabla_X\xi,Y)\xi-r\eta(X)\varphi Y. \tag{7.5}$$

Olszak [179] showed that an almost contact Riemannian 3-manifold M is normal if and only if $\nabla\xi\circ\varphi=\varphi\circ\nabla\xi$ or, equivalently,

$$\nabla_X\xi=-\alpha\varphi X+\beta(X-\eta(X)\xi),\quad X\in\Gamma(\mathrm{T}M). \tag{7.6}$$

We call the pair (α,β) the *type* of a normal almost contact Riemannian 3-manifold M.

A normal almost contact Riemannian 3-manifold M is said to be (*cf.* Definition 5.9)

- *quasi-Sasakian* if $\beta = 0$.
- a*-Sasakian* if $\alpha = a$ is a non-zero constant and $\beta = 0$
- b*-Kenmotsu* if $\beta = b$ is a non-zero constant and $\alpha = 0$.

Note that Sasakian 3-manifolds [resp. Kenmotsu 3-manifold] are 1-Sasakian 3-manifolds [1-Kenmotsu 3-manifolds]. A coKähler 3-manifold is a normal almost contact Riemannian 3-manifold of type $(0, 0)$.

Now let M be a contact Riemannian 3-manifold, then its associated almost CR-structure is automatically integrable. Hence we have

$$\nabla_X \xi = -\varphi(\mathrm{I} + h)X, \quad X \in \Gamma(\mathrm{T}M).$$

The covariant derivative φ is given by

$$(\nabla_X \varphi)Y = g((\mathrm{I} + h)X, Y)\xi - \eta(Y)(\mathrm{I} + h)X.$$

Hence we obtain the following result.

Proposition 7.2. *Let $(M, \varphi, \eta, \xi, g)$ be a contact Riemannian 3-manifold. Then the following three conditions are mutually equivalent.*

- *M is a Sasakian manifold,*
- *ξ is a Killing vector field,*
- $\nabla \xi = -\varphi$,
- *$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$ for any vector fields X and Y on M.*
- *$h = 0$ on M.*

Remark 7.1. Analogous to the table of possible types of homogeneous Riemannian structures due to Tricerri and Vanhecke [222], Chinea and Gonzalez gave a table of possible types of almost contact Riemannian structures [45, 46]. Martín Cabrera [160] proved the non-existence of 132 Chinea and González-Dávila types of almost contact Riemannian structures is proved in case the dimension is greater than 3.

8. Homogeneous contact Riemannian structures

8.1. Perrone's classification

According to Boothby and Wang [28], a contact manifold (M, η) is said to be a *homogeneous contact manifold* if there exists a Lie group of strictly contact transformations acting transitively (and effectively) on M.

Theorem 8.1 ([28])**.** *Let (M, η) be a homogeneous contact manifold, then its Reeb vector field is regular.*

Corollary 8.1 ([164])**.** *Let M be a compact homogeneous contact manifold, then its associated almost contact structure (φ, ξ, η) is normal.*

Let M be a contact Riemannian manifold. Then M is said to be a *homogeneous contact Riemannian manifold* if there exists a Lie group G of isometries which preserves the contact form and acts transitively on M. Theorem 5.1 implies that elements of G are automorphisms.

Perrone classified simply connected homogeneous contact Riemannian 3-manifolds. Such spaces are classified by the *Webster scalar curvature* $W = (\mathrm{s} - \mathrm{Ric}(\xi, \xi) + 4)/8$ and the *torsion invariant* $|\tau|^2 := |\pounds_\xi g|^2 = 8 - 4\,\mathrm{Ric}(\xi, \xi)$.

Theorem 8.2 ([190])**.** *Let $(M, \eta, \xi, \varphi, g)$ be a simply connected homogeneous contact Riemannian 3-manifold. Then M is a Lie group G equipped with a left invariant contact metric structure. If G is unimodular, then G is one of the following*:

1. *the Heisenberg group if $W = |\tau|^2 = 0$,*
2. *the special unitary group $\mathrm{SU}(2)$ if $4\sqrt{2}W > |\tau|$,*
3. *the universal covering $\widetilde{\mathrm{SE}}(2)$ of the Euclidean motion group if $4\sqrt{2}W = |\tau| > 0$,*
4. *the universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ if $-|\tau| \neq 4\sqrt{2}W < |\tau|$,*
5. *the Minkowski motion group $\mathrm{SE}(1, 1)$ if $4\sqrt{2}W = -|\tau| < 0$.*

If G is non-unimoduar, then the Lie algebra $\mathfrak{g}$ satisfies the commutation relations:

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = \gamma e_2,$$

where $\xi = e_3$, $e_1, e_2 \in \Gamma(\mathcal{D})$, $e_2 = \varphi e_1$, $\alpha \neq 0$ and $4\sqrt{2}W < |\tau|$. If $\gamma = 0$, then G is Sasakian.

Here we compute the fundamental quantities of 3-dimensional unimodular Lie group classified in Perrone's classification (see [99]).

Proposition 8.1. *Let G be a 3-dimensional unimodular Lie group equipped with a left invariant contact Riemannian structure (φ, ξ, η, g). Take a unimodular basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$, then the endomorphism field h, the Webster scalar curvature and the torsion invariant are given by*

$$he_1 = \lambda e_1, \quad he_2 = -\lambda e_1, \quad \lambda = -\frac{1}{2}(c_1 - c_2), \quad W = \frac{1}{4}(c_1 + c_2), \quad |\tau|^2 = (c_1 - c_2)^2.$$

The holomorphic sectional curvature of G is

$$-3 + \frac{1}{4}(c_1 - c_2)^2 + c_1 + c_2.$$

The unimodular Lie group G is Sasakian if and only if $c_1 = c_2$. In such a case, G is of constant holomorphic sectional curvature $c = -3 + c_1 + c_2$.

Corollary 8.2. *If a unimodular Lie group G is non-Sasakian, i.e., $c_1 \neq c_2$, then G is a contact (κ, μ)-space with*

$$\kappa = 1 - \frac{1}{4}(c_1 - c_2)^2, \;\; \mu = 2 - (c_1 + c_2).$$

Remark 8.1. A 3-dimensional non-unimodular Lie group G equipped with a left invariant homogeneous contact metric structure with $\gamma = 0$ is a Sasakian space form of constant holomorphic sectional curvature $-3 - \alpha^2$ (see [99]).

8.2. Homogeneous almost contact Riemannian structures

Let $(M, \eta, \xi, \varphi, g)$ be an almost contact Riemannian manifold. A *homogeneous almost contact Riemannian structure* is a homogeneous Riemannian structure S which satisfies the additional condition $\tilde{\nabla}\varphi = 0$ (*cf.* [142]).
Chinea and Gonzalez obtained the following fundamental result.

Lemma 8.1 ([44]). *Let S be a homogeneous almost contact Riemannian structure on an almost contact Riemannian manifold M. Then the Ambrose-Singer connection $\tilde{\nabla}$ satisfies*

$$\tilde{\nabla}\eta = 0, \quad \tilde{\nabla}\xi = 0.$$

As a direct consequence of this Lemma, we have:

Theorem 8.3 ([44]). *Let $(M, \eta, \xi, \varphi, g)$ be a homogeneous contact Riemannian manifold. Then there exists a homogeneous contact Riemannian structure S on M. Conversely, let M be a simply connected and complete contact Riemannian manifold with a homogeneous contact Riemannian structure S then M is a homogeneous contact Riemannian manifold.*

Here we quote the following fact.

Proposition 8.2 ([18, 23]). *A non-Sasakian contact (κ, μ)-space is a locally homogeneous contact Riemannian manifold with homogeneous contact Riemannian structure*

$$S^{\mathsf{B}}(X)Y = -g(\varphi(\mathrm{I} + h)X, Y)\xi + \eta(Y)\varphi(\mathrm{I} + h)X + \frac{\mu}{2}\eta(X)\varphi Y. \tag{8.1}$$

For more information on homogeneous almost contact Riemannian structures, we refer to [146, 69, 70, 83, 227].

8.3. Homogeneous contact Riemannian structures on 3-dimensional Lie groups

8.3.1. The unimodular Lie groups Tricerri and Vanhecke studied Cartan-Schouten's $(-)$-connections and corresponding homogeneous Riemannian structures on 3-dimensional Lie groups [222]. Calviño-Louzao, Ferreiro-Subrido, García-Río and Vázquez-Lorenzo classified homogeneous Riemannian structures of 3-dimensional unimodular Lie groups under the assumption the homogeneous Riemannian structures are *left invariant* [39]. Ohno and the present author proved that the classification result due to [39] is true without the left invariance assumption.

Proposition 8.3 ([39, 124]). *Let G be a 3-dimensional unimodular Lie group with a unimodular basis $\{e_1, e_2, e_3\}$. Denote by $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ the metrically dual coframe field. Assume that c_1, c_2 and c_3 are all distinct, then the only homogeneous Riemannian structure of G is given by*

$$S_\flat = -(c_1 + c_2 - c_3)\eta \otimes (\vartheta^1 \wedge \vartheta^2) + (c_1 - c_2 - c_3)\vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) - (c_1 - c_2 + c_3)\vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1).$$

The homogeneous Riemannian structure S is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. In particular it is of type $\mathcal{T}_2$ if and only if $c_1 + c_2 + c_3 = 0$. The connection $\nabla + S$ is the Cartan-Schouten's $(-)$-connection.

It should be remarked that on unimodular Lie groups whose Lie algebras are isomorphic to $\mathfrak{sl}_2\mathbb{R}$, there exists a *non left invariant* homogeneous Riemannian structure. We will describe this phenomena in Theorem 15.3 (see also [124]).

Let us apply this classification to the unimodular Lie group equipped with a left invariant homogeneous contact Riemannian structure, then $S_\flat$ is rewritten as

$$S_\flat = \mu\eta \otimes (\vartheta^1 \wedge \vartheta^2) - 2(1+\lambda)\vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) - 2(1-\lambda)\vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1).$$

On the other hand, the almost contact connections ∇^r is described as

$$A^r_\flat = -2r\eta \otimes (\vartheta^1 \wedge \vartheta^2) - 2(1+\lambda)\vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) - 2(1-\lambda)\vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1).$$

Thus we obtain the following result.

Corollary 8.3. *Let G be a 3-dimensional unimodular Lie group equipped with a left invariant contact Riemannian structure. Assume that the structure constants c_1, c_2 and $c_3 = 2$ are all distinct, then G is a contact (κ, μ)-space with $\kappa = 1 - (c_1 - c_2)^2/4$ and $\mu = 2 - (c_1 + c_2)$. The only homogeneous Riemannian structure of G is given by*

$$S_\flat = A^r_\flat = S^{\mathsf{B}}_\flat, \quad r = -\frac{\mu}{2} = -1 + \frac{1}{2}(c_1 + c_2).$$

The homogeneous Riemannian structure coincides with the one given by (8.1). Moreover, the connection $\nabla + S$ is the Cartan-Schouten's $(-)$-connection. The homogeneous Riemannian structure S is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. In particular it is of type $\mathcal{T}_2$ if and only if $c_1 + c_2 = -2$. In particular,

1. *$\nabla^{(-)} = \nabla + S$ coincides with Tanaka-Webster connection when and only when $c_1 + c_2 = 0$. In such a case, $\mathfrak{g} \cong \mathfrak{sl}_2\mathbb{R} \cong \mathfrak{su}(1,1)$. The Lie group G is a contact $(\kappa, 2)$-space with $\kappa = 1 - c_1^2 < 1$. In case $(c_1, c_2) = (\pm 2, \mp 2)$, then G is locally isometric to $\mathrm{U}\mathbb{H}^2$ equipped with the standard contact Riemannian structure.*
2. *When $c_1 = 0$ and $c_2 \neq 0, 2$, then G is locally isomorphic to $\mathrm{SE}(2)$ if $c_2 > 0$ and $\mathrm{SE}(1,1)$ if $c_2 < 0$, respectively. The fundamental quantities are $\lambda = c_2/2$, $W = c_2/4$, $|\tau|^2 = c_2^2$. The holomorphic sectional curvature is $-3 + c_2 + c_2^2/4$.*

Note that when $c_1 = 0$ and $c_2 = 2$, then G is locally isomorphic to $\mathrm{SE}(2)$. The left invariant metric of G is flat, $\lambda = 1$, $W = 1/4$ and $|\tau|^2 = 4$. We notice that the universal covering group $\widetilde{\mathrm{SE}}(2)$ is isometric to Euclidean 3-space (but *not* isomorphic as a Lie group). For later use, we describe (almost) contact Riemannian structures on $\mathrm{SE}(2)$ in detail.

8.3.2. The Euclidean motion group $\mathrm{SE}(2)$ The rigid motion group $\mathrm{SE}(2)$ of $\mathbb{E}^2$ is the semi-direct product of rotation group $\mathrm{SO}(2)$ and translation group $(\mathbb{R}^2, +)$. The semi-direct product structure of $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ is

$$(A, \boldsymbol{p}) \cdot (B, \boldsymbol{q}) := (AB, \boldsymbol{p} + A\boldsymbol{q}), \quad A, B \in \mathrm{SO}(2), \ \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^2. \tag{8.2}$$

The semi-direct product $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ is isomorphic to the following closed subgroup of $\mathrm{GL}_3\mathbb{R}$ (Example 4.9):

$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R}, \ 0 \leq \theta < 2\pi \right\} \tag{8.3}$$

We may regard (x, y, θ) as a global coordinate system of $\mathrm{SE}(2)$. Thus $\mathrm{SE}(2)$ is $\mathbb{R}^2(x, y) \times S^1$ with multiplication rule:

$$(x, y, \theta) * (x', y', \theta') = (x + \cos\theta\, x' - \sin\theta\, y', y + \sin\theta\, x' + \cos\theta\, y', \theta + \theta'). \tag{8.4}$$

The Lie algebra $\mathfrak{e}(2)$ corresponds to

$$\mathfrak{se}(2) = \left\{ \begin{pmatrix} 0 & -w & u \\ w & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \middle|\ u, v, w \in \mathbb{R} \right\} \tag{8.5}$$

We take a basis $\{V_1, V_2, V_3\}$ of $\mathfrak{se}(2)$:

$$V_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the left translated vector fields of V_1, V_2 and V_3 are

$$v_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad v_2 = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}, \quad v_3 = \frac{\partial}{\partial \theta}.$$

The left invariant Riemannian metric determined by the condition $\{V_1, V_2, V_3\}$ is orthonormal is

$$g = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}\theta^2.$$

Namely, as a Riemannian manifold, $\mathrm{SE}(2)$ is a Riemanian product of $\mathbb{E}^2(x, y)$ and $\mathbb{S}^1$. In other words, $\mathrm{SE}(2)$ is identified with the unit tangent sphere bundle $\mathrm{U}\mathbb{E}^2$. Since

$$[v_1, v_2] = 0, \quad [v_2, v_3] = v_1, \quad [v_3, v_1] = v_2,$$

$\{v_1, v_2, v_3\}$ is a unimodular basis. The Levi-Civita connection ∇ of $\mathrm{SE}(2)$ is described as follows (Table 3):

$$\nabla_{v_1} v_1 = 0, \quad \nabla_{v_1} v_2 = 0, \quad \nabla_{v_1} v_3 = 0,$$

$$\nabla_{v_2} v_1 = 0, \quad \nabla_{v_2} v_2 = 0, \quad \nabla_{v_2} v_3 = 0,$$

$$\nabla_{v_3} v_1 = v_2, \quad \nabla_{v_3} v_2 = -v_1, \quad \nabla_{v_3} v_3 = 0.$$

Define a linear endomorphism ψ on $\mathrm{SE}(2)$ by

$$\psi v_1 = v_2, \quad \psi v_2 = -v_1, \quad \psi v_3 = 0$$

and set $\zeta = v_3$ and $\omega = g(v_3, \cdot)$. Then the triplet (ψ, ζ, ω) gives a left invariant almost contact structure compatible to g. Since ω is exact, this almost contact structure is *non-contact*. Denote by Ψ the fundamental 2-form. Then one can see that $\mathrm{d}\omega = 0$ and $\mathrm{d}\Psi = 0$. hence the structure (ψ, ζ, ω, g) is almost coKähler satisfying $\nabla\zeta = 0$ and hence $(\mathrm{SE}(2), \psi, \zeta, \omega, g)$ is coKähler space form.

In the next subsection, we exhibit canonical contact structure of $\mathrm{SE}(2)$.

8.3.3. Canonical contact structure on $\mathrm{SE}(2)$ The universal covering group $\widetilde{\mathrm{SE}}(2)$ of $\mathrm{SE}(2)$ is Cartesian 3-space $\mathbb{R}^3(x, y, z)$ with multiplication:

$$(x, y, z) * (x', y', z') = (x + \cos z\, x' - \sin z\, y', y + \sin z\, x' + \cos z\, y', z + z'). \tag{8.6}$$

On $\widetilde{\mathrm{SE}}(2) = (\mathbb{R}^3, *)$, the discrete subgroup $\Gamma_{\mathsf{E}} = 2\pi\mathbb{Z}$ of $(\mathbb{R}^3, *)$ acts on $\widetilde{\mathrm{SE}}(2)$ by translation:

$$\widetilde{\mathrm{SE}}(2) \times \Gamma_{\mathsf{E}} \to \widetilde{\mathrm{SE}}(2); \ \ (x, y, z) \cdot 2\pi m = (x, y, z + 2\pi m). \tag{8.7}$$

This action is properly discontinuous. The factor space of $\widetilde{\mathrm{SE}}(2)$ is diffeomorphic to $\mathbb{R}^2(x, y) \times \mathbb{S}^1$ and identified with $\mathrm{SE}(2)$. Let us denote by p_{E} the projection $p_{\mathsf{E}} : \widetilde{\mathrm{SE}}(2) \to \mathrm{SE}(2)$. To adapt with contact Riemannian condition $\Phi = \mathrm{d}\eta$, we take a contact form

$$\tilde{\eta} = \frac{1}{2}(\cos z\, \mathrm{d}x + \sin z\, \mathrm{d}y)$$

and a Riemannian metric

$$\tilde{g} = \frac{1}{4}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2).$$

Then the Reeb vector field is

$$\tilde{\xi} = 2\left(\cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}\right).$$

Let us introduce the endomorphism field $\tilde{\varphi}$ by

$$\mathrm{d}\tilde{\eta}(X,Y) = \tilde{g}(X, \tilde{\varphi}Y), \quad X, Y \in \Gamma(\mathrm{T}\widetilde{\mathrm{SE}}(2)).$$

Since

$$\mathrm{d}\tilde{\eta} = -\frac{\cos z}{2}\,\mathrm{d}y \wedge \mathrm{d}z - \frac{\sin z}{2}\,\mathrm{d}z \wedge \mathrm{d}x,$$

we have

$$\tilde{\varphi}\partial_x = -\sin z\,\partial_z, \quad \tilde{\varphi}\partial_y = \cos z\,\partial_z, \quad \tilde{\varphi}\partial_z = \sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y}.$$

Set

$$\tilde{e}_1 = 2\left(-\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}\right), \quad \tilde{e}_2 = 2\frac{\partial}{\partial z}, \quad \tilde{e}_3 = \tilde{\xi}.$$

Then we have

$$[\tilde{e}_1, \tilde{e}_2] = 2\,\tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3] = 0, \quad [\tilde{e}_3, \tilde{e}_1] = 2\,\tilde{e}_2.$$

Hence $(\widetilde{\mathrm{SE}}(2), \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a simply connected unimodular Lie group equipped with a left invariant contact Riemannian structure. From Corollary 8.3, we deduce that the homogeneous contact Riemannian 3-manifold $(\widetilde{\mathrm{SE}}(2), \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a contact $(0,0)$-space. One can confirm that the almost contact connection ∇^r defined by (5.16) with respect to the contact Riemannian structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an Ambrose-Singer connection when and only when $r = 0$ (Sasaki-Hatakeyama's (φ, ξ, η)-connection). Moreover, since $\widetilde{\mathrm{SE}}(2)$ is a contact $(0,0)$-space, the Sasaki-Hatakeyama's (φ, ξ, η)-connection coincides with Boeckx's connection (8.1). It should be emphasized that the Sasaki-Hatakeyama's (φ, ξ, η)-connection of $\widetilde{\mathrm{SE}}(2)$ is nothing but the $(-)$-connection. The $(-)$-connection of $\widetilde{\mathrm{SE}}(2)$ is described as (*cf.* (16.6)):

$$\nabla^{(-)}_{\partial_z}\partial_x = -\partial_y, \quad \nabla^{(-)}_{\partial_z}\partial_y = \partial_x. \tag{8.8}$$

The contact form $\tilde{\eta}$ induces a contact form η on $\mathrm{SE}(2)$ so that $p_{\mathsf{E}}^*\eta = \tilde{\eta}$. As a contact manifold, the Euclidean motion group $\mathrm{SE}(2)$ is isomorphic to $(\mathbb{R}^3/2\pi\mathbb{Z}, \eta)$.

8.3.4. Standard contact structure on tori The canonical contact structure on $\widetilde{\mathrm{SE}}(2)$ induces a contact structure on 3-dimensional tori. In this section, we exhibit the induced contact structure on flat tori.

The contact form $\tilde{\eta}$ is invariant under the action of discrete subgroup $\Gamma_{\mathsf{T}} = 2\pi\,\mathbb{Z}^3$ of the abelian group $(\mathbb{R}^3, +)$ defined by

$$(x, y, z) + 2\pi(l, m, n), \quad (l, m, n) \in \mathbb{Z}^3.$$

Furthermore the Euclidean metric $\tilde{g} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is invariant under Γ_{T}. Hence $\tilde{\eta}$ induces a contact structure η_{T} on the (flat) torus $\mathbb{T}^3 = \mathbb{R}^3/\Gamma_{\mathsf{T}}$. Thus $(\mathbb{T}^3, \eta_{\mathsf{T}})$ is a compact flat 3-manifold which admits a contact structure [95, 113].

Proposition 8.4. *The 3-torus $\widetilde{\mathrm{SE}}(2)/2\pi\mathbb{Z}^3$ admits a contact structure.*

Note that on Γ_{T}, two multiplications "+" and "$*$" given by (8.6) coincide. Hence the factor space $(\mathbb{R}^3, *)/2\pi\mathbb{Z}^3$ is a 3-torus with "noncommutative" Lie group structure.

We can see that this contact manifold $\mathbb{T}^3$ is *not* regular. The integral curve $\psi(t)$ of the Reeb vector field $\tilde{\xi}$ through $(0, 0, \pi/3)$ is

$$\psi(t) = \mathrm{Exp}(t\tilde{\xi})(0, 0, \pi/3) = \left(\frac{t}{2}, \frac{\sqrt{3}t}{2}, \frac{\pi}{3}\right).$$

Hence ξ induces an irrational flow on 2-torus in $\mathbb{T}^3$ defined by $z = \pi/3$. Thus the 3-torus $\mathbb{T}^3$ is not regular contact manifold. More generally every 3-torus can not admit regular contact structure. The contact Riemanian structure on $\mathbb{E}^3$ and $\mathbb{T}^3$ are not homogeneous with respect to the additive group structure but homogeneous with respect to the group structure of $\widetilde{\mathrm{SE}}(2)$. In particular, $\mathrm{SE}(2)$ itself is a homogeneous contact Riemannian manifold.

According to this observation, it seems to be natural that the contact structure determined by $\tilde{\eta}$ on $\mathbb{R}^3$ is regarded as a contact structure on the covering group $\widetilde{\mathrm{SE}}(2)$ of the Euclidean motion group from the group theoretical view.

8.3.5. The model space Sol_3 Let us realize the Minkowski plane $\mathbb{E}^{1,1}$ as Cartesian plane $\mathbb{R}^2(x,y)$ equipped with the scalar product $\mathrm{d}x \odot \mathrm{d}y$ relative to the global null coordinates (x,y). Then the identity component of the isometry group is realized as

$$\mathrm{SE}(1,1) = \left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \ x,y,z \in \mathbb{R} \right\}$$

and called the Minkowski motion group (Remark 4.2, Table 3). The Lie algebra $\mathfrak{se}(1,1)$ is given by

$$\mathfrak{se}(1,1) = \left\{ \begin{pmatrix} -w & 0 & u \\ 0 & w & v \\ 0 & 0 & 0 \end{pmatrix} \middle| \ u,v,w \in \mathbb{R} \right\}.$$

We can take a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{se}(1,1)$. By left translation, we obtain a left invariant vector fields:

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_1 = e^{z}\frac{\partial}{\partial y}, \quad e_1 = \frac{\partial}{\partial z},$$

$$[e_1,e_2] = 0, \quad [e_2,e_3] = -e_2, \quad [e_3,e_1] = -e_1.$$

The left invariant Riemannian metric on $\mathrm{SE}(1,1)$ determined by the condition $\{e_1,e_2,e_3\}$ is orthonormal with respect to it is

$$g = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2.$$

The homogeneous Riemannian 3-space $\mathrm{Sol}_3 := (\mathbb{R}^3, g) = \mathrm{SE}(1,1)/\{E_3\}$ is the model space of *solvegeometry* in the sense of Thurston [220]. According to the classification of Kowalski [149, Theorem VI.2], the only simply connected proper generalized symmetric Riemannian 3-space is $\mathrm{Sol}_3 = \mathrm{Sol}_3/\{\mathsf{e}\}$. The Riemannian 4-symmetric space representation is $\mathrm{Sol}_3 = \mathrm{Sol}_3/\{\mathsf{e}\}$ associated with the automorphism

$$\tau(x,y,z) = (-y,x,-z).$$

It should be remarked that $\mathrm{Sol}_3/\{e\}$ is irreducible. Tsunero Takahashi [210] showed that Sol_3 can be isometrically embedded in hyperbolic 4-space $\mathbb{H}^4$ of constant curvature -1. He proved that simply connected homogeneous Riemannian 3-space (M^3,g) can be isometrically immersed in $\mathbb{H}^4$ with type number 2 if and only if M^3 is isometric to Sol_3.

8.4. The standard contact structure on $\mathrm{SE}(1,1)$

Let us start to study homogeneous contact structures on the space Sol_3. To adapt with the convention $\mathrm{d}\eta = \Phi$, the standard contact form of Sol_3 is rescaled as

$$\omega = \frac{1}{2\sqrt{2}}(e^{z}\mathrm{d}x + e^{-z}\mathrm{d}y).$$

The Reeb vector field ζ of ω is

$$\zeta = \sqrt{2}\left(e^{-z}\frac{\partial}{\partial x} + e^{z}\frac{\partial}{\partial x}\right).$$

The associated Riemannian metric is

$$\tilde{g} = \frac{1}{4}g = \frac{1}{4}(e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2).$$

Note that the Levi-Civita connection of $\tilde{g}$ coincides with ∇ of g.

Take a left invariant orthonormal frame field $\{u_1, u_2, u_3\}$ of $(\mathrm{Sol}_3, \tilde{g})$ defined by

$$u_1 = 2\frac{\partial}{\partial z}, \quad u_2 = -\sqrt{2}\left(e^{-z}\frac{\partial}{\partial x} - e^{z}\frac{\partial}{\partial y}\right), \quad u_3 = \sqrt{2}\left(e^{-z}\frac{\partial}{\partial x} + e^{z}\frac{\partial}{\partial y}\right) = \zeta.$$

Then the Levi-Civita connection of $\tilde{g}$ is described as

$$\nabla_{u_1}u_1 = \nabla_{u_1}u_2 = \nabla_{u_1}u_3 = 0,$$

$$\nabla_{u_2}u_1 = -2u_3, \quad \nabla_{u_2}u_2 = 0, \quad \nabla_{u_2}u_3 = 2u_1,$$
$$\nabla_{u_3}u_1 = -2u_2, \quad \nabla_{u_3}u_2 = 2u_1, \quad \nabla_{u_3}u_3 = 0.$$

The left invariant orthonormal frame field $\{u_1, u_2, u_3\}$ satisfies

$$[u_1, u_2] = 2u_3, \quad [u_2, u_3] = 0, \quad [u_3, u_1] = -2u_1.$$

Hence $\{u_1, u_2, u_3\}$ is a unimodular basis of $\mathfrak{sol}_3$.

The endomorphism field ψ determined by the formula

$$\mathrm{d}\omega(X, Y) = \tilde{g}(X, \psi Y)$$

is computed as

$$\psi u_1 = u_2, \ \ \psi u_2 = -u_1, \ \ \psi u_3 = 0.$$

The endomorphism field ψ is described as

$$\psi e_1 = \frac{1}{\sqrt{2}}e_3, \quad \psi e_2 = -\frac{1}{\sqrt{2}}e_3, \quad \psi e_3 = -\frac{1}{\sqrt{2}}(e_1 - e_2)$$

relative to the orthonormal frame field $\{e_1, e_2, e_3\}$.

From Corollary 8.3, we deduce that the homogeneous contact Riemannian 3-manifold $(\mathrm{Sol}_3, \tilde{g})$ is a contact $(0, 4)$-space. The only homogeneous (contact) Riemannian structure is $S = A^{-2}$. This connection will be reinterpreted in Corollary 16.1.

9. Sasakian φ-symmetric spaces

Okumura [175] proved that locally symmetric Sasakian manifolds are of constant curvature 1. Tanno generalized Okumura's result to K-contact manifolds [216]. Boeckx and Cho [26] proved that locally symmetric contact Riemannian $(2n+1)$-manifolds are locally isomorphic to $\mathbb{S}^{2n+1}$ or the unit tangent sphere bundle $\mathrm{U}\mathbb{E}^{n+1}$ equipped with a normalized Sasaki-lift metric (thus it is isometric to $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$). Thus the local symmetry is a too strong restriction for contact Riemannian manifolds.

9.1. The canonical connection

Let M be a Sasakian manifold. Then the almost contact connection (5.16) has a particular form:

$$\nabla^r_X Y = \nabla_X Y + A^r_X Y, \quad A^r_X Y = \mathrm{d}\eta(X, Y)\xi - r\eta(X)\varphi Y + \eta(Y)\varphi X, \quad r \in \mathbb{R}. \tag{9.1}$$

Since Sasakian manifolds are strongly pseudo-convex, the connection $\nabla^r|_{r=-1}$ coincides with Tanaka-Webster connection $\hat{\nabla}$. This 1-parameter family of linear connection coincides with the one introduced by Okumura [175]. The connection $\nabla^r|_{r=1}$ was investigated by Motomiya [165] (see also [136]). Some authors called $\nabla^r|_{r=1}$, the *Okumura connection* (*e.g.*, [21]). Toshio Takahashi [209] called it the M-*connection*.

The connection $\nabla^r|_{r=1}$ is called the *characteristic connection* by Friedrich and Ivanov [73].

Proposition 9.1 ([175, 209])**.** *The linear connection ∇^r satisfies*

$$\nabla^r g = 0, \quad \nabla^r \varphi = 0, \quad \nabla^r \eta = 0, \quad \nabla^r \xi = 0, \quad \nabla^r A^r = 0.$$

This Proposition implies that if a Sasakian manifold M satisfies $\nabla^r R = 0$ then it is a locally homogeneous Sasakian manifold with homogeneous contact Riemannian structure A^r.

Proposition 9.2 (*cf.* [126, 209])**.** *The curvature R^r and torsion T^r of the connection ∇^r satisfies the following formulas:*

$$\begin{aligned} T^r(X,Y) =& 2\mathrm{d}\eta(X,Y)\xi - (r+1)\{\eta(X)\varphi Y - \eta(Y)\varphi X\}, \\ R^r(X,Y)Z =& R(X,Y)Z + \{\eta(Y)g(Z,X) - \eta(X)g(Y,Z)\}\xi + \eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X \\ &+ \mathrm{d}\eta(Y,Z)\varphi X + \mathrm{d}\eta(Z,X)\varphi Y - 2r\mathrm{d}\eta(X,Y)\varphi Z, \\ R^r(X,Y)\xi =& R^r(\xi,X)Y = 0, \quad \eta(R^r(X,Y)Z) = 0, \\ R^r(\varphi X, \varphi Y)\varphi Z =& \varphi R^r(X,Y)Z. \end{aligned}$$

for all X,Y,$Z \in \Gamma(\mathrm{T}M)$.

Proposition 9.3 ([209])**.** *On a Sasakian manifold M, $\nabla^r R^r = 0$ holds if and only if $\nabla^r R = 0$.*

Here we recall a more geometric interpretation of the parallelism $\nabla^r R^r = 0$. For this purpose we use the following conservation law:

Lemma 9.1. *Let M be a K-contact manifold. A geodesic γ initially orthogonal to ξ remain orthogonal to ξ.*

The following notion was originally introduced by Toshio Takahashi [209]. Here is a reformulated one due to Bueken and Vanhecke [32].

Definition 9.1 ([32, 209])**.** Let M be a K-contact manifold.

1. A geodesic $\gamma(s)$ parametrized by arc length in M is said to be a *φ-geodesic* if $\eta(\gamma') = 0$.
2. A local diffeomorphism s_p is said to be a *φ-geodesic symmetry* with base point $p \in M$ if for each φ-geodesic $\gamma(s)$ such that $\gamma(0)$ lies in the trajectory of ξ passing through p, $s_p\gamma(s) = \gamma(-s)$ for each s.
3. M is said to be a *locally φ-symmetric space* if its φ-geodesic symmetries are isometric.

Since the points of the Reeb flow through p is are fixed by s_p, one can see that s_p is represented as a polar map (see (4.3)):

$$s_p = \exp_p \circ (-\mathrm{I}_p + 2\eta_p \otimes \xi_p) \circ \exp_p^{-1}.$$

Bueken and Vanhecke proved the following important fact.

Proposition 9.4 ([32])**.** *Let M be a locally φ-symmetric space, then M is Sasakian and all the local φ-geodesic symmetries are local automorphisms.*

A K-contact manifold M whose local φ-geodesic symmetries are extendable to global isometries is called a *φ-symmetric space*. Takahashi proved that φ-symmetric spaces are homogeneous Sasakian manifolds on which the automorphism group acts transitively. Moreover, φ-symmetric spaces are regular Sasakian manifolds and principal bundles over Hermitian symmetric spaces.

Takahashi characterized locally φ-symmetric spaces in terms of almost contact connections as follows:

Proposition 9.5 ([209])**.** *A Sasakian manifold is locally φ-symmetric if and only if the curvature tensor R^r of the almost contact connections ∇^r is parallel with respect to ∇^r, i.e., $\nabla^r R^r = 0$.*

Since the difference tensor A^1 satisfies $A^1_X X = 0$, the following interesting result is deduced [21].

Proposition 9.6. *Let M be a locally φ-symmetric space. Then A^1 defines a homogeneous Riemannian structure of type $\mathcal{T}_3$ on M.*

Three-dimensional φ-symmetric spaces are classified by Blair and Vanhecke [20] (as a consequence of the classification of 3-dimensional naturally reductive homogeneous spaces). Next, Kowalski and Węgrzynowski [155] classified 5-dimensional φ-symmetric spaces. Jiménez and Kowalski [135] had done a full classification of φ-symmetric spaces (see also Tamaru [211]).

Here we specialize that $\dim M = 3$. Take a local orthonormal frame field $\{e_1, e_2, e_3\}$ satisfying

$$e_2 = \varphi e_1, \quad e_3 = \xi.$$

Denote by $\{\vartheta^1, \vartheta^2, \vartheta^3\}$ the dual orthonormal coframe field to $\{e_1, e_2, e_3\}$, then A^r is locally expressed in the following form:

Proposition 9.7. *The almost contact connection ∇^r on a Sasakian 3-manifold M is expressed as $\nabla^r = \nabla + A^r$ with*

$$A^r_\flat = -2r\vartheta^3 \otimes (\vartheta^1 \wedge \vartheta^2) - 2\vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) - 2\vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1).$$

In particular $A^1_\flat = -\mathrm{d}v_g$.

Proof. Since $A^r_\flat$ satisfies

$$A^r_\flat(X,Y,Z) + A^r_\flat(X,Z,Y) = 0,$$

$A^r_\flat$ is expressed as

$$A^r_\flat = 2A^r_\flat(X,e_1,e_2)(\vartheta^1 \wedge \vartheta^2) + 2A^r_\flat(X,e_2,e_3)(\vartheta^2 \wedge \vartheta^3) + 2A^r_\flat(X,e_3,e_1)(\vartheta^3 \wedge \vartheta^1).$$

By the definition of A^r, we get the required result. □

Corollary 9.1. *Let M be a 3-dimensional Sasakian φ-symmetric space, then A^r is a homogeneous Riemannian structure of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. Moreover A^r is of type $\mathcal{T}_2$ if and only if $r = -2$.*

Proof. Some calculation show that $c_{12}(A^r_\flat) = 0$ and

$$\underset{X,Y,Z}{\mathfrak{S}} A^r_\flat(X,Y,Z) = -4(r+2)\{\vartheta^3 \otimes (\vartheta^1 \wedge \vartheta^2) + \vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) + \vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1)\}.$$

□

Under the *Sasakian assumption*, Bueken and Vanhecke proved the following characterization.

Proposition 9.8 ([31]). *A Sasakian manifold M is locally φ-symmetric if and only if all the local φ-geodesic symmetries are harmonic maps.*

Remark 9.1 (Reflections around curves). Local φ-geodesic symmetry is a particular example of reflection around a curve. Let (M,g) be a Riemannian manifold and γ be an embedded curve in M with tubular neighborhood U. For any point $p \in M$ we set

$$p = \exp_{\gamma(t)}(ru), \quad u \in \mathrm{T}^\perp_{\gamma(t)}\gamma, \quad \|u\| = 1, \quad t \in [a,b],$$

where r is the Riemannian distance between p and $\gamma(t)$. Then the map $s_\gamma : U \to U$ defined by

$$s_\gamma(\exp_{\gamma(t)}(ru)) = \exp_{\gamma(t)}(-ru)$$

is called a *local reflection around* γ [224]. More generally, Nicolodi and Vanhecke [167, 168, 169] introduced the notion of rotation around a curve in the following manner. Take a field S of linear endomorphisms defined along γ satisfying

$$\mathsf{S}(t)\dot\gamma(t) = \dot\gamma(t), \quad g(\mathsf{S}(t)X, \mathsf{S}(t)Y) = g(X,Y), \quad X,Y \in \mathrm{T}^\perp_{\gamma(t)}M.$$

Then the polar map s_γ

$$s_\gamma = \exp_\gamma \circ \mathsf{S} \circ \exp_\gamma^{-1}$$

is called the *rotation around* γ. In case $\mathsf{S} - \mathrm{I}$ is non-singular in the normal bundle of γ, then s_γ is called a *free rotation*. One can see that rotation aroud γ with $\mathsf{S} = -\mathrm{I}$ is noting but the reflection around γ. On a K-contact manifold M, a reflection with $\mathsf{S} = -\mathrm{I} + 2\eta \otimes \xi$ is a local φ-geodesic symmetry. On an almost Hermitian manifold (M,g,J), Nicolodi and Vanhecke [167] studied rotations $j_\gamma = \exp_\gamma \circ J \circ \exp_\gamma^{-1}$ which are called *J-rotations*. The present author [126] and Bueken and Vanhecke [33] studied rotations of the form $s_\gamma = \exp_\gamma \circ (\varphi + \eta \otimes \xi) \circ \exp_\gamma^{-1}$ (called *φ-rotations*). Bueken and Vanhecke proved that a real analytic Sasakian manifold M is locally φ-symmetric if and only if all the φ-rotations are isometric. Moreover a φ-rotation is harmonic if and only if it is isometric. On the other hand, the present author [126] proved that a K-contact manifold is locally φ-symmetric if and only if all the φ-rotations are isometric. Another kind of generalization of φ-geodesic symmetry on K-contact manifolds was proposed in [128].

For more information on Sasakian φ-symmetric spaces, we refer to [22, 30, 135, 155]. Recently Ohnita [172, 173] used the connection ∇^r with $r = -1/2$

9.2. $\mathrm{SU}(n+1)$*-invariant metric connections*

Here we point out that Okumura connection $\nabla^1 = \nabla + A^1$ on $\mathbb{S}^{2n+1}$ can be discovered by the classification of $\mathrm{SU}(n+1)$-invariant metric connections [62, Theorem 4.9].

Theorem 9.1. *For every $\mathrm{SU}(n+1)$-invariant metric linear connection D on the unit sphere $\mathbb{S}^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ with $n \geq 4$ equipped with a $\mathrm{SU}(n+1)$-invariant Sasakian structure, there exit $q \in \mathbb{C}$ and $t \in \mathbb{R}$ such that the difference tensor field $S = D - \nabla$ is expressed as*

$$S(X)Y = (\mathrm{Re}\, q - 1)(g(X, \varphi Y)\xi + \eta(Y)\varphi X) + \mathrm{Im}\, q\, (\nabla_X \varphi)Y + \left(t + \frac{1}{n}\right)\eta(X)\varphi Y.$$

In particular, every $\mathrm{SU}(n+1)$-invariant metric linear connection D with totally skew symmetric torsion is expressed as

$$D_X Y = \nabla_X Y + t\, A^1(X)Y$$

for some $t \in \mathbb{R}$.

In case $n > 1$, the canonical connection ∇^{c} of the second kind (in the sense of Nomizu) of $\mathbb{S}^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ is given by (see [62, Example 4.11]):

$$\nabla^{\mathrm{c}}_X Y = \nabla_X Y - g(X, \varphi Y)\xi - \eta(Y)\varphi X + \frac{1}{n}\eta(X)\varphi Y.$$

Thus ∇^{c} does not have totally skew-symmetric torsion. Note that when $n = 1$, the isotropy algebra is $\{0\}$.

9.3. CR-symmetry

Now let M be a strongly pseudo-convex CR-manifold. A local diffeomorphism σ_p defined around a point $p \in M$ is said to be a *local* CR-*symmetry* at p if p is an isolated fixed point of it and satisfies $(\mathrm{d}\sigma_p)_p\big|_{\mathcal{D}_p} = -\mathrm{I}_p\big|_{\mathcal{D}_p}$.

A strongly pseudo-convex CR-manifold M is said to be locally CR-*symmetric* if all the local CR-symmetries are local CR-automorphism. When all the local CR-symmetry of a locally CR-symmetric space M are extendable to global ones, then M is said to be a CR-*symmetric space* ([57, 138]). One can see that the local CR-symmetry satisfies

$$(\mathrm{d}\sigma_p)_p = -\mathrm{I}_p + 2\eta_p \otimes \xi_p.$$

Thus we obtain the following CR-geometric interpretation of local φ-symmetry [57]:

Proposition 9.9. *A Sasakian manifold M is locally CR-symmetric if and only if it is locally φ-symmetric.*

A *spherical CR-manifold* is a strongly pseudo-convex CR-manifold locally CR-equivalent to $\mathbb{S}^{2n+1}$. Spherically CR-manifolds are characterized as strongly pseudo-convex CR-manifolds with vanishing Chern-Moser-Tanaka invariant. The unit sphere $\mathbb{S}^{2n+1}$ $(n \geq 2)$ is characterized as the unique simply connected and compact spherically CR-symmetric space up to homothety.

Theorem 9.2 ([57]). *Let M be a strongly pseudo-convex CR-manifold of dimension $2n+1 > 3$. Assume that M is non-Sasakian. Then M is locally CR-symmetric if and only if it is a contact (κ, μ)-space.*

Theorem 9.3 ([57]). *Let M be a locally CR-symmetric strongly pseudo-convex CR-manifold of dimension $2n+1 > 3$. Then the following properties are mutually equivalent*:

- *M is spherical.*
- *$\mu = 2$.*
- *The Webster curvature of M vanishes.*

Example 9.1. Let $M^n(\varepsilon c^2)$ be an n-dimensional Riemannian space form of curvature εc^2. Here $\varepsilon = 0$ or ± 1 and c is a positive constant. Then its unit tangent bundle $\mathrm{U}M^n(\varepsilon c^2)$ equipped with the standard contact Riemannian structure is a contact (κ, μ)-space with

$$\kappa = \varepsilon c^2(2 - \varepsilon c^2), \quad \mu = -2\varepsilon c^2.$$

In particular, the unit tangent sphere bundle $\mathrm{U}\mathbb{H}^n$ is a contact $(-3, 2)$-space with vanishing Boeckx invariant.

Remark 9.2. Boeckx and Cho [27] investigate two classes of contact Riemannian manifolds.

1. Contact Riemannian manifolds whose generalized Tanaka-Webster connection ${}^*\nabla$ satisfying
$$ {}^*\nabla\,{}^*T = 0, \quad {}^*\nabla\,{}^*R = 0. $$
Here ${}^*T$ and ${}^*R$ are torsion and curvature of ${}^*\nabla$. Note that the parallelism ${}^*\nabla\,{}^*T = 0$ implies the integrability of the associated almost CR-structures. They showed that those contact Riemannian manifolds are Sasakian locally φ-symmetric or non-Sasakian contact $(\kappa, 2)$-spaces.

2. A strongly pseudo-convex CR-manifold M is said to be a *weakly locally pseudo-Hermitian symmetric space* in the sense of Boeckx-Cho if
$$ L((\hat{\nabla}_X \hat{R})(Y,Z)U,V) = 0 $$
for any X, Y, Z, U, $V \in \Gamma(\mathcal{D})$. Here L is the Levi-form and $\hat{R}$ is the curvature of the Tanaka-Webster connection $\hat{\nabla}$.
They showed that locally φ-symmetric spaces are weakly locally pseudo-Hermitian symmetric spaces. Moreover non-Sasakian contact (κ,μ)-spaces are also weakly locally pseudo-Hermitian symmetric.

On the other hand, a strongly pseudo-convex CR-manifold M is said to be a *strongly locally pseudo-Hermitian symmetric space* in the sense of Boeckx-Cho if all $\hat{\nabla}$-reflections around the Reeb flow are locally affine mappings. It is known that M is strongly locally pseudo-Hermitian symmetric if and only if [14, 27, 65]
$$ \hat{\nabla}_X \hat{T} = 0, \quad \hat{\nabla}_X \hat{R} = 0 $$
for any $X \in \Gamma(\mathcal{D})$.

Moreover, a strongly pseudo-convex CR manifold is a strongly locally pseudo-Hermitian symmetric if and only if M is locally φ-symmetric or a non-Sasakian contact (κ,μ)-space ([27, Theorem 14]).

The local φ-symmetry can be interpreted as sub-Riemannian symmetry. We recommend [3, 14, 65, 66, 67, 208] for interested readers.

9.4. *Almost contact connections*

In [225], Vezzoni considered linear connections on contact manifolds satisfying the following properties:

1. The contact distribution is invariant, that is, for any $X \in \Gamma(\mathrm{T}M)$, $D_X\Gamma(\mathcal{D}) \subset \Gamma(\mathcal{D})$.
2. $D_\xi Y = [Y,\xi]$ for any $Y \in \Gamma(\mathcal{D})$.
3. $D_X\xi = 0$ for any $X \in \Gamma(\mathcal{D})$.
4. $(D_Y(\mathrm{d}\eta))(Y_1,Y_2) = 0$ for any $Y_1, Y_2 \in \Gamma(\mathcal{D})$.

Such a linear connection D is called an *almost contact connection* in [225]. The *transverse torsion* $T^D_{\mathcal{D}}$ of D is defined by
$$ T^D_{\mathcal{D}}(X,Y) = D_XY - D_YX - [X,Y]_{\mathcal{D}}, \quad X,Y \in \Gamma(\mathcal{D}). $$
An almost contact connection D in the sense of Vezzoni is said to be a *contact connection* if its transverse torsion vanishes.

Vezzoni proved that every contact Riemannian manifold M admits such a linear connection. One can confirm the following propositions.

Proposition 9.10. *The generalized Tanaka-Webster connection ${}^*\nabla$ of a contact Riemannian manifold M can be a contact connection in the sense of Vezzoni when and only when M is Sasakian.*

Proposition 9.11. *On a Sasakian manifold M, the linear connection ∇^r is a contact connection in the sense of Vezzoni if and only if $r = -1$, that is, it is the Tanaka-Webster connection.*

Note that the natural contact connection of a Sasakian manifold given in [225, Example 2.4] coincides with the Tanaka-Webster connection.

10. Sasakian space forms

10.1. Three dimensional Sasakian space forms

Let M be a $(2n+1)$-dimensional Sasakian space form of constant holomorphic sectional curvature c. Then M is said to be an *elliptic* [resp. *parabolic* or *hyperbolic*] Sasakian space form if $c > -3$ [resp. $c = -3$ or $c < -3$].

Tanno [218] classified simply connected Sasakian space forms. In 3-dimensional case, Tanno's classification is reformulated as follows (*cf.* [10, 20]) :

Proposition 10.1. ([218]) *Let $\mathcal{M}^3(c)$ be a 3-dimensional simply connected Sasakian space form of constant holomorphic sectional curvature c. Then $\mathcal{M}^3(c)$ is isomorphic to Heisenberg group Nil_3 with canonical Sasakian structure if $c = -3$, and $\mathcal{M}^3(c)$ is isomorphic to the universal covering group $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$ equipped with Sasakian structure if $c < -3$.*

It is known that every Sasakian manifold of constant holomorphic sectional curvature is locally φ-symmetric [209]. Conversely, every 3-dimensional Sasakian φ-symmetric space is of constant holomorphic sectional curvature [20].

10.2. Elliptic Sasakian space forms

Now we recall Tanno's explicit construction of simply connected elliptic Sasakian space form $\mathcal{M}^{2n+1}(c),\ c > -3$.

Let us denote by $(\eta_1, \xi_1, \varphi_1, g_1)$ the canonical contact structure Riemannian of unit sphere $\mathbb{S}^{2n+1}$. For any constant $c > -3$, we perform the transversally homothetic deformation (5.5) to $\mathbb{S}^{2n+1}$ with $a = 4/(c+3)$. The resulting structure is described as

$$\eta := \frac{4}{c+3}\eta_1, \quad \xi := \frac{c+3}{4}\xi_1, \quad \varphi = \varphi_1, \quad g := \frac{4}{c+3}g_1 + \frac{4(1-c)}{(c+3)^2}\eta_1 \otimes \eta_1. \tag{10.1}$$

One can easily check that the $\mathcal{D}$-homothetic deformation of $\mathbb{S}^{2n+1}$ is a Sasakian manifold of constant holomorphic sectional curvature $c > -3$. Tanno classified the simply connected elliptic Sasakian space forms.

Proposition 10.2. ([218]) *Every simply connected 3-dimensional elliptic Sasakian space form $\mathcal{M}^{2n+1}(c)$ is isomorphic to a $\mathcal{D}$-homothetic deformation of the unit sphere $\mathbb{S}^{2n+1}$ with $c > -3$.*

As a Riemannian $(2n+1)$-manifold, $\mathcal{M}^{2n+1}(c)$ with $c > -3$ and $c \neq 1$ is the so-called *Berger sphere* (up to homothety) [11, 12]. We give explicit models of 3-dimensional Sasakian space forms in Section 10, 11 and 12 (and 14) according as $c = -3$, $c > -3$ and $c < -3$, respectively.

11. Parabolic Sasakian space forms

In this section we give an explicit linear Lie group model of the Sasakian space form $\mathcal{M}^3(-3)$.

11.1. The Heisenberg group Nil_3

The 3-dimensional *Heisenberg group* Nil_3 is $\mathbb{R}^3(x, y, z)$ together with the group structure:

$$(x, y, z) \cdot (x', y', z') := (x + x', y + y', z + z' + (xy' - x'y)/2\ ).$$

We define a left invariant Riemannian metric g by

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{4} + \frac{1}{4}\left(\mathrm{d}z + \frac{y\mathrm{d}x - x\mathrm{d}y}{2}\right)^2. \tag{11.1}$$

Then the homogeneous Riemannian 3-manifold (Nil_3, g) has 4-dimensional isometry group. In fact the identity component of the isometry group of (Nil_3, g) is isomorphic to the semi-direct product $\mathrm{Nil}_3 \ltimes \mathrm{SO}(2)$ [61]. The action of $\mathrm{Nil}_3 \ltimes \mathrm{SO}(2)3$ on Nil_3 is

$$\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \frac{1}{2}(a\sin\theta - b\cos\theta) & \frac{1}{2}(a\cos\theta + b\sin\theta) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Note that the action of the subgroup

$$\left\{ \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \middle| (a,b,c) \in \mathrm{Nil}_3 \right\}$$

of $\mathrm{Nil}_3 \ltimes \mathrm{SO}(2)$ on Nil_3 is the left translations.

The Heisenberg group (Nil_3, g) is represented by $\mathrm{Nil}_3 \ltimes \mathrm{SO}(2)/\mathrm{SO}(2)$. This is a naturally reductive homogeneous space representation for (Nil_3, g). Note that $(\mathrm{Nil}_3, 4g)$ is the model space of *nilgeometry* in the sense of Thurston [220].

The additive group $(\mathbb{R}, +)$ acts isometrically and freely on Nil_3:

$$\mathrm{Nil}_3 \times \mathbb{R} \to \mathrm{Nil}_3; \ \ (x,y,z) \cdot a = (x, y, z + a).$$

The natural projection $\pi : \mathrm{Nil}_3 \to \mathrm{Nil}_3/\mathbb{R} = \mathbb{R}^2(x,y)$ defines a principal line bundle over $\mathbb{R}^2(x,y)$. The metric g induces a flat Riemannian metric $(\mathrm{d}x^2 + \mathrm{d}y^2)/4$ on $\mathbb{R}^2(x,y)$. Furthermore π is a Riemannian submersion.

Taking a left invariant orthonormal frame field $\mathcal{E} = (e_1, e_2, e_3)$:

$$e_1 = 2\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \ \ e_2 = 2\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \ \ e_3 = 2\frac{\partial}{\partial z}, \tag{11.2}$$

the commutation relations of $\mathcal{E}$ are

$$[e_1, e_2] = 2e_3, \ [e_2, e_3] = [e_3, e_1] = 0. \tag{11.3}$$

The dual coframe field $\vartheta = (\vartheta^1, \vartheta^2, \vartheta^3)$ is given by

$$\vartheta^1 = \frac{1}{2}\mathrm{d}x, \ \vartheta^2 = \frac{1}{2}\mathrm{d}y, \ \vartheta^3 = \frac{1}{2}\mathrm{d}z - \frac{1}{4}(x\mathrm{d}y - y\mathrm{d}x).$$

Note that the 1-form $\eta := \theta^3$ is a contact form on Nil_3. The Levi-Civita connection ∇ of (Nil_3, g) is given by

$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = e_3, \ \ \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -e_2, \ \ \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = e_1. \tag{11.4}$$

The Riemannian curvature R of (Nil_3, g) is described by

$$R_{1212} = -3, \quad R_{1313} = R_{2323} = 1.$$

The sectional curvatures are

$$K_{12} = -3, \quad K_{13} = K_{23} = 1.$$

Define an endomorphism field φ by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \varphi\xi = 0, \quad \xi = e_3.$$

Then (η, ξ, φ) is a left invariant almost contact structure on Nil_3. Since the metric g is related to this almost contact structure by

$$\mathrm{d}\eta(X, Y) = g(X, \varphi Y).$$

Hence $(\mathrm{Nil}_3, \eta, \xi, \varphi, g)$ is a contact Riemannian manifold. Moreover the holomorphic sectional curvature of Nil_3 is constant -3. From Tanno's classification we then obtain the following result.

Proposition 11.1. *The simply connected parabolic Sasakian space form* $\mathcal{M}^3(-3)$ *is isomorphic to the Heisenberg group* Nil_3.

Remark 11.1. The Heisenberg group $\mathrm{Nil}_3(x,y,z)$ is isomorphic to the following linear Lie group:

$$\mathrm{H}_3(x,y,t) = \left\{ \begin{pmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} ; x, y, t \in \mathbb{R} \right\}.$$

In fact, the mapping $\iota : \mathrm{Nil}_3 \to \mathrm{GL}_3\mathbb{R}$ defined by

$$\iota(x,y,z) = \begin{pmatrix} 1 & y & t \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad t = z + \frac{xy}{2}$$

is a Lie group isomorphism between Nil_3 and H_3. The homogeneous Sasakian metric on H_3 induced by g is written as

$$g_{\mathrm{H}} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{4} + \frac{1}{4}(\mathrm{d}t - y\mathrm{d}x)^2. \tag{11.5}$$

The contact form η corresponds to

$$\eta_{\mathrm{H}} = \frac{1}{2}(\mathrm{d}t - y\mathrm{d}x).$$

Hence the Sasakian manifold $(\mathrm{H}_3, \eta_{\mathrm{H}}, g_{\mathrm{H}})$ coincides with the *standard model* of $\mathcal{M}^3(-3)$ given as Example A in [16, p. 29 and p. 81].

On the other hand, Tricerri and Vanhecke [222, Chapter 7] used the following model:

$$(\mathbb{R}^3(\bar{x},\bar{y},\bar{z}), \bar{g}), \quad \bar{g} = \mathrm{d}\bar{x}^2 + \mathrm{d}\bar{z}^2 + (\mathrm{d}\bar{y} - \bar{x}\mathrm{d}\bar{x})^2.$$

This model is homothetic to our Nil_3. Indeed, $(x,y,z) = (\bar{x}, \bar{z}, \bar{y} - \bar{z}\bar{x}/2)$ is an isometry from $(\mathbb{R}^3(\bar{x},\bar{y},\bar{z}), \bar{g})$ to $(\mathrm{Nil}_3, 4g)$.

For more information on Nil_3, we refer to [105, 106].

11.2. The homogeneous Riemannian structures on Nil_3

In [222, Chapter 9], homogeneous Riemannian structures on $(\mathrm{Nil}_3, 4g)$ are classified. Their classification result is adjusted to our setting in the following way:

Theorem 11.1. ([222, Theorem 7.1]) *All the homogeneous Riemannian structures on* $\mathcal{M}^3(-3) = \mathrm{Nil}_3$ *are given by the following* $(0,3)$*-tensor fields*:

$$S^{\mu}_{\flat} = 4\mu\vartheta^3 \otimes (\vartheta^1 \wedge \vartheta^2) - 2\vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) - 2\vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1), \quad \mu \in \mathbb{R}.$$

The homogeneous Riemannian structure S^{μ} is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. In particular,

- *S^{μ} is of type $\mathcal{T}_2$ if and only if $\mu = 1$*
- *S^{μ} is of type $\mathcal{T}_3$ if and only if $\mu = -1/2$.*

The corresponding coset space representations of S^{μ} are

$$\mathrm{Nil}_3 = \begin{cases} \mathrm{Nil}_3 \ltimes \mathrm{SO}(2)/\mathrm{SO}(2) & \mu \neq 1/2 \\ \mathrm{Nil}_3/\{\mathsf{e}\} & \mu = 1/2. \end{cases}$$

Comparing this result with Proposition 9.7, we conclude that $A^r_{\flat} = S^{-r/2}_{\flat}$. Hence we obtain the following result.

Theorem 11.2. *The set $\mathcal{S}$ of all homogeneous Riemannian structures on Nil_3 is given by $\{A^r \mid r \in \mathbb{R}\}$. Moreover $\mathcal{S}$ coincides with the set of all homogeneous almost contact Riemannian structures on Nil_3.*

From this classification, we have the following characterizations of Tanaka-Webster connection.

Theorem 11.3. *The coset space representations of Nil_3 are given by*

- *$\mathrm{Nil}_3/\{\mathsf{e}\}$ with respect to the Tanaka-Webster connection $\hat{\nabla} = \nabla^{-1}$.*
- *$\mathrm{Nil}_3 \ltimes \mathrm{SO}(2)/\mathrm{SO}(2)$ for other connection ∇^r $(r \neq -1)$.*

Namely, Tanaka-Webster connection is the only Ambrose-Singer connection on Nil_3 which has the representation $\mathrm{Nil}_3/\{\mathsf{e}\}$.

One can check that Tanaka-Webster connection coincides with Cartan-Schouten's $(-)$-connection of Nil_3. Theorem 11.1 motivates us to describe explicitly the homogeneous Riemannian structures on Sasakian space forms $\mathcal{M}^3(c)$ with $c \neq -3$. The universal covering group of $\mathrm{Nil}_3 \ltimes \mathrm{SO}(2)$ is the so-called *oscillator group*, see [119].

12. Elliptic Sasakian space form

In this section, we recall an explicit matrix group model of a simply connected elliptic Sasakian space form $\mathcal{M}^3(c)$ ([10, 112]).

12.1. *The unit* 3*-sphere*

As is well known, the unit 3-sphere $(\mathbb{S}^3, \varphi_1, \xi_1, \eta_1, g_1)$ is identified with the special unitary group $\mathrm{SU}(2)$ with bi-invariant metric. In this section we give an $\mathrm{SU}(2)$-model of $\mathcal{M}^3(c)$.

The bi-invariant metric g_1 of constant curvature 1 on $\mathrm{SU}(2)$ is induced by the following inner product $\langle\cdot,\cdot\rangle_1$ on $\mathfrak{su}(2)$:

$$\langle X, Y\rangle_1 = -\frac{1}{2}\,\mathrm{tr}(XY), \quad X, Y \in \mathfrak{su}(2).$$

We take a quaternionic basis of $\mathfrak{su}(2)$:

$$\boldsymbol{i} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \boldsymbol{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{k} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

By using this basis, the Lie group $\mathrm{SU}(2)$ is described as

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} x_0 + \sqrt{-1}x_3 & -x_2 + \sqrt{-1}x_1 \\ x_2 + \sqrt{-1}x_1 & x_0 - \sqrt{-1}x_3 \end{pmatrix} \;\middle|\; x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

We identify $\mathfrak{su}(2)$ with Euclidean 3-space $\mathbb{E}^3$ via the correspondence

$$(x_1, x_2, x_3) \longleftrightarrow x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}.$$

Denote the left translated vector fields of $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ by $\{E_1, E_2, E_3\}$. Then a left invariant Sasakian structure of G is given by

$$\xi_1 := E_3, \quad \eta_1 = g_1(E_3, \cdot),$$
$$\varphi_1(E_1) = E_2, \quad \varphi_1(E_2) = -E_1, \quad \varphi_1(E_3) = 0.$$

Note that the commutation relations of $\{E_1, E_2, E_3\}$ are

$$[E_1, E_2] = 2E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 2E_2.$$

The Lie group $\mathrm{SU}(2)$ acts isometrically on the Lie algebra $\mathfrak{su}(2)$ by the Ad-action.

$$\mathrm{Ad} : \mathrm{SU}(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2); \quad \mathrm{Ad}(a)X = aXa^{-1}, \quad a \in \mathrm{SU}(2),\ X \in \mathfrak{su}(2).$$

It should be remarked that Ad is regarded as a Lie group homomorphism from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$. The kernel of the Lie group homomorphism $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is $\{\pm\mathbf{1}\} \cong \mathbb{Z}_2$. Thus $\mathrm{SU}(2)$ is a double covering of $\mathrm{SO}(3)$.

The Ad-orbit of $\boldsymbol{k}/2$ is a sphere $\mathbb{S}^2(4)$ of radius $1/2$ in the Euclidean 3-space $\mathbb{E}^3 = \mathfrak{su}(2)$. The Ad-action of $\mathrm{SU}(2)$ on $\mathbb{S}^2(4)$ is isometric and transitive. The isotropy subgroup of $\mathrm{SU}(2)$ at $\boldsymbol{k}/2$ is

$$\mathrm{U}(1) = \left\{ \begin{pmatrix} e^{\sqrt{-1}t} & 0 \\ 0 & e^{-\sqrt{-1}t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Hence $\mathbb{S}^2(4)$ is represented by $\mathrm{SU}(2)/\mathrm{U}(1)$ as a homogeneous Riemannian space. The natural projection

$$\pi_1 : \mathbb{S}^3 \to \mathbb{S}^2(4), \quad \pi_1(a) = \mathrm{Ad}(a)(\boldsymbol{k}/2)$$

is a Riemannian submersion and defines a principal $\mathrm{U}(1)$-bundle over $\mathbb{S}^2(4)$. The tangent space $\mathrm{T}_{\boldsymbol{k}/2}\mathbb{S}^2(4)$ of $\mathbb{S}^2(4)$ at the origin $\boldsymbol{k}/2$ is identified with the linear subspace $\overline{\mathfrak{m}} = \mathbb{R}\boldsymbol{i} \oplus \mathbb{R}\boldsymbol{j}$ of $\mathfrak{su}(2)$. Thus we have the reductive decomposition $\mathfrak{su}(2) = \mathfrak{u}(1) \oplus \overline{\mathfrak{m}}$. Since $[\overline{\mathfrak{m}}, \overline{\mathfrak{m}}] \subset \mathfrak{u}(1)$, $\mathbb{S}^2(4) = \mathrm{SU}(2)/\mathrm{U}(1)$ is a Riemannian symmetric space.

The product Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathbb{S}^3 = \mathrm{SU}(2)$ via the action (3.2). As we saw in Section 3.5, the isotropy subgroup at $\mathbf{1}$ is the diagonal subgroup $\Delta\mathrm{SU}(2)$. The inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ induced from the product metric of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is

$$\langle (X, Y), (V, W)\rangle = \langle X, V\rangle + \langle Y, W\rangle.$$

The orthogonal complement $\Delta\mathfrak{su}(2)^{\perp}$ of $\Delta\mathfrak{su}(2)$ in $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ is $\mathfrak{m}_0$ given in Section 3.5. The reductive decomposition $\mathfrak{su}(2)\oplus\mathfrak{su}(2)=\Delta\mathfrak{su}(2)\oplus\mathfrak{m}_0$ satisfies $[\mathfrak{m}_0,\mathfrak{m}_0]\subset\Delta\mathfrak{su}(2)$. Thus $\mathbb{S}^3=(\mathrm{SU}(2)\times\mathrm{SU}(2))/\Delta\mathrm{SU}(2)$ is a Riemannian symmetric space. Hereafter we denote this Riemannian symmetric space as $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{SU}(2)$ for simplicity. However it should be remarked that $\dim\mathrm{Iso}(\mathbb{S}^3)=\dim(\mathrm{SU}(2)\times\mathrm{SU}(2))=6$. But $\dim\mathrm{Aut}(\mathbb{S}^3)=4$. Thus the Riemannian symmetric space representation $\mathbb{S}^3=(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{SU}(2)=\mathrm{SO}(4)/\mathrm{SO}(3)$ is *not* homogeneous contact. As we will see in the next subsection, $\mathbb{S}^3$ is represented as $\mathbb{S}^3=(\mathrm{SU}(2)\times\mathrm{U}(1))/\mathrm{U}(1)$ as a homogeneous contact Riemannian manifold. This representation is not Riemannian symmetric, but naturally reductive.

12.2. The Berger sphere

Since the Sasakian structure $(\eta_1,\xi_1,\varphi_1,g_1)$ is left invariant, its $\mathcal{D}$-homothetic deformation is also left invariant. Hence the elliptic Sasakian space form $\mathcal{M}^3(c)$ is identified with $\mathrm{SU}(2)$ with the left invariant contact Riemannian structure (φ,ξ,η,g) defined by (5.5). The Reeb vector field ξ generates a one parameter group of transformations on $\mathcal{M}^3(c)$ isomorphic to $\mathrm{U}(1)$. Since ξ is a Killing vector field, the one parameter group $\mathrm{U}(1)$ acts isometrically on $\mathcal{M}^3(c)$. The factor space $\mathcal{M}^3(c)/\mathrm{U}(1)$ is nothing but the 2-sphere $\mathbb{S}^2(c+3)$.

The Sasakian metric g is not only left $\mathrm{SU}(2)$-invariant but also right $\mathrm{U}(1)$-invariant. Hence the elliptic Sasakian space form $\mathcal{M}^3(c)$ is represented by $\mathcal{M}^3(c)=(\mathrm{SU}(2)\times\mathrm{U}(1))/\mathrm{U}(1)$ as a reductive homogeneous space. For $c\neq 1$, $\mathcal{M}^3(c)$ has 4-dimensional isometry group.

In particular g is bi-invariant if and only if $c=1$. In this case $\mathcal{M}^3(1)$ is represented by $(\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{SU}(2)$ as a Riemannian symmetric space. Note that $\mathcal{M}^3(1)$ has 6-dimensional isometry group as we mentioned before.

In this article, we call the Sasakian 3-manifold $\mathcal{M}^3(c)$ with $c\neq 1$ by the name *Berger sphere* (see also [159]). Precisely speaking, the original one due to Berger [12] is $(\mathbb{S}^3,(c+3)g/4)$ with $c>1$. Under the limit $c\to\infty$ in Gromov-Hausdorff sense, $(\mathbb{S}^3,(c+3)g/4)$ converges to $\mathbb{S}^3$ equipped with the Carnot-Carathéodory metric. On the other hand, under the limit $c\to 1$, $(\mathbb{S}^3,(c+3)g/4)$ collapses to $\mathbb{S}^2(4)$. Another geometric property of $\mathcal{M}^3(c)$ is a relation to the geometry of isoparametric hypersurfaces. One can see that $\mathcal{M}^3(-2)$ is isometric to the universal covering of the minimal Cartan hypersurface of the unit 4-sphere (see [68]). The minimal Cartan hypersurface is realized as $\mathrm{SO}(3)/\mathbb{Z}_2\times\mathbb{Z}_2=\mathcal{M}^3(-2)/\Gamma$, where $\Gamma=\{\pm\mathbf{1},\pm\boldsymbol{i},\pm\boldsymbol{j},\pm\boldsymbol{k}\}$.

Now we take an orthonormal frame field $\{e_1,e_2,e_3\}$ of $\mathcal{M}^3(c)$ by

$$e_1:=\frac{\sqrt{c+3}}{2}E_1,\quad e_2:=\frac{\sqrt{c+3}}{2}E_2,\quad e_3:=\frac{c+3}{4}\xi_1.$$

Then the commutation relations of this basis are

$$[e_1,e_2]=2e_3,\quad [e_2,e_3]=\frac{c+3}{2}e_1,\quad [e_3,e_1]=\frac{c+3}{2}e_2. \tag{12.1}$$

Note that if we take the limit $c\to -3$, then (12.1) converges to (11.3).

The Levi-Civita connection ∇ of $(\mathcal{M}^3(c),g)$ is described by

$$\begin{aligned}\nabla_{e_1}e_1=0,\quad \nabla_{e_1}e_2=e_3,\quad \nabla_{e_1}e_3=-e_2,\\ \nabla_{e_2}e_1=-e_3,\quad \nabla_{e_2}e_2=0,\quad \nabla_{e_2}e_3=e_1,\\ \nabla_{e_3}e_1=\frac{c+1}{2}e_2,\quad \nabla_{e_3}e_2=-\frac{c+1}{2}e_1,\quad \nabla_{e_3}e_3=0.\end{aligned} \tag{12.2}$$

The Riemannian curvature tensor field R of $(\mathcal{M}^3(c),g,\nabla)$ is described by

$$R_{1212}=c,\quad R_{1313}=R_{2323}=1 \tag{12.3}$$

and the sectional curvatures are:

$$K_{12}=c,\quad K_{13}=K_{23}=1. \tag{12.4}$$

The Ricci tensor Ric and the scalar curvature s are computed to be

$$R_{11}=R_{22}=c+1,\quad R_{33}=2,\quad \mathrm{s}=2(c+2).$$

The Ricci tensor field has the form

$$\mathrm{Ric}=(c+1)g+(1-c)\eta\otimes\eta. \tag{12.5}$$

The Sasakian space form $\mathcal{M}^3(c) = (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathrm{U}(1)$ is naturally reductive. The product Lie group $\mathrm{SU}(2) \times \mathrm{U}(1)$ acts on $\mathcal{M}^3(c)$ via the action

$$(\mathrm{SU}(2) \times \mathrm{U}(1)) \times \mathcal{M}^3(c) \to \mathcal{M}^3(c); \quad (a,k) \cdot \mathsf{x} = a\,\mathsf{x}\,k^{-1}, \quad a \in \mathrm{SU}(2),\ k \in \mathrm{U}(2),\ \mathsf{x} \in \mathcal{M}^3(c).$$

The isotropy subgroup at $\mathbf{1}$ is $\Delta\mathrm{U}(1) = \{(k,k) \mid k \in \mathrm{U}(1)\}$ with Lie algebra $\Delta\mathfrak{u}(1) = \{(V,V) \mid V \in \mathfrak{u}(1)\}$. The Berger sphere $\mathcal{M}^3(c)$ admits a Lie subspace

$$\mathfrak{m} = \left\{ \left(V + W, \frac{1-c}{4} W \right) \,\middle|\, V \in \overline{\mathfrak{m}}, \quad W \in \mathfrak{u}(1) \right\}$$

of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Thus $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathrm{U}(1)$ is a naturally reductive homogeneous space. Note that if we choose $c = 1$, then we obtain a reductive decomposition

$$\mathfrak{su}(2) \oplus \mathfrak{u}(1) = \Delta\mathfrak{u}(1) \oplus \mathfrak{m}, \quad \mathfrak{m} = \{(X,0) \mid X \in \mathfrak{su}(2)\}.$$

Every element $(X,Y) \in \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ is decomposed as $(X,Y) = (Y,Y) + (X-Y,0)$ along this reductive decomposition. The corresponding reductive homogeneous space is the homogeneous contact Riemannian manifold representation of the unit 3-sphere $\mathbb{S}^3$.

12.3. The normal metric

In case, $c > 1$, the Berger sphere $\mathcal{M}^3(c)$ is represented as $\mathcal{M}^3(c) = \mathrm{U}(2)/\mathrm{U}(1)$. The Riemannian metric of $\mathcal{M}^3(c)$ is derived from the following inner product on the Lie algebra $\mathfrak{u}(2)$:

$$\langle X, Y \rangle = \frac{4}{c+3} \left(-\frac{1}{2} \mathrm{tr}(\boldsymbol{x}\boldsymbol{y}) + \frac{1}{\sqrt{c-1}}\, \mathrm{tr}(X)\, \overline{\mathrm{tr}(Y)} \right), \quad X, Y \in \mathfrak{u}(2),$$

where

$$\boldsymbol{x} = X - \frac{\mathrm{tr}(X)}{2}\mathbf{1}, \quad \boldsymbol{x} = X - \frac{\mathrm{tr}(X)}{2}\mathbf{1} \in \mathfrak{su}(2).$$

In particular when $c = 5$, we have

$$\langle X, Y \rangle = -\frac{1}{4}\mathrm{tr}(XY), X, Y \in \mathfrak{u}(2).$$

One can confirm that this inner product induces a bi-invariant Riemannian metric on $\mathrm{U}(2)$. Hence $\mathcal{M}^3(c) = \mathrm{U}(2)/\mathrm{U}(1)$ is a normal homogeneous space. For more detailed discussion, see [118].

Remark 12.1. As we mentioned before $\mathcal{M}^3(-2)/\Gamma$ is isometrically immersed in the unit 4-sphere $\mathbb{S}^4$ as an isoparametric hypersurface. Kim and Tsunero Takahashi proved the following results.

Proposition 12.1 ([140]). *A hypersurface M of a space form $M^{n+1}(\varepsilon\, c^2)$ of constant curvature $\varepsilon\, c^2$ is isoparametric if and only if there exits a metric connection D such that the shape operator is parallel with respect to D.*

Proposition 12.2 ([140]). *If a hypersurface M of a space form $M^{n+1}(\varepsilon\, c^2)$ admits a homogeneous Riemannian structure S and the type number of M is not equal to 1 and 2, then the shape operator is parallel with respect to the Ambrose-Singer connection $\nabla + S$ and M is isoparametric.*

13. The $\mathrm{SU}(1,1)$-model of the hyperbolic Sasakian space forms

13.1. The Poincaré disc model

Let us start with recalling the Poincaré disc model of the hyperbolic 2-space $\mathbb{H}^2(-c^2)$ of curvature $\kappa = -c^2 < 0$:

$$\mathbb{H}^2(-c^2) = (\{z = x + \sqrt{-1}y \in \mathbb{C} \mid |z|^2 = x^2 + y^2 < 4/c^2\}, \bar{g}),$$

where the Poincaré metric $\bar{g}$ is defined by

$$\bar{g} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left\{1 - \frac{c^2}{4}(x^2+y^2)\right\}^2}.$$

The indefinite special unitary group

$$\mathrm{SU}(1,1)=\left\{\left(\begin{array}{cc} x_0+\sqrt{-1}x_1 & x_3-\sqrt{-1}x_2 \\ x_3+\sqrt{-1}x_2 & x_0-\sqrt{-1}x_1 \end{array}\right)\ \middle|\ x_0^2+x_1^2-x_2^2-x_3^2=1\right\}.$$

acts isometrically and transitively on $\mathbb{H}^2(-c^2)$ via the linear fractional transformation:

$$\mathrm{T}:\mathrm{SU}(1,1)\times\mathbb{H}^2(-c^2)\to\mathbb{H}^2(-c^2);\ (\mathsf{A},z)\mapsto \mathrm{T}_{\mathsf{A}}(z),$$

where $\mathrm{T}_{\mathsf{A}}(z)$ is defined by

$$\mathrm{T}_{\mathsf{A}}(z)=\frac{az+b}{cz+d},\quad \mathsf{A}=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$$

as before. The isotropy subgroup of $\mathrm{SU}(1,1)$ at the origin 0 is the unitary group

$$\mathrm{U}(1)=\left\{\left(\begin{array}{cc} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{array}\right)\ \middle|\ 0\leq\theta<2\pi\right\}.$$

Thus $\mathbb{H}^2(-c^2)$ is represented by $\mathrm{SU}(1,1)/\mathrm{U}(1)$ as a homogeneous Riemannian space. The Lie algebra $\mathfrak{su}(1,1)$ is spanned by the *split-quaternionic basis* (see [75]):

$$\boldsymbol{i}=\left(\begin{array}{cc} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{array}\right),\ \boldsymbol{j}'=\left(\begin{array}{cc} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{array}\right),\ \boldsymbol{k}'=\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right).$$

Take a scalar product $\langle\cdot,\cdot\rangle^{(-)}$ of $\mathfrak{su}(1,1)$ by

$$\langle X,Y\rangle^{(-)}=-\frac{2}{c^2}\mathrm{tr}(XY),\quad X,Y\in\mathfrak{su}(1,1).$$

Then $\langle\cdot,\cdot\rangle^{(-)}$ induces a bi-invariant Lorentz metric on $\mathrm{SU}(1,1)$. Thus $\mathrm{SU}(1,1)$ is identified with the *anti de Sitter spacetime* $\mathrm{AdS}_3(-c^2+3)$ of constant curvature $-c^2+3$. The projection $\mathrm{AdS}_3(-c^2+3)\to\mathbb{H}^2(-c^2)$ defines a principal circle bundle. This fibering is called the *hyperbolic Hopf fibering*.

Denote the left-translated vector fields of $\boldsymbol{j}',\boldsymbol{k}',\boldsymbol{i}$ by E_1,E_2,E_3. Note that the commutation relations of $\{E_1,E_2,E_3\}$ are

$$[E_1,E_2]=-2E_3,\quad [E_2,E_3]=2E_1,\quad [E_3,E_1]=2E_2.$$

We can take an orthonormal frame field $\{e_1,e_2,e_3\}$ by

$$e_1^-=\frac{c}{2}E_1,\quad e_2^-=\frac{c}{2}E_2,\quad e_3^-=\frac{c}{2}E_3.$$

Note that e_3^- is timelike. On the other hand e_1^- and e_2^- are spacelike. The isotropy algebra $\mathfrak{u}(1)$ is spanned by e_3. The tangent space $\overline{\mathfrak{m}}=\mathrm{T}_0\mathbb{H}^2(-c^2)$ is spanned by e_1 and e_2 and hence it is spacelike. The decomposition $\mathfrak{su}(1,1)=\mathfrak{u}(1)+\overline{\mathfrak{m}}$ is reductive and orthogonal. Moreover $[\overline{\mathfrak{m}},\overline{\mathfrak{m}}]\subset\mathfrak{u}(1)$ holds. Thus $\mathbb{H}^2(-c^2)=\mathrm{SU}(1,1)/\mathrm{U}(1)$ is a Riemannian symmetric space.

13.2. The unit tangent sphere bundle

The tangent bundle $\mathrm{T}\mathbb{H}^2(\kappa)$ and the unit tangent bundle $\mathrm{U}\mathbb{H}^2(-c^2)$ are given explicitly by

$$\begin{aligned}\mathrm{T}\mathbb{H}^2(-c^2)=&\mathbb{H}^2(-c^2)\times\mathbb{R}^2=\{(z,\boldsymbol{v})\mid z\in\mathbb{H}^2(\kappa),\ \boldsymbol{v}=(v_1,v_2)\in\mathbb{R}^2\},\\ \mathrm{U}\mathbb{H}^2(-c^2)=&\{(z,\boldsymbol{v})\in\mathrm{T}\mathbb{H}^2(-c^2)\mid (v_1)^2+(v_2)^2=1-c^2|z|^2/4\}.\end{aligned}$$

This action of $\mathrm{SU}(1,1)$ on $\mathbb{H}^2(\kappa)$ induces a transitive action on $\mathrm{U}\mathbb{H}^2(-c^2)$. The isotropy subgroup of $\mathrm{SU}(1,1)$ of the induced action on $\mathrm{U}\mathbb{H}^2(-c^2)$ at $(0,(1,0))$ is $\mathbb{Z}^2=\{\pm\mathbf{1}\}$. Hence $\mathrm{U}\mathbb{H}^2(-c^2)$ is represented by $\mathrm{U}\mathbb{H}^2(-c^2)=\mathrm{SU}(1,1)/\mathbb{Z}^2=\mathrm{PSU}(1,1)$ as a homogeneous space.

Let us define a Riemannian metric on $\mathrm{U}\mathbb{H}^2(-c^2)$. Take a curve $\gamma(t)=(z(t),\boldsymbol{v}(t))$ in $\mathrm{U}\mathbb{H}^2(-c^2)$. Then we define a Riemannian metric g on $\mathrm{U}\mathbb{H}^2(-c^2)$ by

$$g(\gamma'(t_0),\gamma'(t_0)):=\bar{g}_{t_0}(z'(t_0),z'(t_0))+\left(\frac{2\tau}{-c^2}\right)^2\bar{g}\left(\nabla_{z'(t)}\boldsymbol{v}(t),\nabla_{z'(t)}\boldsymbol{v}(t)\right)|_{t=t_0}.$$

Here τ is a non-zero constant. In particular, when $\tau=\pm c/2$, g coincides with the *Sasaki lift metric* of $\mathrm{U}\mathbb{H}^2(-c^2)$.

Now let us introduce a global coordinate system (x,y,θ) on $\mathrm{U}\mathbb{H}^2(-c^2)$:

$$\mathbb{H}^2(-c^2)\times\mathbb{S}^1\to\mathrm{U}\mathbb{H}^2(-c^2);$$

$$((x,y),\theta)\longmapsto\left((x,y),(1+\frac{\kappa}{4}(x^2+y^2))\left\{\cos\left(\frac{-c^2\,\theta}{2\tau}\right)\frac{\partial}{\partial x}+\sin\left(\frac{-c^2\,\theta}{2\tau}\right)\frac{\partial}{\partial y}\right\}\right).$$

With respect to this coordinate system, the metric g is computed as

$$g=\frac{\mathrm{d}x^2+\mathrm{d}y^2}{(1+\frac{\kappa}{4}(x^2+y^2))^2}+\left(\mathrm{d}\theta+\frac{\tau(y\mathrm{d}x-x\mathrm{d}y)}{1+\frac{\kappa}{4}(x^2+y^2)}\right)^2,\quad\kappa=-c^2<0.$$

This formula shows that $(\mathrm{U}\mathbb{H}^2(-c^2),g)$ is isometric to the Bianchi-Cartan-Vrănceanu space of base curvature $\kappa=-c^2$ and bundle curvature τ (see Example 5.3).

Now we equip a Sasakian structure compatible to $(\mathrm{U}\mathbb{H}^2(\kappa),g)$. Choose $\tau=1$. Then we have a Sasakian structure (η,ξ,φ,g) as follows (see Example 5.3).

Define a contact form η by

$$\eta=\mathrm{d}\theta+\frac{y\mathrm{d}x-x\mathrm{d}y}{1-\frac{c^2}{4}(x^2+y^2)}.$$

Then the Reeb vector field is $\partial/\partial\theta$. Choose an orthonormal frame field

$$u_1=\left(1-\frac{c^2}{4}(x^2+y^2)\right)\frac{\partial}{\partial x}-y\frac{\partial}{\partial\theta},\quad u_2=\left(1-\frac{c^2}{4}(x^2+y^2)\right)\frac{\partial}{\partial x}+x\frac{\partial}{\partial\theta},\quad u_3=\xi.$$

Define an endomorphism field φ by

$$\varphi u_1=u_2,\quad\varphi u_2=-u_1,\quad\varphi u_3=0.$$

Then we have $\mathrm{d}\eta(X,Y)=g(X,\varphi Y)$ for all vector fields X and Y on $\mathrm{U}\mathbb{H}^2(-c^2)$. The resulting contact Riemannian manifold $(\mathrm{U}\mathbb{H}^2(\kappa),g)$ is a Sasakian manifold of constant holomorphic sectional curvature $-3-c^2$.

As we saw before the Sasaki lift metric is determined by the condition $\tau=\pm c^2/2$. Under this choice, the almost contact structure satisfies

$$(\nabla_X\varphi)Y=\tau\{g(X,Y)-\eta(Y)X\}.$$

Thus under the normalization (see Example 5.3)

$$\tilde\eta=\tau\,\eta,\quad\tilde\xi=\frac{1}{\tau}\xi,\quad\tilde\varphi=\varphi,\quad\tilde g=\tau^2\,g.$$

Then $(\mathrm{U}\mathbb{H}^2(\kappa),\hat\eta,\hat\xi,\hat\varphi,\hat g)$ is a Sasakian manifold of constant holomorphic sectional curvature $-3+4\kappa$.

In particular, if $\kappa=-1$, then metric $\tilde g$ is $g/4$. Thus the metric $\tilde g$ coincides with the associated metric of the standard contact Riemannian structure (5.14).

13.3. *Homogeneous space representation*

In the hyperbolic Hopf fibering $\mathrm{SU}(1,1)\to\mathbb{H}^2(\kappa)$, we equipped a Lorentz metric on $\mathrm{SU}(1,1)$. The hyperbolic Hopf fibering is a homothetic submersion.

In this subsection we equip a *Riemannian metric* on $\mathrm{SU}(1,1)$ so that the hyperbolic Hopf fibering is still a homothetic submersion.

Choose non-zero real constants λ_1, λ_2, λ_3 and define

$$e_1^+=-\frac{1}{\lambda_2\lambda_3}E_1,\quad e_2^+=\frac{1}{\lambda_3\lambda_1}E_2,\quad e_3^+=\frac{1}{\lambda_1\lambda_2}E_3.$$

Then we have

$$[e_1^+,e_2^+]=c_3e_3^+,\quad[e_2^+,e_3^+]=c_1e_1^+,\quad[e_3^+,e_1^+]=c_2e_2^+,$$

with

$$c_1=-\frac{2}{\lambda_1^2}<0,\quad c_2=-\frac{2}{\lambda_2^2}<0,\quad c_3=\frac{2}{\lambda_3^2}>0.$$

The left-invariant metric $\overline{g}_{(c_1,c_2,c_3)}$, defined by the condition that $\{e_1^+, e_2^+, e_3^+\}$ is an orthonormal basis, is

$$\overline{g}_{(c_1,c_2,c_3)} = 4\left\{-\frac{1}{c_2c_3}\omega_1^2 - \frac{1}{c_3c_1}\omega_2^2 + \frac{1}{c_1c_2}\omega_3^2\right\},$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual coframe field of $\{E_1, E_2, E_3\}$.

Proposition 13.1 ([188])**.** *Any left-invariant Riemannian metric on* $\mathrm{SU}(1,1)$ *is isometric to one of the metrics* $\overline{g}_{(c_1,c_2,c_3)}$ *with* $c_1 \leq c_2 < 0 < c_3$*. Moreover, this metric gives rise to an isometry group of dimension* 4 *if and only if* $c_1 = c_2$*.*

Since $\{e_1^+, e_2^+, e_3^+\}$ is a unimodular basis, the Levi-Civita connection ∇ of $\mathrm{SU}(1,1)$ is described as follows:

Proposition 13.2. *The Levi-Civita connection is given by*

$$\begin{array}{ccc} \nabla_{e_1^+}e_1^+ = 0, & \nabla_{e_2^+}e_1^+ = \mu_1 e_3^+, & \nabla_{e_1^+}e_3^+ = -\mu_1 e_2^+, \\ \nabla_{e_2^+}e_1^+ = -\mu_2 e_3^+, & \nabla_{e_2^+}e_2^+ = 0, & \nabla_{e_2^+}e_3^+ = \mu_2 e_1^+, \\ \nabla_{e_3^+}e_1^+ = \mu_3 e_2^+, & \nabla_{e_3^+}e_2^+ = -\mu_3 e_1^+ & \nabla_{e_3^+}e_3^+ = 0, \end{array}$$

where the constants $\{\mu_1, \mu_2, \mu_3\}$ *are given by* (4.17)*.*

The Riemannian curvature tensor R *is determined by the following sectional curvatures:*

$$\begin{aligned} K_{12} = \langle R(e_1^+, e_2^+)e_2^+, e_1^+\rangle =& c_3\mu_3 - \mu_1\mu_2, \\ K_{23} = \langle R(e_2^+, e_3^+)e_3^+, e_2^+\rangle =& c_1\mu_1 - \mu_2\mu_3, \\ K_{13} = \langle R(e_1^+, e_3^+)e_3^+, e_1^+\rangle =& c_2\mu_2 - \mu_1\mu_3. \end{aligned}$$

Now we choose $c_1 = c_2 = \kappa/2 < 0$ and $c_3 = 2$ (equivalently $|\lambda_1| = |\lambda_2| = 2/\sqrt{-\kappa}$ and $|\lambda_3| = 1$). Note that $\tau = 1$. Then the metric $g(\kappa/2, \kappa/2, 2)$ is a Sasakian metric of constant holomorphic sectional curvature $-3+\kappa$. The associated contact metric structure is given by

$$\varphi e_1^+ = e_2^+, \ \varphi e_2^+ = -e_1^+, \varphi\xi = 0, \ \ \xi = e_3^+.$$

Note that in this case,

$$e_1^+ = -\frac{\sqrt{-\kappa}}{2}E_1, \quad e_2^+ = \frac{\sqrt{-\kappa}}{2}E_2, \quad e_3^+ = \frac{-\kappa}{4}E_3.$$

If we set $c := -3+\kappa$, then the commutation relations are rewritten as

$$[e_1^+, e_2^+] = 2e_3^+, \quad [e_2^+, e_3^+] = \frac{c+3}{2}e_1^+, \quad [e_3^+, e_1^+] = \frac{c+3}{2}e_2^+.$$

This is the same form to (12.1). The Sasakian space form $(\mathrm{SU}(1,1), \eta, \xi, \varphi, g)$ is isomorphic to the BCV-space with base curvature κ and bundle curvature $\tau = 1$. Under this choice the table of Levi-Civita connection given in Proposition 13.2 coincides with (12.2). Thus we can study homogeneous contact Riemannian structures on 3-dimensional Sasakian space forms in a unified way. We will carry out the classification in Section 15.

Remark 13.1 ($\kappa = -4$)*.* In the case of $\kappa = -4$, we have

$$e_1^+ = -E_1, \quad e_2^+ = E_2, \quad e_3^+ = E_3.$$

The inner product $\langle \cdot, \cdot \rangle$ at the identity has a simple formula (see [96]):

$$\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}\,({}^tXY), \quad X, Y \in \mathfrak{su}(1,1).$$

As we saw before the bi-invariant Lorentzian metric $\langle \cdot, \cdot \rangle^{(-)}$ of constant curvature -1 on $\mathrm{SU}(1,1)$ is given by

$$\langle X, Y\rangle^{(-)} = \frac{1}{2}\mathrm{tr}\,(XY), \quad X, Y \in \mathfrak{su}(1,1).$$

With respect to this bi-invariant metric, $\mathrm{SU}(1,1)$ is identified with the anti de Sitter spacetime AdS_3 of curvature -1. Compare with the case $\mathrm{SU}(2) = \mathbb{S}^3$.

14. The $\mathrm{SL}_2\mathbb{R}$-model

In Section 4.6, we mentioned the upper half plane model of the hyperbolic plane

$$\mathbb{H}^2(-c^2)=(\{z=x+yi\in\mathbb{C}\mid y>0\},\bar{g}),\quad \bar{g}=\frac{\mathrm{d}x^2+\mathrm{d}y^2}{c^2y^2}$$

of curvature $-c^2$. In this section we exhibit $\mathrm{SL}_2\mathbb{R}$-model of the hyperbolic Sasakian space form (see [94, 96, 107, 108, 114, 117]). Note that we have already given a model $\mathbb{H}^2(-c^2)\times\mathbb{R}$ in Example 5.4.

14.1. *Iwasawa decomposition, revisited*

The Iwasawa decomposition $\mathrm{SL}_2\mathbb{R}=NAK$ of $\mathrm{SL}_2\mathbb{R}$ exhibited in Section 4.6 allows to introduce the following global coordinate system (x,y,θ) of $\mathrm{SL}_2\mathbb{R}$:

$$(x,y,\phi)\longmapsto\left(\begin{array}{cc}1&x\\0&1\end{array}\right)\left(\begin{array}{cc}\sqrt{y}&0\\0&1/\sqrt{y}\end{array}\right)\left(\begin{array}{cc}\cos\phi&\sin\phi\\-\sin\phi&\cos\phi\end{array}\right).\tag{14.1}$$

The mapping (14.1) is a diffeomorphism onto $\mathrm{SL}_2\mathbb{R}$. Hereafter, we refer (x,y,ϕ) as a global coordinate system of $\mathrm{SL}_2\mathbb{R}$. Hence $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to $\mathbb{R}\times\mathbb{R}^+\times\mathbb{S}^1$ and hence diffeomorphic to $\mathbb{R}^3\setminus\mathbb{R}$. Since $\mathbb{R}\times\mathbb{R}^+$ is diffeomorphic to open unit disk $\mathbb{D}$, then $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to open solid torus $\mathbb{D}\times\mathbb{S}^1$.

14.2. *The standard Riemannian metric*

We remark that every Sasakian space form $\mathcal{M}^3(c)$ of constant holomorphic sectional curvature $c<-3$ is transversally homothetic to $\mathcal{M}^3(-7)$.

On the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$, we can take the following basis:

$$\mathsf{E}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \mathsf{F}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \mathsf{H}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

This basis satisfies the commutation relations:

$$[\mathsf{E},\mathsf{F}]=\mathsf{H},\quad [\mathsf{F},\mathsf{H}]=2\mathsf{F},\quad [\mathsf{H},\mathsf{E}]=2\mathsf{E}.$$

The Lie algebra $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ of the closed subgroups N, A and K are given by

$$\mathfrak{n}=\mathbb{R}\mathsf{E},\quad \mathfrak{a}=\mathbb{R}\mathsf{H},\quad \mathfrak{k}=\mathbb{R}(\mathsf{E}-\mathsf{F}).$$

The Lie algebra $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}_2\mathbb{R}$. Moreover $\mathfrak{n}$ and $\mathbb{R}\mathsf{F}$ are root spaces with respect to $\mathfrak{h}$. The decomposition $\mathfrak{sl}_2\mathbb{R}=\mathfrak{h}\oplus\mathfrak{n}\oplus\mathbb{R}F$ is the root space decomposition of $\mathfrak{sl}_2\mathbb{R}$.

The split-quaternionic basis (4.8) is related to $\{\mathsf{E},\mathsf{F},\mathsf{H}\}$ by

$$\boldsymbol{i}=\left(\begin{array}{cc}0&-1\\1&0\end{array}\right)=-\mathsf{E}+\mathsf{F},\quad \boldsymbol{j}'=\mathsf{E}+\mathsf{F}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right),\quad \boldsymbol{k}'=\left(\begin{array}{cc}-1&0\\0&1\end{array}\right)=-\mathsf{H}.$$

The split-quaternionic basis satisfies

$$[\boldsymbol{i},\boldsymbol{j}']=2\boldsymbol{k}',\quad [\boldsymbol{j}',\boldsymbol{k}']=-2\boldsymbol{i},\quad [\boldsymbol{k}',\boldsymbol{i}]=2\boldsymbol{j}'.$$

The split-quaternionic basis $\{\boldsymbol{i},\boldsymbol{j}',\boldsymbol{k}'\}$ is an orthonormal basis with respect to the Lorentz scalar product

$$\langle X,Y\rangle^{(-)}=\frac{1}{2}\mathrm{tr}(XY),\quad X,Y\in\mathfrak{sl}_2\mathbb{R}.$$

Note that $\boldsymbol{i}$ is timelike. On the other hand, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are spacelike. The Lorentz scalar product $\langle\cdot,\cdot\rangle^{(-)}$ induces a bi-invariant Lorentz metric of constant curvature -1. Hence we have again an identification $\mathrm{SL}_2\mathbb{R}=\mathrm{AdS}_3$. But our interest is the Riemannian metric which makes $\widetilde{\mathrm{SL}}_2\mathbb{R}$ to be a model space of Thurston geometry.

Hereafter we use the left invariant frame field

$$e_1=\mathsf{E}+\mathsf{F}=\boldsymbol{j}',\quad e_2=\mathsf{H}=-\boldsymbol{k}',\quad e_3=\mathsf{E}-\mathsf{F}=-\boldsymbol{i}.$$

These left invariant vector fields are given explicitly by

$$\begin{aligned} e_1 =& \cos(2\phi)\left(2y\frac{\partial}{\partial x}-\frac{\partial}{\partial \phi}\right)+\sin(2\phi)\left(2y\frac{\partial}{\partial y}\right), \\ e_2 =& -\sin(2\theta)\left(2y\frac{\partial}{\partial x}-\frac{\partial}{\partial \phi}\right)+\cos(2\theta)\left(2y\frac{\partial}{\partial y}\right), \\ e_3 =& \frac{\partial}{\partial \phi}. \end{aligned}$$

Define an inner product $\langle\cdot,\cdot\rangle$ so that $\{e_1,e_2,e_3\}$ is orthonormal with respect to $\langle\cdot,\cdot\rangle$. By left-translating this inner product, we equip a left invariant Riemannian metric

$$g=\frac{\mathrm{d}x^2+\mathrm{d}y^2}{4y^2}+\left(\mathrm{d}\phi+\frac{\mathrm{d}x}{2y}\right)^2.$$

The 1-form

$$\eta=\mathrm{d}\phi+\frac{\mathrm{d}x}{2y}$$

is a globally defined contact form on $\mathrm{SL}_2\mathbb{R}$ with Reeb vector field $\xi=e_3$.

The universal covering space of $(\mathrm{SL}_2\mathbb{R},g)$ is one of the model space of Thurston geometry [220].

14.3. The hyperbolic Hopf fibering

As we saw in Section 4.6, $\mathrm{SL}_2\mathbb{R}$ acts isometrically and transitively on the upper half plane model $\mathbb{H}^2(-4)$ via the linear fractional action. The isotropy subgroup of $\mathrm{SL}_2\mathbb{R}$ at $\bar{o}=(0,1)$ is $K=\mathrm{SO}(2)$. The natural projection $\pi:(\mathrm{SL}_2\mathbb{R},g)\to\mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2)=\mathbb{H}^2(-4)$ is given explicitly by

$$\pi(x,y,\phi)=(x,y)\in\mathbb{H}^2(-4)$$

in terms of the global coordinate system (14.1). The projection is a Riemannian submersion with totally geodesic fibres and called the *hyperbolic Hopf fibering* of $\mathbb{H}^2(-4)$.

The tangent space $\mathrm{T}_{\bar{o}}\mathbb{H}^2(-4)$ at the origin $\bar{o}=(0,1)$ is identified with the linear subspace $\overline{\mathfrak{m}}_0$ of $\mathfrak{sl}_2\mathbb{R}$ spanned by $\{\boldsymbol{j}',\boldsymbol{k}'\}$. The Lie subspace $\overline{\mathfrak{m}}_0$ is rewritten as

$$\overline{\mathfrak{m}}_0=\{X\in\mathfrak{sl}_2\mathbb{R}\mid {}^tX=X\}.$$

The splitting $\mathfrak{g}=\mathfrak{k}\oplus\overline{\mathfrak{m}}_0$ is orthogonal direct sum. This splitting can be carried out explicitly as

$$X=X_{\mathfrak{k}}+X_{\mathfrak{m}},\quad X_{\mathfrak{k}}=\frac{1}{2}(X-{}^tX),\quad X_{\mathfrak{m}}=\frac{1}{2}(X+{}^tX).$$

Under the identification $\mathfrak{k}\cong\mathbb{R}$, the contact form η is regarded as a connection form of the principal circle bundle $\mathrm{SL}_2\mathbb{R}\to\mathbb{H}^2(-4)$.

14.4. The naturally reductive structure

On the Lie algebra $\mathfrak{sl}_2\mathbb{R}$, the inner product $\langle\cdot,\cdot\rangle$ at the identity induced from g is written as

$$\langle X,Y\rangle=\frac{1}{2}\,\mathrm{tr}\,({}^tXY),\quad X,Y\in\mathfrak{sl}_2\mathbb{R}.$$

One can see that the metric g is not only invariant by $\mathrm{SL}_2\mathbb{R}$-left translation but also right translations by $\mathrm{SO}(2)$. Hence the Lie group $\mathcal{G}=\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$ with multiplication:

$$(a,b)(a',b')=(aa',bb')$$

acts isometrically on $\mathrm{SL}_2\mathbb{R}$ via the action:

$$(\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))\times\mathrm{SL}_2\mathbb{R}\to\mathrm{SL}_2\mathbb{R};\quad (a,b)\cdot X=aXb^{-1}.$$

Furthermore, this action of $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$ on $\mathrm{SL}_2\mathbb{R}$ is transitive, hence $\mathrm{SL}_2\mathbb{R}$ is a homogeneous Riemannian space of $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$. The isotropy subgroup H of $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$ at the identity matrix $\mathbf{1}$ is the diagonal subgroup

$$\Delta K = \{(k,k)\mid k\in K\}\cong K$$

of $K\times K$. The coset space $(\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))/\mathrm{SO}(2)$ is a reductive homogeneous space. The Lie algebra of the product group $\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2)$ is $\mathfrak{sl}_2\mathbb{R}\oplus\mathfrak{k}$. On the other hand the Lie algebra of ΔK is

$$\Delta\mathfrak{k} = \{(W,W)\mid W\in\mathfrak{k}\}\cong\mathfrak{k}.$$

The tangent space $\mathrm{T}_{\mathbf{1}}\mathrm{SL}_2\mathbb{R}$ of $(\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))/\Delta K$ is the Lie algebra $\mathfrak{g}=\mathfrak{sl}_2\mathbb{R}$. This tangent space is identified with the linear subspace $\mathfrak{m}$ of $\mathfrak{sl}_2\mathbb{R}\oplus\mathfrak{k}$ defined by (see [87]):

$$\mathfrak{m} = \{(V+W,2W)\mid V\in\overline{\mathfrak{m}}_0,\ W\in\mathfrak{k}\}.$$

The Lie algebra $\mathfrak{g}\oplus\mathfrak{k}$ is decomposed as $\mathfrak{g}\oplus\mathfrak{k}=\Delta\mathfrak{k}\oplus\mathfrak{m}$. One can see that this decomposition is reductive. Every $(X,Y)\in\mathfrak{g}\oplus\mathfrak{k}$ is decomposed as

$$(X,Y) = (2X_{\mathfrak{k}}-Y,2X_{\mathfrak{k}}-Y)+(X_{\mathfrak{m}}+(Y-X_{\mathfrak{k}}),2(Y-X_{\mathfrak{k}})).$$

One can see that $(\mathrm{SL}_2\mathbb{R}\times\mathrm{SO}(2))/\mathrm{SO}(2)$ is naturally reductive with respect to the decomposition $\mathfrak{sl}_2\oplus\mathfrak{k}=\Delta\mathfrak{k}\oplus\mathfrak{m}$.

14.5. Curvatures

The commutation relations of $\{e_1,e_2,e_3\}$ are

$$[e_1,e_2]=-2e_3,\quad [e_2,e_3]=2e_1,\quad [e_3,e_1]=2e_2.$$

The Levi-Civita connection ∇ of is given by

$$\begin{array}{ccc}\nabla_{e_1}e_1=0, & \nabla_{e_1}e_2=-e_3, & \nabla_{e_1}e_3=e_2\\ \nabla_{e_2}e_1=e_3, & \nabla_{e_2}e_2=0, & \nabla_{e_2}e_3=-e_1\\ \nabla_{e_3}e_1=3e_2, & \nabla_{e_3}e_2=-3e_1 & \nabla_{e_3}e_3=0.\end{array}$$

The Riemannian curvature R is given by

$$R(e_1,e_2)e_1=7e_2,\quad R(e_1,e_2)e_2=-7e_1,$$

$$R(e_2,e_3)e_2=e_3,\quad R(e_2,e_3)e_3=-e_2,$$

$$R(e_1,e_3)e_1=-e_3,\quad R(e_1,e_3)e_3=e_1.$$

The basis $\{e_1,e_2,e_3\}$ diagonalizes the Ricci tensor field. The principal Ricci curvatures are given by

$$\rho_1=\rho_2=-6,\quad \rho_3=2.$$

The bilinear form U defined by (4.2) is given by

$$\mathsf{U}(e_1,e_3)=2e_2,\quad \mathsf{U}(e_2,e_3)=-2e_1. \tag{14.2}$$

All the other components are zero.

From these we obtain

$$\mathsf{U}(X,Y)=[X_{\mathfrak{k}},Y_{\mathfrak{m}}]+[Y_{\mathfrak{k}},X_{\mathfrak{m}}],\quad X,Y\in\mathfrak{g}.$$

The Levi-Civita connection is rewritten as

$$\nabla_XY=\frac{1}{2}[X,Y]+\mathsf{U}(X,Y)=\frac{1}{2}[X,Y]+[X_{\mathfrak{k}},Y_{\mathfrak{m}}]+[Y_{\mathfrak{k}},X_{\mathfrak{m}}],\quad X,Y\in\mathfrak{g}. \tag{14.3}$$

14.6. The canonical Sasakian structure

The almost contact structure associated to η and compatible to the metric g is determined by the endomorphism field φ defined by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0.$$

One can see that $\nabla\xi = -\varphi$. Hence $(\mathrm{SL}_2\mathbb{R}, \varphi, \xi, \eta, g)$ is a Sasakian space form of constant holomorphic sectional curvature -7. The left invariant Sasakian structure (φ, ξ, η) is called the *canonical Sasakian structure* of $\mathrm{SL}_2\mathbb{R}$.

By performing transversally homothetic change of the structure:

$$g \longmapsto g_c := -\frac{4}{c+3}g + \frac{4(c+7)}{(c+3)^2}\eta\otimes\eta, \quad \xi \longmapsto \xi_c := -\frac{c+3}{4}\xi, \quad \eta \longmapsto \eta_c := -\frac{4}{c+3}\eta$$

we obtain a left invariant Sasakian structure of constant holomorphic sectional curvature $c < -3$ on $\mathrm{SL}_2\mathbb{R}$.

The Sasakian space form $\mathrm{SL}_2\mathbb{R}$ with $c < -3$ admits a Lie subspace

$$\mathfrak{m} = \left\{ \left(V + W, \frac{1-c}{4}W \right) \,\middle|\, V \in \overline{\mathfrak{m}}_0, \quad W \in \mathfrak{u}(1) \right\}$$

of $\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(2)$. Thus $(\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2))/\mathrm{SO}(2)$ is a naturally reductive homogeneous space. For more discussion of the homogeneous geometry of $\mathrm{SL}_2\mathbb{R}$, see [116].

14.7. The double covering $\mathrm{SL}_2\mathbb{R} \to \mathrm{U}\mathbb{H}^2(-c^2)$

Let us consider the unit tangent sphere bundle $\mathrm{U}\mathbb{H}^2(-c^2)$, where we realize $\mathbb{H}^2(-c^2)$ as the upper half plane. Then the tangent bundle $\mathrm{T}\mathbb{H}^2(-c^2)$ is realized as

$$\mathrm{T}\mathbb{H}^2(-c^2) = \{(x, y, u, v) \in \mathbb{H}^2(-c^2) \times \mathbb{R}^2 \mid u, v \in \mathbb{R}\}.$$

The unit tangent sphere bundle $\mathrm{U}\mathbb{H}^2(-c^2)$ is realized as

$$\mathrm{T}\mathbb{H}^2(-c^2) = \{(x, y, u, v) \in \mathrm{T}\mathbb{H}^2(-c^2) \mid u^2 + v^2 = c^2y^2\}.$$

Introducing the fiberwise polar coordinates (r, θ) of $\mathrm{T}\mathbb{H}^2(-c^2)$, $\mathrm{U}\mathbb{H}^2(-c^2)$ is expressed as

$$\mathrm{U}\mathbb{H}^2(-c^2) = \{(x, y, cy\cos\theta, cy\sin\theta) \mid (x, y) \in \mathbb{H}^2(-c^2),\ 0 \leq \theta < 2\pi\}.$$

The Sasaki lift metric g^{s} is expressed as

$$g^{\mathsf{s}} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{c^2y^2} + \left(\mathrm{d}\theta + \frac{\mathrm{d}x}{y}\right)^2. \tag{14.4}$$

Let us take the following orthonormal frame field on $\mathrm{U}\mathbb{H}^2(-c^2)$:

$$\epsilon_1 = cy\frac{\delta}{\delta x}, \quad \epsilon_2 = cy\frac{\delta}{\delta y}, \quad \epsilon_3 = \frac{\partial}{\partial\theta},$$

where

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} - \frac{1}{y}\frac{\partial}{\partial\theta}, \quad \frac{\delta}{\delta y} = \frac{\partial}{\partial y}.$$

Then the Levi-Civita connection ∇ of $\mathrm{U}\mathbb{H}^2(-c^2)$ is computed as

$$\nabla_{\epsilon_1}\epsilon_1 = c\epsilon_2, \quad \nabla_{\epsilon_1}\epsilon_2 = -c\epsilon_1 - \frac{c^2}{2}\epsilon_3, \quad \nabla_{\epsilon_1}\epsilon_3 = \frac{c^2}{2}\epsilon_2,$$

$$\nabla_{\epsilon_2}\epsilon_1 = \frac{c^2}{2}\epsilon_3, \quad \nabla_{\epsilon_2}\epsilon_2 = 0, \quad \nabla_{\epsilon_2}\epsilon_3 = -\frac{c^2}{2}\epsilon_1,$$

$$\nabla_{\epsilon_3}\epsilon_1 = \frac{c^2}{2}\epsilon_2, \quad \nabla_{\epsilon_3}\epsilon_2 = -\frac{c^2}{2}\epsilon_1, \quad \nabla_{\epsilon_3}\epsilon_3 = 0.$$

The geodesic spray ξ^{s} is given by

$$\xi^{\mathsf{s}} = \cos\theta\,\epsilon_1 + \sin\theta\,\epsilon_2. \tag{14.5}$$

Let us relate the fiberwise angle function θ of $\mathrm{U}\mathbb{H}^2(-c^2)$ and the angle function ϕ of $\mathrm{SL}_2\mathbb{R}$. To this end we need to fix the origin of $\mathrm{U}\mathbb{H}^2(-c^2)$.

1. Take a unit tangent vector

$$W = c\frac{\partial}{\partial x}\Big|_i \in \mathrm{T}_i\mathbb{H}^2(-c^2)$$

and choose it as the origin of $\mathrm{U}\mathbb{H}^2(-c^2)$. Then the projection $\pi : \mathrm{SL}_2\mathbb{R} \to \mathrm{U}\mathbb{H}^2(-c^2)$ is described as

$$\pi(\mathsf{A}) = (\mathrm{T}_{\mathsf{A}}(i); c(a_{21}\mathrm{T}_{\mathsf{A}}(i) - a_{11})^2).$$

Here we identified W with the real scalar $c \in \mathbb{R}$. Let us choose

$$\mathsf{A} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \in K,$$

then we get

$$\pi \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \left(i; e^{2\phi i}c\right) = (i; c\cos(2\phi), c\sin(2\phi)).$$

Thus we obtain the formula (*cf.* [88, Proposition 2.1]):

$$\theta = 2\phi.$$

This relation implies that the geodesis spray ξ^{s} is derived from the left invariant vector field $\frac{c}{2}(E+F)$ on $\mathrm{SL}_2\mathbb{R}$. The geodesic spray generates ξ^{s} the 1-parameter transformation group

$$\mathcal{G} = \{\exp(t\xi)\}_{t\in\mathbb{R}} = \left\{ \begin{pmatrix} \cosh(ct/2) & \sinh(ct/2) \\ \sinh(ct/2) & \cosh(ct/2) \end{pmatrix} \Bigg| \ t \in \mathbb{R} \right\} = \mathrm{SO}^+(1,1). \tag{14.6}$$

2. Take a unit tangent vector

$$W = c\frac{\partial}{\partial y}\Big|_i \in \mathrm{T}_i\mathbb{H}^2(-c^2)$$

and choose it as the origin of $\mathrm{U}\mathbb{H}^2(-c^2)$. Then the projection $\pi : \mathrm{SL}_2\mathbb{R} \to \mathrm{U}\mathbb{H}^2(-c^2)$ is described as

$$\pi(\mathsf{A}) = (\mathrm{T}_{\mathsf{A}}(i); ci(a_{21}\mathrm{T}_{\mathsf{A}}(i) - a_{11})^2).$$

Here we identified W with the pure imaginary scalar $ci \in i\mathbb{R}$. The projection π is described as

$$\pi \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \left(i; ie^{2\phi i}c\right) = (i; -c\sin(2\phi), c\cos(2\phi)).$$

Thus we obtain the formula

$$\theta = 2\phi + \frac{\pi}{2}.$$

Hence the geodesic spray ξ^{s} is derived from the left invariant vector field $\frac{c}{2}H$ on $\mathrm{SL}_2\mathbb{R}$. The geodesic spray generates the 1-parameter transformation group

$$\{\exp(t\xi_2)\}_{t\in\mathbb{R}} = \left\{ \begin{pmatrix} e^{ct/2} & 0 \\ 0 & e^{-ct/2} \end{pmatrix} \Bigg| \ t \in \mathbb{R} \right\} \tag{14.7}$$

which is the abelian part A of $\mathrm{SL}_2\mathbb{R}$ (see [6, pp. 26–27]). One can see that the abelian part $A \subset \mathrm{SL}_2\mathbb{R}$ is conjugate with $\mathrm{SO}^+(1,1)$.

Here we describe the almost contact structure $(\varphi^{\mathsf{s}}, \xi^{\mathsf{s}}, \eta^{\mathsf{s}})$ of $\mathrm{U}\mathbb{H}^2(-c^2)$ compatible to the metric g explicitly. The contact form η^{s} dual to the geodesic spray ξ^{s} is

$$\eta^{\mathsf{s}} = \frac{\cos\theta\,\mathrm{d}x + \sin\theta\,\mathrm{d}y}{cy}.$$

The associated endomorphism field φ^{s} is given by

$$\varphi\epsilon_1 = -\sin\theta\epsilon_3, \quad \varphi\epsilon_2 = \cos\theta\epsilon_3, \quad \varphi\epsilon_3 = \sin\theta\epsilon_1 - \cos\theta\epsilon_2.$$

If one wishes to work in the contact Riemannian context, the following normalization for the structure tensors on $\mathrm{U}\mathbb{H}^2(-c^2)$:

$$\tilde{\eta} = \frac{1}{2}\eta^{\mathsf{s}}, \quad \tilde{\xi} = 2\xi^{\mathsf{s}}, \quad \tilde{\varphi} = \varphi^{\mathsf{s}}, \quad \tilde{g} = \frac{1}{4}g^{\mathsf{s}}.$$

Then $(\mathrm{U}\mathbb{H}^2(-c^2), \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a homogeneous contact (κ, μ)-space with $\kappa = -c^2(c^2+2) < 0$ and $\mu = 2c^2 > 0$.

In both cases $\theta = 2\phi$ or $\theta = 2\phi + \pi/2$, the metric on the double covering $\mathrm{SL}_2\mathbb{R}$ of $\mathrm{U}\mathbb{H}^2(-c^2)$ induced from the Sasaki lift metric is

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{cy^2} + 4\left(\mathrm{d}\phi + \frac{\mathrm{d}x}{2y}\right)^2$$

with respect to the coordinate system (x, y, ϕ). In particular, the metric g_1 on $\mathrm{SL}_2\mathbb{R}$ induced from $\mathrm{U}\mathbb{H}^2(-c^2)$ is

$$g_1 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} + 4\left(\mathrm{d}\phi + \frac{\mathrm{d}x}{2y}\right)^2.$$

Comparing this with the *Sasakian metric* of constant holomorphic sectional curvature -4 on $\mathrm{SL}_2\mathbb{R}$, we obtain the following result (see also [196, 197, 198]).

Proposition 14.1. *Let $\tilde{g}$ be the Riemannian metric of the standard contact Riemannian structure of $\mathrm{U}\mathbb{H}^2(-1)$. Then the Riemannian metric on $\mathrm{SL}_2\mathbb{R}$ induced from $\tilde{g}$ coincides with the Sasakian metric of constant holomorphic sectional curvature -7 on $\mathrm{SL}_2\mathbb{R}$.*

In this article we study homogeneous Riemannian spaces. Thus we consider homogeneous Riemannian metrics on $\mathrm{U}\mathbb{H}^2(-c^2)$. From homogeneity viewpoint, it is natural to induce an indefinite metric on $\mathrm{U}\mathbb{H}^2(-c^2)$ induced from the Killing form of $\mathrm{SL}_2\mathbb{C}$. The resulting homogeneous space $\mathrm{U}\mathbb{H}^2(-c^2) = \mathrm{SL}_2\mathbb{C}/\mathrm{U}(1)$ is an indefinit normal homogeneous space. For more detail, see [59].

15. The homogeneous contact Riemannian structures on Sasakian space forms

15.1. *Sasakian space forms of constant holomorphic sectional curvature $c \geq -3$ and $c \neq 1$*

Tricerri-Vanhecke classified 3-dimensional naturally reductive homogeneous spaces ([222, Theorem 6.5]):

Theorem 15.1. *Let (M, g) be a simply connected and complete Riemannian 3-manifold. If M admits a non-trivial homogeneous Riemannian structure of type $\mathcal{T}_3$, then M is isometric to one of the following naturally reductive homogeneous spaces*:

- $\mathbb{S}^3(c^2) = \mathrm{SO}(4)/\mathrm{SO}(3)$, $\mathbb{S}^3(c^2) = \mathrm{SU}(2)/\{\mathbf{1}\}$, $\mathbb{E}^3 = \mathrm{SE}(3)/\mathrm{SO}(3)$, $\mathbb{H}^3(-c^2) = \mathrm{SO}^+(1,3)/\mathrm{SO}(3)$,
- $\mathrm{Nil}_3 = (\mathrm{Nil}_3 \ltimes \mathrm{SO}(2))/\mathrm{SO}(2)$.
- $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathrm{U}(1)$.
- $\widetilde{\mathrm{SL}}_2\mathbb{R} = (\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathrm{SO}(2))/\mathrm{SO}(2)$.

As we saw before, Sasakian space forms are naturally reductive homogeneous spaces. In 1983, Tricerri and Vanhecke classified homogeneous Riemannian structures on the Heisenberg group Nil_3 in [222, Theorem 7.1]. In 2009 (September), the present author noticed that the one-parameter family of Ambrose-Singer connections on Nil_3 explicitly given by Tricerri and Vanhecke coincides with the Okumura's 1-parameter family of almost contact connections. In 2010, Gadea and Oubiña classified homogeneous Riemannian structure on the Berger sphere [79]. Motivated by [222, 79], the present author confirmed that the one-parameter family of Ambrose-Singer connections on the Berger sphere oincides with the Okumura's 1-parameter family of almost contact connections (December, 2012).

Let us represent $\mathcal{M}^3(c)$ as $\mathcal{M}^3(c) = (G \ltimes K)/K$, where

$$G = \begin{cases} \mathrm{SU}(2), & c > -3 \\ \mathrm{Nil}_3, & c = 3, \\ \widetilde{\mathrm{SL}}_2\mathbb{R}, & c < -3 \end{cases} \qquad K = \mathrm{SO}(2) \cong \mathrm{U}(1).$$

Precisely speaking, when $c \neq -3$, $G \ltimes K$ is just a direct product $G \times K$.

Note that the set of all the homogeneous Riemannian structures on the 3-sphere $\mathbb{S}^3$ was determined by Abe [1] (see section 15.3), we concentrate our attention to Sasakian space forms of constant holomorphic sectional curvature $c \neq 1$. First we consider the case $c \geq -3$ and $c \neq 1$.

Theorem 15.2 ([124]). *The set $\mathcal{S}$ of all homogeneous Riemannian structures on a Sasakian space form $\mathcal{M}^3(c) = (G \ltimes K)/K$ with $c \geq -3$ and $c \neq 1$ is given by $\{A^r \mid r \in \mathbb{R}\}$. Namely the set of all the Ambrose-Singer connections coincides with the one-parameter family of linear connections due to Okumura.*

Moreover $\mathcal{S}$ coincides with the set of all homogeneous almost contact Riemannian structures on $\mathcal{M}^3(c)$. The corresponding coset space representations are given as follows:

$$\mathcal{M}^3(c) = \begin{cases} (G \ltimes K)/K & r \neq (c+1)/2 \\ G/\{\mathsf{e}\} & r = (c+1)/2. \end{cases}$$

When $r = (c+1)/2$, the Ambrose-Singer connection $\nabla^r = \nabla + A^r$ is the Cartan-Schouten's $(-)$-connection. Every homogeneous Riemannian structure is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$.

The homogeneous Riemannian structure A^r is of type $\mathcal{T}_2$ if and only if $r = -2$. On the other hand, A^r is of type $\mathcal{T}_3$ if and only if $r = 1$. In this case $A^1 = -\mathrm{d}V$.

15.2. Sasakian space forms of constant holomorphic sectional curvature $c < -3$

Here we exhibit an explicit model of the simply connected non-unimodular Lie group equipped with a left invariant Sasakian structure appeared in Theorem 8.2 (see also [99, 122]). Let $\tilde{G}(c)$ be a 3-dimensional simply connected non-unimodular Lie group equipped with a left invariant Sasakian structure (η, φ, ξ, g). Set $c = -3 - \alpha^2$ for some non-zero constant α, then there exits an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}(c)$ of $\tilde{G}(c)$ satisfying (see *e.g.* [190]):

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = [e_3, e_1] = 0. \tag{15.1}$$

The left invariant Sasakian structure is described as

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad \xi = e_3. \tag{15.2}$$

One can see that $\tilde{G}(c)$ is isomorphic to the following solvable linear Lie group (*cf.* [122, 163]):

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^{\alpha x} & 0 & y \\ 0 & \frac{2}{\alpha}(e^{\alpha x} - 1) & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\} \tag{15.3}$$

The orthonormal basis $\{e_1, e_2, e_3\}$ defines a left invariant vector fields:

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = e^{\alpha x} \frac{\partial}{\partial y}, \quad e_3 = \frac{2}{\alpha}(e^{\alpha x} - 1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Generally speaking, generic non-symmetric 3-dimensional Riemannian Lie group $G = (G, \langle \cdot, \cdot \rangle)$ (Lie group equipped with left invariant metrics) has unique expression. Here the uniqueness means that if a 3-dimensional Riemannian Lie group $(G', \langle \cdot, \cdot \rangle')$ is isometric to $(G, \langle \cdot, \cdot \rangle)$ as a Riemannian 3-manifold, then G' is isomorphic to G as a Lie group. However as like $\widetilde{\mathrm{SL}}_2\mathbb{R}$ and $\tilde{G}(c)$, non-isomorphic 3-dimensional Lie groups might admit left invariant metrics which make them isometric as Riemannian 3-manifolds. In 3-dimensional Lie group theory, other than $\widetilde{\mathrm{SL}}_2\mathbb{R}$ and $\tilde{G}(c)$, Euclidean 3-space $\mathbb{E}^3$ and hyperbolic 3-space $\mathbb{H}^3$ does not satisfy the uniqueness. For more detail, see [162].

Now we can state our classification on homogeneous contact Riemannian structures on a Sasakian space form $\mathcal{M}^3(c)$ of constant holomorphic sectional curvature $c < -3$ obtained a recent work with Ohno.

Theorem 15.3 ([124]). *The set $\mathcal{S}$ of all homogeneous Riemannian structures on a Sasakian space form $\mathcal{M}^3(c) = (G \ltimes K)/K$ with $c < -3$ is given by $\mathcal{S} = \{A^r \mid r \in \mathbb{R}\} \cup \{\nabla^{(-)}\}$, where $\nabla^{(-)}$ is the Cartan-Schouten's $(-)$-connection of the non-unimodular Sasakian Lie group $G(c)$ diffeomorphic to G corresponding to the Lie algebra $\mathfrak{g}$. The set $\mathcal{S}$ coincides with the set of all homogeneous almost contact Riemannian structures on $\mathcal{M}^3(c)$. The corresponding coset space representations are given as follows:*

$$\mathcal{M}^3(c) = \begin{cases} (G \ltimes K)/K & \nabla + A^r, \quad r \neq (c+1)/2 \\ G/\{\mathsf{e}\} & \nabla + A^r, \quad r = (c+1)/2 \\ G(c)/\{\mathsf{e}\} & \nabla^{(-)}. \end{cases}$$

When $r = (c+1)/2$, the Ambrose-Singer connection $\nabla^r = \nabla + A^r$ is the Cartan-Schouten's $(-)$-connection for G. Every homogeneous Riemannian structure A^r is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. The homogeneous Riemannian structure A^r is of type $\mathcal{T}_2$ if and only if $r = -2$. On the other hand, A^r is of type $\mathcal{T}_3$ if and only if $r = 1$. In this case $A^1 = -\mathrm{d}V$. The Cartan-Schouten's $(-)$-connection for $\tilde{G}(c)$ is of type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$.

15.3. The unit 3-sphere

Finally we discuss the homogeneous Riemannian structures of the unit 3-sphere $\mathbb{S}^3$. In our setting and notation, Abe's classification is reformulated in the following manner (see also [79]):

Theorem 15.4 ([1])**.** *The homogeneous Riemannian structures on the unit 3-sphere are classified as follows:*

1. $S(X)Y = rA^1_X Y = -r\,\mathrm{d}V(X,Y)$ *for some constant* $r \geq 0$, $r \neq 1$. *The corresponding coset space representation is*
$$\mathbb{S}^3 = \mathrm{SO}(4)/\mathrm{SO}(3) = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{SU}(2).$$
The homogeneous Riemannian structure is of type $\mathcal{T}_3$.
2. $S(X)Y = A^r_X Y$ *for some* $r \in \mathbb{R}$ *with* $r \neq 1$. *The homogeneous Riemannian structure is of type* $\mathcal{T}_2 \oplus \mathcal{T}_3$. *It is of type* $\mathcal{T}_2$ *if and only if* $r = -2$. *The corresponding coset space representation is*
$$\mathbb{S}^3 = (\mathrm{SU}(2)\times\mathrm{U}(1))/\mathrm{U}(1) = \mathrm{U}(2)/\mathrm{U}(1).$$
3. $S(X)Y = A^1_X Y = -\mathrm{d}V(X,Y)$. *The homogeneous Riemannian structure is of type* $\mathcal{T}_3$. *The corresponding coset space representation is*
$$\mathbb{S}^3 = \mathrm{SU}(2)/\{\mathbf{1}\}.$$

Remark 15.1. The set of all $\mathrm{SU}(2)$-invariant metric linear connections with totally skew-symmetric torsion on $\mathbb{S}^3 = \mathrm{SU}(2)/\{\mathbf{1}\}$ is given by $\{\nabla + r\,A^1\}_{r\in\mathbb{R}}$ [62, Theorem 7.1].

Although the homogeneous Riemannian structure A^r is a homogeneous contact Riemannian structure for any $r \in \mathbb{R}$, the homogeneous Riemannian structure $-r\mathrm{d}V$ is a homogeneous contact Riemannian structure when and only when $r = 1$. Note that $\dim \mathrm{SO}(4) = \dim(\mathrm{SU}(2)\times\mathrm{SU}(2)) = 6$. On the other hand, $\dim \mathrm{Aut}(\mathbb{S}^3) = 4$ [217]. Now we retrive the following classification [79, Theorem 5.3] due to Gadea and Oubiña.

Corollary 15.1 ([79])**.** *The set of all homogeneous contact Riemannian structures on the unit 3-sphere* $\mathbb{S}^3$ *is given by* $\{A^r \mid r \in \mathbb{R}\}$.

Combining this classification with our main theorem, we obtain the following corollary.

Corollary 15.2. *The set of all homogeneous contact Riemannian structures on the 3-dimensional Sasakian space form* $\mathcal{M}^3(c)$ *is given by* $\{A^r \mid r \in \mathbb{R}\}$ *if* $c \geq -3$. *In case* $c < -3$, $\mathcal{S}$ *is given by* $\mathcal{S} = \{A^r \mid r \in \mathbb{R}\} \cup \{\nabla^{(-)}\}$. *Here* $\nabla^{(-)}$ *is the Cartan-Schouten's* $(-)$*-connection of the non-unimodular Sasakian Lie group* (15.3).

The corresponding coset space representations are given as follows:

$$\mathcal{M}^3(c) = \begin{cases} (G \ltimes K)/K & \nabla + A^r, \quad r \neq (c+1)/2 \\ G/\{\mathsf{e}\} & \nabla + A^r, \quad r = (c+1)/2 \\ \tilde{G}(c)/\{\mathsf{e}\} & \nabla^{(-)}, \quad c < -3. \end{cases}$$

Every homogeneous Riemannian structure other than $\nabla^{(-)}$ *of* $\tilde{G}(c)$ *is of type* $\mathcal{T}_2 \oplus \mathcal{T}_3$.

The homogeneous Riemannian structure A^r *is of type* $\mathcal{T}_2$ *if and only if* $r = -2$. *On the other hand,* A^r *is of type* $\mathcal{T}_3$ *if and only if* $r = 1$. *In this case* $A^1 = -\mathrm{d}V$. *The Cartan-Schoutren's* $(-)$*-connection of* $\tilde{G}(c)$ *is of type* $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$ *and can not be of type* $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$.

15.4. Non-umimodular Lie groups

The hyperbolic Sasakian space form $\mathcal{M}^3(c)$ of constant holomorphic sectional curvature $c < -3$ admits another homogeneous space representation. Indeed $\mathcal{M}^3(c)$ is realized as a non-unimodular Lie groups. In this subsection we discuss this homogeneous space representation. Now let us consider 3-dimensional non-unimodular Lie groups equipped with left invariant contact Riemannian structure. Here we recall Perrone's construction [190].

Let G be a Lie group of arbitrary dimension with Lie algebra $\mathfrak{g}$. Denote by ad the *adjoint representation* of $\mathfrak{g}$,

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}); \quad \mathrm{ad}(X)Y = [X,Y].$$

Then one can see that $\mathrm{tr}\,\mathrm{ad}$;

$$X \longmapsto \mathrm{tr}\,\mathrm{ad}(X)$$

is a Lie algebra homomorphism into the commutative Lie algebra $\mathbb{R}$. The kernel

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \mathrm{tr}\,\mathrm{ad}(X) = 0\}$$

of $\mathrm{tr}\,\mathrm{ad}$ is an ideal of $\mathfrak{g}$ which contains the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Now we equip a left invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on G. Denote by $\mathfrak{u}$ the orthogonal complement of $\mathfrak{u}$ in $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$. Then the homomorphism theorem implies that $\dim \mathfrak{u}^\perp = \dim \mathfrak{g}/\mathfrak{u} \leq 1$. Moreover, Milnor's criterion impolies that G is unimodular if and only if $\mathfrak{u} = \mathfrak{g}$. Based on this criterion, the ideal $\mathfrak{u}$ is called the *unimodular kernel* of $\mathfrak{g}$. In particular, for a 3-dimensional non-unimodular Lie group G, its unimodular kernel $\mathfrak{u}$ is commutative and of 2-dimension.

Now let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure (φ, ξ, η, g). Then one can easily check that $\xi \in \mathfrak{u}$. We take an orthonormal basis $\{e_2, e_3 = \xi\}$ of $\mathfrak{u}$. Then $e_1 = -\varphi e_2 \in \mathfrak{u}^\perp$ and hence $\mathrm{ad}(e_1)$ preserves $\mathfrak{u}$. Express $\mathrm{ad}(e_1)$ as

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3$$

over $\mathfrak{u}$. The compatibility condition $\mathrm{d}\eta = \Phi$ implies that $\beta = 2$. Next, $\nabla_\xi \xi = 0$ implies that $\delta = 0$. Moreover one can deduce that $[e_2, e_3] = 0$ from the Jacobi identity.

Remark 15.2. Milnor [163] chose the following orthonormal basis $\{u_1, u_2, u_3\}$ for a non-unimodular Lie group G with left invariant Riemannian metric.

$$u_1 \in \mathfrak{u}^\perp, \quad \langle \mathrm{ad}(u_1)u_2, \mathrm{ad}(u_1)u_3\rangle = 0. \tag{15.4}$$

This orthonormal basis $\{u_1, u_2, u_3\}$ satisfies

$$[u_1, u_2] = \alpha u_2 + \beta u_3, \quad [u_2, u_3] = 0, \quad [u_1, u_3] = \gamma u_2 + \delta u_3$$

with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. Moreover $\{u_1, u_2, u_3\}$ diagonalizes the Ricci tensor field. On the other hand, the basis $\{e_1, e_2, e_3\}$ constructed for a non-unimodular homogeneous contact Riemannian 3-manifold G does not satisfy the orthogonality condition $\langle \mathrm{ad}(u_1)u_2, \mathrm{ad}(u_1)u_3\rangle = 0$. In fact, $\{e_1, e_2, e_3\}$ satisfies this orthogonality condition if and only if $\gamma = 0$.

Theorem 15.5 ([190]). *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact Riemannian structure. Then the Lie algebra $\mathfrak{g}$ satisfies the commutation relations*

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma e_2,$$

with $e_3 = \xi$, $e_1 = -\varphi e_2 \in \mathfrak{u}^\perp$ and $\alpha \neq 0$. The Webster scalar curvature and the torsion invariant satisfy the relation:

$$4\sqrt{2}W < |\tau|.$$

The Levi-Civita connection of G is given by the following table:

Proposition 15.1 ([190]).

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -\frac{1}{2}(\gamma - 2)e_3, & \nabla_{e_1} e_3 = \frac{1}{2}(\gamma - 2)e_2 \\ \nabla_{e_2} e_1 = -\alpha e_2 - \frac{1}{2}(\gamma + 2)e_3, & \nabla_{e_2} e_2 = \alpha e_1, & \nabla_{e_2} e_3 = \frac{1}{2}(\gamma + 2)e_1 \\ \nabla_{e_3} e_1 = -\frac{1}{2}(\gamma + 2)e_2, & \nabla_{e_3} e_2 = \frac{1}{2}(\gamma + 2)e_1 & \nabla_{e_3} e_3 = 0. \end{array}$$

The endomorphism field $h = (\pounds_\xi \varphi)/2$ is given by

$$he_1 = -\frac{1}{2}\gamma e_1, \quad he_2 = \frac{1}{2}\gamma e_2.$$

The Riemannian curvature R is given by

$$
\begin{aligned}
R(e_1,e_2)e_1 =& -\left\{\frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2\right\}e_2+\alpha\gamma e_3,\\
R(e_1,e_2)e_2 =& \left\{\frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2\right\}e_1,\\
R(e_1,e_3)e_1 =& \alpha\gamma e_2+\frac{1}{4}(3\gamma^2+4\gamma-4)e_3,\\
R(e_1,e_3)e_3 =& -\frac{1}{4}(3\gamma^2+4\gamma-4)e_1,\\
R(e_2,e_3)e_2 =& -\frac{1}{4}(\gamma+2)^2e_3,\\
R(e_2,e_3)e_3 =& \frac{1}{4}(\gamma+2)^2e_2,\\
R(e_1,e_2)e_3 =& -\alpha\gamma e_1.
\end{aligned}
$$

$$
K_{12}=\frac{1}{4}(\gamma^2-4\gamma-12)-\alpha^2,\quad K_{13}=-\frac{1}{4}(3\gamma^2+4\gamma-4),\quad K_{23}=\frac{1}{4}(\gamma+2)^2.
$$

The Ricci tensor field has non-trivial components

$$
R_{11}=-\alpha^2-2-2\gamma-\frac{\gamma^2}{2},\quad R_{22}=-\alpha^2-2+\frac{\gamma^2}{2},\quad R_{33}=2-\frac{\gamma^2}{2},\quad R_{23}=-\alpha\gamma.
$$

The bilinear form U *is given by*

$$
\mathsf{U}(e_1,e_2)=-\frac{1}{2}(\alpha e_2+\gamma e_3),\quad \mathsf{U}(e_1,e_3)=-e_2,\quad \mathsf{U}(e_2,e_2)=\alpha e_1,\quad \mathsf{U}(e_2,e_3)=\frac{1}{2}(\gamma+2)e_1.
$$

The Lie algebra $\mathfrak{g}$ *is classified by the* Milnor invariant $\mathcal{D}=-8\gamma/\alpha^2$.

As we can see in [190], G satisfies $\gamma=0$ if and only if it is isometric to a Sasakian space form $\mathcal{M}^3(c)$ of constant holomorphic sectional curvature $c=-3-\alpha^2<-3$. More precisely G is (locally) isomorphic to $\tilde{G}(c)$ given by (15.3).

On the other hand, homogeneous Riemannian structures on 3-dimensional non-unimodular Lie group are classified in [39] under the *left invariance* and the orthogonality assumption (15.4) (see Remark 15.2). The classification due to [39] is improved by Ohno and the present author [125].

Let us concentrate on Sasakian space forms. In this case we have $\gamma=\delta=0$ and hence $\{e_1,e_2,e_3\}$ satisfies the orthogonality condition (15.4). Thus we can apply the classification due to [39, Theorem 1.3] (see also [124]).

Proposition 15.2. *The homogeneous Riemannian structures on the non-unimodular Lie group G equipped with a left invariant Sasakian structure of constant holomorphic sectional curvature $c=-3-\alpha^2<-3$ are given by*

- *$S=A^r$ or*
- *S is associated with the Cartan-Schouten $(-)$-connection*:

$$
S=\mathrm{d}\eta(X,Y)\xi+\eta(Y)\varphi X+(\alpha g(X,e_2)+\eta(X))\,\varphi Y.
$$

The homogeneous Riemannian structure $S=\nabla^{(-)}-\nabla$ is of type $\mathcal{T}_1\oplus\mathcal{T}_2\oplus\mathcal{T}_3$.

15.5. The reduction of the homogeneous structures

Let $M=(M,\varphi,\xi,\eta,g)$ be a regular Sasakian manifold fibered over a Kähler manifold $\overline{M}=(\overline{M},\bar{g},J)$. The complex structure J on $\overline{M}$ is related to φ by the fundamental relation (5.13). Motivated by this fundamental relation we may introduce a tensor field $\bar{S}$ on $\overline{M}$ by ([36, 42]):

$$
\bar{S}(\overline{X})\overline{Y}=\pi_*(S(\overline{X}^{\mathsf{h}})\overline{Y}^{\mathsf{h}}),\quad \overline{X},\overline{Y}\in\Gamma(\mathrm{T}\overline{M}). \tag{15.5}
$$

Here the superscript h means the horizontal lift operations of tangent vectors as well as vector fields.

Every complete Sasakian space form $\mathcal{M}^3(c)$ is a regular Sasakian manifold fibered over a space form $\overline{\mathcal{M}}(c+3)$. We know that homogeneous Sasakian structures are exhausted by the 1-parameter family $\{A^r\}_{r\in\mathbb{R}}$. However one can see that the induced homogeneous Riemannian structure on $\overline{\mathcal{M}}(c+3)$ are trivial ones for every $r \in \mathbb{R}$. On the other hand, the Lie algebra $\mathfrak{g}(c)$ of the non-unimodular Sasakian Lie group (15.3) is a 1-dimensional solvable extension of the Lie algebra of the solvable Lie group model of $\mathbb{H}^2(-\alpha^2)$. Indeed, the solvable Lie group model $\overline{\mathcal{S}}$ of $\mathbb{H}^2(-\alpha^2)$ is isomorphic and isometric to the Lie group $\overline{\mathcal{M}}$ given in (4.9). The Lie algebra $\bar{\mathfrak{m}}$ of $\overline{\mathcal{M}}$ is spanned by the basis

$$\bar{e}_1 = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad \bar{e}_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we may set

$$e_1 = \bar{e}_2, \quad e_2 = \bar{e}_1$$

and consider the linear space $\bar{\mathfrak{m}} \oplus \mathbb{R}e_3$ and define a Lie bracket by (15.1). Then the resulting Lie algebra is nothing but $\mathfrak{g}(c)$. The Ambrose-Singer connection of the homogeneous Kähler structure of type $\mathcal{T}_1$ on $\mathbb{H}^2(-\alpha^2) = \overline{\mathcal{M}}/\{E_2\}$ is the Cartan-Schouten's $(-)$-connection of $\overline{\mathcal{M}}$. The Cartan-Scouten's $(-)$-connection of $\tilde{G}(c)$ is understood as the extension of that of $\mathbb{H}^2(-\alpha^2) = \overline{\mathcal{M}}$ to $\tilde{G}(c)$. Thus the difference of $c \geq -3$ and $c < -3$ in the classification of homogeneous Riemannian structures on $\mathcal{M}^3(c)$ is caused by Proposition 4.6.

16. Solvable Lie groups

16.1. Some fundamental facts in solvable Lie groups

In this section we give explicit formulas for homogeneous Riemannian structures on certain homogeneous Riemannian 3-spaces. In particular we give explicit formulas for homogeneous structures on hyperbolic 3-space and Euclidean 3-space in terms of almost contact structures. First we recall the following theorem due to Heintze.

Theorem 16.1 ([91])**.** *Let M be a connected homogeneous Riemannian space of non-positive sectional curvature, then M admits a solvable Lie group structure equipped with a left invariant metric.*

Note that Kobayashi [144] proved that a connected homogeneous Riemannian space of non-positive sectional curvature and negative definite Ricci operator is simply connected.

Azencott and Wilson proved the following fact.

Theorem 16.2 ([7])**.** *Let M be a simply connected homogeneous Riemannian space of non-positive curvature and G a connected Lie subgroup of $\mathrm{Iso}_\circ(M)$ acting transitively on M. Then there exits a solvable Lie subgroup $\mathcal{S}$ of G acting simply transitively on M.*

Here we collect some notions given in [8, pp. 41–48] (see also [1]).

Definition 16.1. Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 16.2. Assume that M has no Euclidean factor in its de Rham decomposition. Then a connected Lie subgroup $\mathcal{S}'$ of $\mathrm{Iso}_\circ(M)$ is said to be a *modification* of $\mathcal{S}$ if S' acts simple transitively on M and the Lie algebra $\mathfrak{s}'$ of $\mathcal{S}'$ is in the normalizer $\mathfrak{n}(\mathfrak{s})$. Here $\mathfrak{s}$ is the Lie algebra of $\mathcal{S}$.

Definition 16.2. Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 16.2. A solvable Lie subgroup $\mathcal{S}$ of $\mathrm{Iso}_\circ(M)$ is said to be in *standard position* if for some point $o \in M$, $\mathsf{B}(V, U) = 0$ for all $V \in [\mathfrak{s}, \mathfrak{s}]^\perp$ and $U \in \mathfrak{h}$, where B is the Killing form of $\mathfrak{iso}(M)$ and $\mathfrak{h}$ is the isotropy algebra at o. The linear subspace $[\mathfrak{s}, \mathfrak{s}]^\perp$ is the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$ with respect to the inner product on $\mathfrak{s}$ induced from the metric g. The Lie algebra $\mathfrak{s}$ is also called in standard position.

Theorem 16.3. *Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 16.2 and has no Euclidean factor in its de Rham decomposition. The for any connected Lie subgroup $\mathcal{S}'$ of $\mathrm{Iso}_\circ(M)$ acting simple transitively on M, there exits a unique modification $\mathcal{S}$ of $\mathcal{S}'$ which is in standard position.*

Theorem 16.4. *Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 16.2 and has no Euclidean factor in its de Rham decomposition. The for any connected Lie subgroups $\mathcal{S}$ and $\mathcal{S}'$ of $\mathrm{Iso}_{circ}(M)$ acting simple transitively on M and are in standard position. Then there exits an element $a \in \mathrm{Iso}_\circ(M)$ such that $a\mathcal{S}a^{-1} = \mathcal{S}'$.*

Conversely, if $\mathcal{S}$ is in standard position, then for any $a \in \mathrm{Iso}_\circ(M)$, $a\mathcal{S}a^{-1}$ is in standard position.

Let M be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 16.2 and has no Euclidean factor in its de Rham decomposition. Take a (solvable) Lie subgroup $\mathcal{S} \subset \mathrm{Iso}_\circ(M)$ which is in standard position. Then for any $a \in \mathrm{Iso}_\circ(M)$, The Lie algebra of $a\mathcal{S}a^{-1}$ is $\mathrm{Ad}(a)\mathfrak{s}$ and isomorphic to $\mathfrak{s}$ as a Lie algebra. Moreover $\mathfrak{s}$ and $\mathrm{Ad}(a)\mathfrak{s}$ are isometric relative to the inner product induced from (T_oM, g_o) and $(\mathrm{T}_{a\cdot o}M, g_{a\cdot o})$. The homogeneous Riemannian structures S determined by $\mathcal{S} = \mathcal{S}/\{\mathsf{e}\}$ and S' determined by $\mathcal{S}' = (\mathrm{Ad}(a)\mathcal{S})/\{\mathsf{e}\}$ are isomorphic.

16.2. The 2-parameter family $G(\gamma_1,\gamma_2)$

Let us define a 2-parameter family $\{G(\gamma_1,\gamma_2)\}_{\gamma_1,\gamma_2\in\mathbb{R}}$ of linear Lie groups by

$$G(\gamma_1,\gamma_2) = \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & z \\ 0 & e^{\gamma_1 z} & 0 & x \\ 0 & 0 & e^{\gamma_2 z} & y \\ 0 & 0 & 0 & 1 \end{array} \right) \middle|\; x,y,z \in \mathbb{R} \right\}.$$

This 2-parameter family can be seen in [222, p. 85] (see also [60, 95, 97, 111]). The Lie group $G(\gamma_1,\gamma_2)$ is identical to $G^2(\gamma_1,\gamma_2)$ in [76, 77].

The Lie algebra $\mathfrak{g}(\gamma_1,\gamma_2)$ of each $G(\gamma_1,\gamma_2)$ is spanned by the basis

$$E_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad E_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad E_3 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

with commutation relations

$$[E_1,E_2] = 0, \quad [E_2,E_3] = -\gamma_2 E_2, \quad [E_3,E_1] = \gamma_1 E_1.$$

These relations show that $G(\gamma_1,\gamma_2)$ is *solvable*. Moreover $G(\gamma_1,\gamma_2)$ is *non-unimodular* unless $\gamma_1+\gamma_2=0$.

The left translated vector fields of E_1, E_2 and E_3 are

$$e_1 = e^{\gamma_1 z}\frac{\partial}{\partial x}, \quad e_2 = e^{\gamma_2 z}\frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial z}.$$

We equip an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}(\gamma_1,\gamma_2)$ so that $\{E_1,E_2,E_3\}$ is orthonormal. Then the left translated Riemannian metric $g = g(\gamma_1,\gamma_2)$ is

$$g(\gamma_1,\gamma_2) = e^{-2\gamma_1 z}\mathrm{d}x^2 + e^{-2\gamma_2 z}\mathrm{d}y^2 + \mathrm{d}z^2.$$

The Levi-Civita connection ∇ of g is computed as

$$\begin{array}{ccc} \nabla_{e_1}e_1 = \gamma_1\, e_3 & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = -\gamma_1\, e_1, \\ \nabla_{e_2}e_1 = 0 & \nabla_{e_2}e_2 = \gamma_2 e_3, & \nabla_{e_2}e_3 = -\gamma_2 e_2, \\ \nabla_{e_3}e_1 = 0 & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0. \end{array}$$

Let us compute the connection form and curvature form. The orthonormal coframe field $\Theta = \{\vartheta^1,\vartheta^2,\vartheta^3\}$ metrically dual to $\mathcal{E} = \{e_1,e_2,e_3\}$ is given by

$$\vartheta^1 = e^{-\gamma_1 z}\,\mathrm{d}x, \quad \vartheta^2 = e^{-\gamma_2 z}\,\mathrm{d}y, \quad \vartheta^3 = \mathrm{d}z.$$

Since

$$\mathrm{d}\vartheta^1 = \gamma_1\, e^{-\gamma_1 z}\mathrm{d}x\wedge\mathrm{d}z, \quad \mathrm{d}\vartheta^2 = \gamma_2\, e^{-\gamma_2 z}\mathrm{d}y\wedge\mathrm{d}z, \quad \mathrm{d}\vartheta^3 = 0,$$

we get

$$\omega = \left(\begin{array}{ccc} 0 & 0 & -\gamma_1\,\vartheta^1 \\ 0 & 0 & -\gamma_2\,\vartheta^2 \\ \gamma_1\,\vartheta^1 & \gamma_2\,\vartheta^2 & 0 \end{array} \right).$$

The curvature form is computed as

$$\Omega = \left(\begin{array}{ccc} 0 & -\gamma_1\gamma_2\,\vartheta^1\wedge\vartheta^2 & -\gamma_1^2\,\vartheta^1\wedge\vartheta^3 \\ \gamma_1\gamma_2\,\vartheta^1\wedge\vartheta^2 & 0 & -\gamma_2^2\,\vartheta^2\wedge\vartheta^3 \\ \gamma_1^2\,\vartheta^1\wedge\vartheta^3 & \gamma_2^2\,\vartheta^2\wedge\vartheta^3 & 0 \end{array} \right).$$

The sectional curvature of $G(\gamma_1,\gamma_2)$ is given by

$$K_{12}=-\gamma_1\gamma_2 \quad K_{13}=\gamma_1^2, \quad K_{23}=\gamma_2^2.$$

The Ricci tensor field and scalar curvature are given by

$$R_{11}=-\gamma_1(\gamma_1+\gamma_2), \quad R_{22}=-\gamma_2(\gamma_1+\gamma_2), \quad R_{33}=-(\gamma_1^2+\gamma_2^2), \quad \mathrm{s}=-2(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2).$$

Now we introduce an almost contact structure (φ,ξ,η) compatible to g by

$$\varphi e_1=e_2, \quad \varphi e_2=-e_1, \quad \varphi e_3=0, \quad \xi=e_3, \ \eta=g(\xi,\cdot).$$

Then we obtain a 2-parameter family of homogeneous almost contact metric manifolds. The almost contact metric manifold $G(\alpha,\beta)$ has constant holomorphic sectional curvature $-\gamma_1\gamma_2$. Then we have

$$\nabla\xi=-\gamma_1\,\vartheta^1\otimes e_1-\gamma_2\,\vartheta^2\otimes e_2.$$

$$(\nabla_X\varphi)Y=\{-\gamma_2\vartheta^2\otimes\eta\otimes e_1+\gamma_1\vartheta^1\otimes\eta\otimes e_2+(-\gamma_1\vartheta^1\otimes\vartheta^2+\gamma_2\vartheta^2\otimes\vartheta^2)\otimes\xi\}(X,Y).$$

From these we have

$$\alpha=\frac{1}{2}\mathrm{tr}(\varphi\nabla\xi)=0, \quad \beta=\mathrm{div}\,\xi=-(\gamma_1+\gamma_2).$$

One can see that $G(\gamma_1,\gamma_2)$ is normal when and only when $\gamma_1=\gamma_2$.

Let $\nabla^r=\nabla+A^r$ the almost contact connections. Then the covariant form $A^r_\flat$ of A^r is computed as

$$A^r_\flat=2\gamma_1\,\vartheta^1\otimes(\vartheta^3\wedge\vartheta^1)-2\gamma_2\,\vartheta^2\otimes(\vartheta^2\wedge\vartheta^3)-2r\,\vartheta^3\otimes(\vartheta^1\wedge\vartheta^2). \tag{16.1}$$

Thus we obtain

$$A^r_XY=-\eta(Y)\nabla_X\xi+g(\nabla_X\xi,Y)\xi-r\,\eta(X)\varphi Y. \tag{16.2}$$

Example 16.1 ($\gamma_1=\gamma_2=0$)**.** The Lie group $G(0,0)$ is isometric and isomorphic to the Euclidean 3-space $\mathbb{E}^3=(\mathbb{R}^3,+)$. The almost contact Riemannian structure is coKähler. In this case, the almost contact connection ∇^r is given by

$$\nabla^r_XY=\nabla_XY+A^r_XY, \quad A^r_XY=-r\eta(X)\varphi Y. \tag{16.3}$$

Example 16.2 ($\gamma_1=\gamma_2=c\neq0$)**.** Let us take $\gamma_1=\gamma_2=c\neq0$. Then $G(c,c)$ is a warped product model (also called the solvable Lie group model) of hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant curvature $-c^2<0$:

$$\mathbb{H}^3(-c^2)=(\mathbb{R}^3(x,y,z),e^{-2cz}(\mathrm{d}x^2+\mathrm{d}y^2)+dz^2).$$

In this case, we have

$$\nabla_X\xi=-c\{X-\eta(X)\xi\}, \ (\nabla_X\varphi)Y=c\{\eta(Y)\varphi X+g(X,\varphi Y)\xi\}.$$

These formulae imply that $G(c,c)=\mathbb{H}^3(-c^2)$ is a $(-c)$-Kenmotsu manifold. In particular $G(-1,-1)=\mathbb{H}^3(-1)$ is a Kenmotsu manifold. In this case, the almost contact connection ∇^r is given by

$$\nabla^r_XY=\nabla_XY+A^r_XY, \quad A^r_XY=g(X,Y)\xi-\eta(Y)X-r\eta(X)\varphi Y. \tag{16.4}$$

Example 16.3 ($\gamma_1=-\gamma_2=-1$)**.** In case $\gamma_1=-\gamma_2=-1$,

$$G(-1,1)=(\mathbb{R}^3(x,y,z),e^{2z}\mathrm{d}x^2+e^{-2z}\mathrm{d}y^2+\mathrm{d}z^2).$$

is the model space Sol_3 of solvgeometry [220]. Note that $G(-1,1)$ is isomorphic to the Minkowski motion group $\mathrm{SE}(1,1)$ discussed in Section 8.3.5. The left invariant almost contact Riemannian structure is non-normal almost coKähler. This structure is referred as to the class C_9 in [45].

Example 16.4 ($\gamma_1=0,\gamma_2\neq0$)**.** Take $(\gamma_1,\gamma_2)=(0,c)$ with $c\neq0$. Then $G(0,c)$ is the Riemannian product of $\mathbb{R}^1(x)$ and hyperbolic 2-space

$$\mathbb{H}^2(-c^2)=(\mathbb{R}^2(y,z),e^{-2cz}\,\mathrm{d}y^2+\mathrm{d}z^2).$$

Under the coordinate change:

$$\bar{x}=cy, \quad \bar{y}=e^{cz}, \quad \bar{z}=x,$$

the metric g is transformed to

$$\frac{\mathrm{d}\bar{x}^2+\mathrm{d}\bar{y}^2}{c\bar{y}^2}+\mathrm{d}z^2.$$

The left invariant almost contact Riemannian structure is non-normal $(-c/2)$-almost Kenmotsu. In particular, $G(0,-2)=\mathbb{R}\times\mathbb{H}^2(-4)$ is an almost Kenmotsu manifold.

Example 16.5 ($\gamma_1 \neq 0, \gamma_2 = 0$)**.** Take $(\gamma_1, \gamma_2) = (c, 0)$ with $c \neq 0$. Then $G(c, 0)$ is the Riemannian product of $\mathbb{R}^1(y)$ and hyperbolic 2-space

$$\mathbb{H}^2(-c^2) = (\mathbb{R}^2(x,z), e^{-2cz}\,\mathrm{d}x^2 + \mathrm{d}z^2).$$

Under the coordinate change:

$$\bar{x} = cx, \quad \bar{y} = c^{cz}, \quad \bar{z} = y,$$

the metric g is transformed to

$$\frac{\mathrm{d}\bar{x}^2 + \mathrm{d}\bar{y}^2}{c\bar{y}^2} + \mathrm{d}z^2.$$

The left invariant almost contact Riemannian structure is non-normal $(-c/2)$-almost Kenmotsu. In particular, $G(-2, 0) = \mathbb{R} \times \mathbb{H}^2(-4)$ is an almost Kenmotsu manifold.

16.3. The homogeneous Riemannian structures

Now let us classify the homogeneous Riemannian structures on $G(\gamma_1, \gamma_2)$. Let $S_\flat$ be a tensor field of type $(0, 3)$ satisfying (4.11). Then $S_\flat$ is expressed as the form of (4.12). Denote by S the $(1, 2)$-tensor field associated to $S_\flat$.

Assume that S is a homogeneous Riemannian structure, then the linear connection $\tilde{\nabla} = \nabla + S$ satisfies $\tilde{\nabla} R = 0$. Since $G(\gamma_1, \gamma_2)$ is 3-dimensional, this condition is equivalent to (4.14). Since the Ricci tensor field of $G(\gamma_1, \gamma_2)$ is given by

$$-\gamma_1(\gamma_1 + \gamma_2)\vartheta^1 \otimes \vartheta^1 - \beta(\gamma_1 + \gamma_2)\vartheta^2 \otimes \vartheta^2 - (\gamma_1^2 + \gamma_2^2)\vartheta^3 \otimes \vartheta^3,$$

we obtain the system

$$\begin{cases} (\gamma_1 - \gamma_2)(\gamma_1 + \gamma_2)S_\flat(X, e_1, e_2) = 0, \\ \gamma_2(\gamma_1 - \gamma_2)(S_\flat(X, e_1, e_3) + \gamma_1\vartheta(X)) = 0, \\ \gamma_1(\gamma_1 - \gamma_2)(S_\flat(X, e_2, e_3) + \gamma_2\vartheta(X)) = 0. \end{cases}$$

Let us assume that $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ and $\gamma_1 \neq \gamma_2$. Then we get

$$S_\flat = 2\{-\gamma_2\vartheta^2 \otimes (\theta^2 \wedge \theta^3) + \gamma_1\vartheta^1 \otimes (\theta^3 \wedge \theta^1) + \sigma) \otimes (\vartheta^1 \wedge \vartheta^2)\},$$

where $\sigma(X) = S_\flat(X, e_1, e_2)$. The homogeneous Riemannian structure S is expressed as

$$S(X)Y = -\eta(Y)\nabla_X\xi + g(\nabla_X\xi, Y)\xi + \sigma(X)\varphi Y.$$

In case $\gamma_1 + \gamma_2 \neq 0$, we have $\sigma = 0$.

Next, we determine the 1-form σ for the case $\gamma_1 + \gamma_2 = 0$. The 1-form σ is determined by the parallelism $\tilde{\nabla} S = 0$, where $\tilde{\nabla} = \nabla + S$. The connection forms of $\tilde{\nabla}$ are computed as

$$\tilde{\omega}_1{}^2 = \sigma, \quad \tilde{\omega}_1{}^3 = \tilde{\omega}_2{}^3 = 0.$$

Hence

$$\tilde{\nabla}_X\vartheta^1 = \sigma \otimes \vartheta^2, \quad \tilde{\nabla}_X\vartheta^2 = -\sigma \otimes \vartheta^1.$$

Hence

$$(\tilde{\nabla}_X S_\flat) = 2(\gamma_2 - \gamma_1)\sigma \otimes \{\vartheta^1 \otimes (\vartheta^2 \wedge \vartheta^3) - \vartheta^2 \otimes (\vartheta^3 \wedge \vartheta^1)\} + 2(\tilde{\nabla}_X\sigma) \otimes (\vartheta^1 \wedge \vartheta^2).$$

Since $\gamma_1 + \gamma_2 = 0$, $\tilde{\nabla} S_\flat = 0$ if and only if $\sigma = 0$. Hence we get

$$S(X)Y = A^0_X Y = -\eta(Y)\nabla_X\xi + g(\nabla_X\xi, Y)\xi.$$

One can confirm that this S is really a homogeneous Riemannian structure, especially a homogeneous almost contact Riemannian structure. Even if the cases, $\gamma_1\gamma_2 = 0$ or $\gamma_1 = \gamma_2$, $S = A^0$ is still a homogeneous almost contact Riemannian structure of $G(\gamma_1, \gamma_2)$.

The following result is a reformulation (and an improvement) of a result due to Tricerri and Vanhecke given in [222, p. 85, pp. 88-89].

Theorem 16.5. *For any γ_1, $\gamma_2 \in \mathbb{R}$, the tensor field*

$$S(X)Y = A^0_X Y = -\eta(Y)\nabla_X\xi + g(\nabla_X\xi, Y)\xi$$

is a homogeneous almost contact Riemannian structure on the solvable Lie group $G(\gamma_1, \gamma_2)$.

- *S is of type $\mathcal{T}_1 \oplus \mathcal{T}_2$.*
- *S is of type $\mathcal{T}_2$ if and only if $\gamma_1 + \gamma_2 = 0$.*
- *S is of type $\mathcal{T}_1$ if and only if $\gamma_1 = \gamma_2 \neq 0$.*

In all the cases, the corresponding coset space representation of $G(\gamma_1, \gamma_2)$ is $G(\gamma_1, \gamma_2)/\{\mathsf{e}\}$. Moreover the Ambrose-Singer connection $\nabla + S$ coincides with the Cartan-Schouten's $(-)$-connection and Sasaki-Hatakeyama's (φ, ξ, η)-connection. In case $\gamma_1 \neq 0$, $\gamma_2 \neq 0$ and $\gamma_1 \neq \pm\gamma_2$, all the homogeneous Riemannian structures is given by the single point set $\{A^0\}$. This coincides with the set of all the homogeneous almost contact Riemannian structures.

In case Sol_3, we retrieve the following uniqueness theorem (see also Section 8.3.5).

Corollary 16.1 ([222])**.** *The model space Sol_3 has the only homogeneous Riemannian structure A^0. The canonical connection is the Cartan-Schouten's $(-)$-connection. The homogeneous Riemannian structure A^0 is a homogeneous almost coKähler structure.*

Next we consider hyperbolic 3-space $\mathbb{H}^3(-c^2) = G(c, c)$. In this case,

$$S(X)Y = -c\{g(X, Y)\xi - \eta(Y)X\}.$$

This formula shows that S is a homogeneous Riemannian structure of type $\mathcal{T}_1$ on the hyperbolic 3-space $\mathbb{H}^3(-c^2)$ for $c \neq 0$. Note that in case $G(0, 0)$, $S = 0$ (see Proposition 16.3).

The existence of homogeneous Riemannian structure of type $\mathcal{T}_1$ characterizes hyperbolic spaces as follows:

Theorem 16.6. ([222, p. 49]) *If a Riemannian manifold (M, g) admits a homogeneous Riemannian structure $S \neq 0$ of type $\mathcal{T}_1$, then M is of constant negative curvature.*

The classifications of homogeneous Riemannian structures on $\mathbb{H}^3$, $\mathbb{E}^3$ and $\mathbb{H}^2(-4) \times \mathbb{E}^1$ will be discussed in Section 16.4, Section 16.5 and Section 18, respectively. To close this subsection we quote the following result.

Theorem 16.7 ([152, 77])**.** *Let M be a simply connected and complete homogeneous Riemannian 3-manifold.*

1. *If M admits a non trivial homogeneous Riemannian structure of type $\mathcal{T}_2$, then M is isometric to one of the following Lie groups*:
   - *The universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$ equipped with a left invariant Riemannian metric satisfying $c_1, c_3 > 0$ and $c_2 = -(c_1 + c_3)$. Note that these metrics have 3-dimensional isometry group and hence these are not the metrics of Thurston geometry.*
   - *The Minkowski motion group $\mathrm{SE}(1, 1)$ with $c_1 + c_2 = 0$. Thus the model space Sol_3 has a homogeneous Riemannian structure of type $\mathcal{T}_2$.*
   - *The universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$ equipped with a left invariant Riemannian metric satisfying $c_1 = c_2 < 0 < c_3$ or $c_2 < 0 < c_1 = c_3$.*
   - *The Heisenberg group equipped with a left invariant Riemannian metric satisfying $c_1 = c_2 = 0$ and $c_3 \neq 0$.*
   - *The special unitary group $\mathrm{SU}(2)$ equipped with a left invariant Riemannian metric satisfying $c_1 = c_2 \neq c_3$.*
2. *If M admits a homogeneous Riemannian structure of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ but not of type $\mathcal{T}_2$, then M is isometric to $G(\gamma_1, \gamma_2)$ with $\gamma_1 + \gamma_2 \neq 0$.*

It should be remarked that if a non-symmetric homogeneous Riemannian 3-space M admits a homogeneous Riemannian structure of type $\mathcal{T}_3$, then it also admits a homogeneous Riemannian structure of type $\mathcal{T}_2$ [222].

16.4. The hyperbolic 3-space

First we recall the Riemannian symmetric space representation of $\mathbb{H}^3$.

16.4.1. Riemannian symmetric space representation To investigate Ambrose-Singer connection on $\mathbb{H}^3(-c^2)$, here we recall the following model (see [59]). For simplicity we normalize the sectional curvature as -1.

Example 16.6 (Hermitian matrix model)**.** Let us denote by $\mathrm{Her}_2\mathbb{C}$ the space of all Hermitian 2 by 2 matrices. Then $\mathrm{Her}_2\mathbb{C}$ is parametrized explicitly as follows:

$$\mathrm{Her}_2\mathbb{C} = \left\{ X = \begin{pmatrix} x_0 + x_1 & x_3 - \sqrt{-1}x_2 \\ x_3 + \sqrt{-1}x_2 & x_0 - x_1 \end{pmatrix} \middle| \ x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

We denote by $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ the natural basis of $\mathrm{Her}_2\mathbb{C}$, *i.e.*,

$$\boldsymbol{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This basis is related to the quaternionic basis with

$$\boldsymbol{e}_0 = \mathbf{1} \quad \boldsymbol{e}_1 = -\sqrt{-1}\boldsymbol{k}, \quad \boldsymbol{e}_2 = \sqrt{-1}\boldsymbol{j}, \quad \boldsymbol{e}_3 = \sqrt{-1}\boldsymbol{i}.$$

One can see that $-\det X = -x_0^2 + x_1^2 + x_2^2 + x_3^2$. Let us introduce a scalar product on $\mathrm{Her}_2\mathbb{C}$ by

$$\langle X, X\rangle = -\det X.$$

Then $\mathrm{Her}_2\mathbb{C}$ is identified with Minkowski 4-space $\mathbb{E}^{1,3}$ with Lorentzian metric $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$. The hyperbolic 3-space $\mathbb{H}^3$ is realized as a hyperquadric

$$\{X \in \mathrm{Her}_2\mathbb{C} \mid \det X = 1,\ \mathrm{tr}\, X > 0\}.$$

The special linear group $\mathrm{SL}_2\mathbb{C}$ acts isometrically on $\mathrm{Her}_2\mathbb{C}$ via the action

$$\mathrm{SL}_2\mathbb{C} \times \mathrm{Her}_2\mathbb{C} \to \mathrm{Her}_2\mathbb{C};\ (g, X) \longmapsto gXg^*.$$

This action induces a Lie group homomorphism from $\mathrm{SL}_2\mathbb{C}$ onto $\mathrm{SO}^+(1,3)$ with kernel $\{\pm\mathbf{1}\} \cong \mathbb{Z}_2$. Thus we get the isomorphism $\mathrm{SL}_2\mathbb{C}/\mathbb{Z}_2 \cong \mathrm{SO}^+(1,3)$. In other words, $\mathrm{SL}_2\mathbb{C}$ is the double covering of $\mathrm{SO}^+(1,3)$.

The special linear group $\mathrm{SL}_2\mathbb{C}$ acts isometrically and transitively on $\mathbb{H}^3$ via the above action. The isotropy subgroup of this action at the identity matrix is $\mathrm{SU}(2)$. The isotropy algebra is spanned by $\{\sqrt{-1}\boldsymbol{e}_1, \sqrt{-1}\boldsymbol{e}_2, \sqrt{-1}\boldsymbol{e}_3\}$. The tangent space $\mathrm{T}_{\boldsymbol{e}_0}\mathbb{H}^3$ is identified with the Lie subspace

$$\mathfrak{m}_0 = \mathrm{Her}_2\mathbb{C} \cap \mathfrak{sl}_2\mathbb{C} = \mathbb{R}\boldsymbol{e}_1 \oplus \mathbb{R}\boldsymbol{e}_2 \oplus \mathbb{R}\boldsymbol{e}_3.$$

The representation $\mathrm{SL}_2\mathbb{C}/\mathrm{SU}(2)$ is a Riemannian symmetric space with Lie subspace $\mathfrak{m}_0$. Note that the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is represented by

$$\mathfrak{sl}_2\mathbb{C} = \mathfrak{m}_0^{\mathbb{C}} = \mathfrak{m}_0 \oplus \sqrt{-1}\mathfrak{m}_0, \quad \mathfrak{su}(2) = \sqrt{-1}\mathfrak{m}_0.$$

Remark 16.1 (Killing form). The Killing form B of $\mathfrak{sl}_2\mathbb{C}$ is given by

$$\mathsf{B}(X, Y) = 4\mathrm{tr}(XY).$$

By using the Killing form, we introduce a normalized Killing metric

$$\langle X, Y\rangle^{\mathsf{K}} = \frac{1}{2}\mathrm{tr}(XY)$$

on $\mathfrak{sl}_2\mathbb{C}$. Then the basis $\{\sqrt{-1}\boldsymbol{e}_1, \sqrt{-1}\boldsymbol{e}_2, \sqrt{-1}\boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3,\}$ is orthonormal with respect to the normalized Killing metric with signature $(-,-,-,+,+,+)$. Thus $\mathfrak{m}_0$ is spacelike. The normalized Killing metric is negative definite on $\mathfrak{su}(2)$. On the other hand, the restriction of the Lorentz metric of $\mathrm{Her}_2\mathbb{C}$ on $\mathfrak{m}_0$ is positive definite and coincides with that of the normalized Killing metric.

The Lie subspace $\mathfrak{m}_0$ is expressed as

$$\mathfrak{m}_0 = \left\{ \begin{pmatrix} t & \overline{w} \\ w & -t \end{pmatrix} \,\middle|\, t \in \mathbb{R},\ w \in \mathbb{C} \right\}. \tag{16.5}$$

Remark 16.2 (Quaternionic basis). By using the quaternionic basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of $\mathfrak{su}(2)$, $\mathfrak{m}_0$ is spanned by $\{\sqrt{-1}\boldsymbol{i}, \sqrt{-1}\boldsymbol{j}, \sqrt{-1}\boldsymbol{k}\}$ and hence it is rewritten as $\sqrt{-1}\mathfrak{su}(2)$.

16.4.2. The stereographic projection The stereographic projection from $\mathbb{H}^3 \subset \mathbb{E}^{1,3}$ onto the upper half space model

$$\mathbb{H}^3 = (\{(x, y, z) \in \mathbb{R}^3\ z > 0\}, g), \quad g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{z^2}.$$

is given by

$$(x_0, x_1, x_2, x_3) \longmapsto \frac{1}{x_0 - x_1}(x_2, x_3, 1).$$

16.4.3. The polar decomposition The Iwasawa decomposition of $\mathrm{SL}_2\mathbb{C}$ is given by

$$\mathrm{SL}_2\mathbb{C} = NA \cdot \mathrm{SU}(2),$$

where

$$N = \left\{ \begin{pmatrix} 1 & x+yi \\ 0 & 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1/\sqrt{z} \end{pmatrix} \middle| \, z > 0 \right\}.$$

The Iwasawa decomposition induces the *polar decomposition* $\mathrm{SL}_2\mathbb{C} = \mathrm{B}_2^+\mathbb{C} \cdot \mathrm{SU}(2)$ of $\mathrm{SL}_2\mathbb{C}$. The solvable part $\mathrm{B}_2^+\mathbb{C} = NA$ is parametrized as

$$\mathrm{B}_2^+\mathbb{C} = \left\{ \begin{pmatrix} \sqrt{z} & (x+yi)/\sqrt{z} \\ 0 & 1/\sqrt{z} \end{pmatrix} \middle| \, x, y \in \mathbb{R}, \, z > 0 \right\}.$$

The multiplication law is described as

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + z_1 x_2, y_1 + z_1 y_2, z_1 z_2).$$

The Maurer-Cartan form of $\mathrm{B}_2^+\mathbb{C}$ is computed as

$$\frac{dx}{z}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{dy}{z}\begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} + \frac{dz}{z}\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Thus the Lie algebra $\mathfrak{b}_2^+\mathbb{C}$ of $\mathrm{B}_2^+\mathbb{C}$ is spanned by

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}.$$

Let us introduce a left invariant metric so that this basis is orthonormal with respect to it. Then $\mathrm{B}_2^+\mathbb{C}$ is isometric to the upper half space model of $\mathbb{H}^3$. It should be remarked that inner product of $\mathfrak{b}_2\mathbb{C}$ is different from the one induced from the normalized Killing metric. Indeed, with respect to the normalized Killing metric,

$$\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} \right\rangle = 0,$$

$$\left\langle \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\rangle = \frac{1}{4}.$$

The possible dimension d of the connected isometry group G acting transitively on $\mathbb{H}^3$ are 3, 4 and 6. More precisely

1. $d = 6$.
2. $d = 4$ and $\dim(G \cap K) = 1$.
3. $d = 3$ and $\dim(G \cap K) = 0$.

Here K is a Lie group whose Lie algebra is isomorphic to $\mathfrak{su}(2)$ (see [1, p. 180]).

16.4.4. The case $d = 6$ From the reductivity $[\mathfrak{su}(2), \mathfrak{m}] \subset \mathfrak{m}$, the possible Lie subspaces are (see [1])

$$\mathfrak{m}_\lambda := (1 + \lambda\sqrt{-1})\mathfrak{m}_0, \quad \lambda \in \mathbb{R}.$$

In other words, $\mathfrak{m}_\lambda$ is spanned by $\boldsymbol{e}_1 + \lambda(\sqrt{-1}\boldsymbol{e}_1)$, $\boldsymbol{e}_2 + \lambda(\sqrt{-1}\boldsymbol{e}_2)$ and $\boldsymbol{e}_3 + \lambda(\sqrt{-1}\boldsymbol{e}_3)$. According to Abe [1], the corresponding homogeneous Riemannian structure is

$$S_3^\lambda(X)Y = -\lambda \mathrm{d}V(X, Y).$$

By using the Kenmotsu structure (φ, ξ, η, g) of $\mathbb{H}^3$, S_3^λ is expressed as

$$S_3^\lambda(X)Y = \lambda\{g(X, \varphi Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X.$$

These homogeneous Riemannian structures are of type $\mathcal{T}_3$. Thus $\mathbb{H}^3$ admits one-parameter family of naturally reductive homogeneous Riemannian structures. Thus we obtain an alternative proof for the existence of one-parameter family of naturally reductive homogeneous representation of $\mathbb{H}^3$ pointed out by Olmos and Reggiani [177, Remark 2.1].

16.4.5. The case $d = 3$ Theorem 16.2 implies that the isometry group G is a solvable Lie subgroup of $\mathrm{SL}_2\mathbb{C}$. The solvable Lie algebra $\mathfrak{g}$ satisfies $\dim\mathfrak{g} = 3$ and $\mathfrak{g}\cap\mathfrak{su}(2) = \{0\}$. The Lie algebra $\mathfrak{b}_2\mathbb{C}$ is in standard position. For any constant λ, the Lie subalgebra $\mathfrak{s}_\lambda$ of $\mathfrak{sl}_2\mathbb{C}$ spanned by

$$\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} (1+\lambda\sqrt{-1})/2 & 0 \\ 0 & -(1+\lambda\sqrt{-1})/2 \end{pmatrix}\right\}$$

is a modification of $\mathfrak{b}_2\mathbb{C}$. Note that $\mathfrak{s}_0 = \mathfrak{b}_2\mathbb{C}$. Abe [1] proved that for two modifications $\mathfrak{s}_\lambda$ and $\mathfrak{s}_\mu$, there exists an isometric Lie algebra isomorphism if and only if $\mu = \pm\lambda$. Let us parametrize $\mathfrak{s}_\lambda$ as

$$\mathfrak{s}_\lambda = \left\{\begin{pmatrix} t(1+\lambda\sqrt{-1})/2 & \zeta \\ 0 & -t(1+\lambda\sqrt{-1})/2 \end{pmatrix} \,\middle|\, t\in\mathbb{R},\ \zeta\in\mathbb{C}\right\}.$$

Then the corresponding Lie group is given by

$$\mathcal{S}_\lambda = \exp\mathfrak{s}_\lambda = \left\{\begin{pmatrix} \exp\{t(1+\sqrt{-1}\lambda)/2\} & w \\ 0 & \exp\{-t(1+\sqrt{-1}\lambda)/2\} \end{pmatrix} \,\middle|\, t\in\mathbb{R},\ w\in\mathbb{C}\right\}.$$

The exponential map $\exp : \mathfrak{s}_\lambda \to \mathcal{S}_\lambda$ is given by

$$\begin{aligned}&\exp\begin{pmatrix} t(1+\sqrt{-1}\lambda)/2 & w \\ 0 & -t(1+\sqrt{-1}\lambda)/2 \end{pmatrix}\\ &= \begin{pmatrix} \exp\{t(1+\sqrt{-1}\lambda)/2\} & w(e^{t(1+\sqrt{-1}\lambda)/2} - e^{-t(1+\sqrt{-1}\lambda)/2})/\{t(1+\sqrt{-1}\lambda)\} \\ 0 & \exp\{-t(1+\sqrt{-1}\lambda)/2\} \end{pmatrix}\end{aligned}$$

for $t \neq 0$. One can see that

$$\lim_{t\to 0}\exp\begin{pmatrix} t(1+\sqrt{-1}\lambda)/2 & \zeta \\ 0 & -t(1+\sqrt{-1}\lambda)/2 \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix}, \quad \zeta\in\mathbb{C}.$$

The hyperbolic 3-space is represented as $\mathbb{H}^3 = \mathcal{S}_\lambda/\{E_2\}$. The corresponding Ambrose-Singer connection is $\nabla + S^\lambda_{123}$, where

$$S^\lambda_{1,2,3}(X)Y = -g(X,Y)\xi + \eta(Y)X + \lambda\eta(X)\varphi Y.$$

The Ambrose-Singer connection is $\nabla + S^\lambda_{123}$ is the Cartan-Schouten's $(-)$-connection of $\mathcal{S}_\lambda$.

16.4.6. The case $d = 4$ Abe proved the following useful fact.

Proposition 16.1. *Let G be a 4-dimensional connected subgroup of $\mathrm{SL}_2\mathbb{C}$ acting transitively on $\mathbb{H}^3$, then there exists a solvable subgroup $\mathcal{S}$ of G acting transitively on $\mathbb{H}^3$ and G is the normalizer $\mathsf{N}(\mathcal{S})$ of $\mathcal{S}$.*

Now let us consider the following 4-dimensional Lie group G

$$G = \left\{\begin{pmatrix} \exp\{t(1+\sqrt{-1}\lambda)/2\} & w \\ 0 & \exp\{-t(1+\sqrt{-1}\lambda)/2\} \end{pmatrix} \,\middle|\, t\in\mathbb{R},\ w\in\mathbb{C},\ \lambda\in\mathbb{R}\right\}.$$

Let us take an element

$$\begin{pmatrix} e^{t(1+\sqrt{-1}\lambda)/2} & w \\ 0 & e^{-t(1+\sqrt{-1}\lambda)/2} \end{pmatrix}$$

of G, then one can see that

$$\begin{pmatrix} e^{t(1+\sqrt{-1}\lambda)/2} & w \\ 0 & e^{-t(1+\sqrt{-1}\lambda)/2} \end{pmatrix} = \begin{pmatrix} e^{t/2} & w(e^{t/2}e^{\sqrt{-1}\lambda t} - e^{-t/2})/(t(1+\lambda\sqrt{-1})) \\ 0 & e^{-t/2} \end{pmatrix}\begin{pmatrix} e^{\sqrt{-1}\lambda/2} & w \\ 0 & e^{-\sqrt{-1}\lambda/2} \end{pmatrix}.$$

Thus we obtain a decomposition $G = \mathrm{B}^2_+\mathbb{C}\cdot\mathrm{U}(1)$. From the Iwasawa decomposition $\mathrm{SL}_2\mathbb{C} = NA\,\mathrm{SU}(2)$, we know that $NA = \mathrm{B}^+_2\mathbb{C}$, thus we have $NA\,\mathrm{U}(1)$.

Abe proved that

$$G = \mathsf{N}(\mathsf{A}\,\mathrm{B}^2_+\mathbb{C}\,\mathsf{A}^{-1}), \quad \mathsf{A}\in\mathrm{SL}_2\mathbb{C}.$$

Compare with [41, p. 567]. Thus we obtain a homogeneous space representation $\mathbb{H}^3 = (\mathrm{B}^2_+\mathbb{C}\cdot\mathrm{U}(1))/\mathrm{U}(1) = (NA\,\mathrm{U}(1))/\mathrm{U}(1)$. Abe proved that the homogeneous Riemannian structure corresponding to $\mathbb{H}^3 = (\mathrm{B}^2_+\mathbb{C}\cdot\mathrm{U}(1))/\mathrm{U}(1)$ is S^λ_{123}.

Proposition 16.2. *Up to isomorphisms, the hyperbolic 3-space $\mathbb{H}^3$ admits two types of homogeneous Riemannian structures up to isomorphims:*

1. *The homogeneous Riemannian structures S_3^λ of type $\mathcal{T}_3$ given by*

$$S_3^\lambda(X)Y = -\lambda \mathrm{d}V(X,Y) = \lambda\{g(X,\varphi Y)\xi - \eta(X)\varphi Y + \eta(Y)\varphi X\}, \quad \lambda \geq 0.$$

The corresponding coset space representation of $\mathbb{H}^3$ is $\mathrm{SL}_2\mathbb{C}/\mathrm{SU}(2)$ with Lie subspace

$$\mathfrak{m}_\lambda = \left\{ (1+\lambda\sqrt{-1}) \begin{pmatrix} t & \bar{w} \\ w & -t \end{pmatrix} \middle|\ t \in \mathbb{R},\ w \in \mathbb{C} \right\}.$$

In particular, $S_3^0 = 0$ defines a Riemannian symmetric space $\mathrm{SL}_2\mathbb{C}/\mathrm{SU}(2)$ with Lie subspace $\mathfrak{m}_0 = \mathfrak{sl}_2\mathbb{C} \cap \mathrm{Her}_2\mathbb{C}$ (see Remark 16.6 below).

2. *$S_{1,2,3}^\lambda(X)Y = -g(X,Y)\xi + \eta(Y)X + \lambda\eta(X)\varphi Y$, $\lambda \geq 0$. The corresponding coset space representations of $\mathbb{H}^3$ are $\mathcal{S}_\lambda/\{E_2\}$ or $(\mathrm{B}_+^2\mathbb{C}\cdot\mathrm{U}(1))/\mathrm{U}(1)$. This homogeneous Riemannian structure is of type $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$. The homogeneous Riemannian structure $S_{1,2,3}^\lambda$ is of type $\mathcal{T}_1$ if and only if $\lambda = 0$. Note that the the Ambrose-Singer connection $\nabla + S_{1,2,3}^\lambda$ coincides with the Cartan-Schouten's $(-)$-connection of $\mathfrak{s}_\lambda$.*

Proof. Let us consider the homogeneous structure $S^\lambda(X)Y = -g(X,Y)\xi + \eta(Y)X + \lambda\eta(X)\varphi Y$. Then we have

$$c_{12}(S_\flat^\lambda)(Z) = -\eta(Z),$$

$$\underset{X,Y,Z}{\mathfrak{S}} S_\flat^\lambda(X,Y,Z) = \lambda\left\{\eta(X)g(\varphi Y,Z) + \eta(Y)g(\varphi Z,X) + \eta(Z)g(\varphi X,Y)\right\},$$

and

$$S_\flat^\lambda(X,Y,Z) + S_\flat^\lambda(Y,X,Z) = \lambda\{\eta(X)g(\varphi Y,Z) + \eta(Y)g(\varphi X,Z)\}.$$

Hence for $\lambda \neq 0$, $S^\lambda \notin \mathcal{T}_2 \oplus \mathcal{T}_3$, $S^\lambda \notin \mathcal{T}_1 \oplus \mathcal{T}_2$ and $S^\lambda \notin \mathcal{T}_1 \oplus \mathcal{T}_3$. □

Remark 16.3. Every $X = (x_{ij}) \in \mathfrak{sl}_2\mathbb{C}$ is decomposed as $X = X_{\mathfrak{s}_\lambda} + X_{\mathfrak{su}(2)}$ where

$$X_{\mathfrak{s}_\lambda} = \begin{pmatrix} (1+\sqrt{-1}\lambda)\,\mathrm{Re}\,x_{11} & x_{12} + \overline{x_{21}} \\ 0 & -(1+\sqrt{-1}\lambda)\,\mathrm{Re}\,x_{11} \end{pmatrix},$$
$$X_{\mathfrak{su}(2)} = \begin{pmatrix} \sqrt{-1}(\mathrm{Im}\,x_{11} - \lambda\,\mathrm{Re}\,x_{11}) & -\overline{x_{21}} \\ x_{21} & -\sqrt{-1}(\mathrm{Im}\,x_{11} - \lambda\,\mathrm{Re}\,x_{11}) \end{pmatrix}$$

along the splitting $\mathfrak{sl}_2\mathbb{C} = \mathfrak{s}_\lambda + \mathfrak{su}(2)$.

Remark 16.4. Hassani and Ahmadi [89, Corollary 3.1] stated that connected Lie groups acting transitively and isometrically on $\mathbb{H}^3$ are (locally) isomorphic to one of the following Lie groups:

$$\mathrm{PSL}_2\mathbb{C}, \quad \mathrm{B}_2^+\mathbb{C}, \quad \mathcal{S}_\lambda.$$

Let us investigate the almost contact homogeneity. Since $\mathbb{H}^3$ is a Kenmotsu 3-manifold, we have

$$(\nabla_X\varphi)Y = -\eta(Y)\varphi X - g(X,\varphi Y)\xi.$$

Next we have

$$S_3^\lambda(X)\varphi Y - \varphi S_3^\lambda(X)Y = -\lambda\{g(X,Y)\xi - \eta(Y)X\}.$$

Thus the covariant derivative $\tilde{\nabla}\varphi$ of φ with respect to $\tilde{\nabla} = \nabla + S_3^\lambda$ is given by

$$(\tilde{\nabla}_X\varphi)Y = -\eta(Y)\varphi X - g(X,\varphi Y)\xi - \lambda\{g(X,Y)\xi - \eta(Y)X\}.$$

Hence S_3^λ is *not* a homogeneous almost contact Riemannian structure. This conclusion is consistent with the fact $\dim \mathrm{Aut}(\mathbb{H}^3) = 4$. Indeed, the homogeneous structure S_3^λ corresponds to $\mathrm{SL}_2\mathbb{C}/\mathrm{SU}(2) = \mathrm{SO}^+(3,1)/\mathrm{SO}(3)$. On the other hand for $S_{1,2,3}^\lambda$, φ is parallel with respect to the connection $\nabla + S_{1,2,3}^\lambda$ for all λ.

Remark 16.5. Rastrepina and Surina [195] considered a linear connection $\tilde{\nabla} = \nabla + S$ on $\mathbb{H}^3$ with

$$S(X)Y = -g(X,Y)\xi + \eta(Y)\xi + g(ae_1 + be_2 + \lambda\xi, X)\varphi Y,$$

where a, b and λ are constants ([195, Theorem 2]). They showed that $\tilde{\nabla}$ satisfies

$$\tilde{\nabla}\varphi = 0, \quad \tilde{\nabla}\xi = 0, \quad \tilde{\nabla}\eta = 0, \quad \tilde{\nabla}g = 0.$$

In case $a = b = 0$, the tensor field S coincides with $S^{\lambda}_{1,2,3}$. It should be remarked that this tensor field S is a homogeneous Riemannian structure if and only if $a = b = 0$. Moreover they studied S^{λ}_3 from the viewpoint of linear connections with skew symmetric torsion (see [195, Remark 3, Theorem 3]).

Remark 16.6 (Flow symmetry). Boeckx, Bueken and Vanhecke [25] introduced the notion of flow-symmetry for Riemannian manifolds. According to [25], a Riemannian manifold (M, g) is said to be *locally flow-symmetric* if there exists a non-vanishing vector field ξ such that the local reflections around the integral curves are local isometries. Under this assumption by virtue of a theorem due to Chen and Vanhecke [43], the integral curves are geodesics. If the vector field ξ of a locally flow symmetric Riemannian manifold M is a unit vector field, then ξ is an eigenvector field of the Ricci operator. The corresponding eigenvalue λ satisfies $\mathrm{d}\lambda = 0$ on the distribution defined by $g(\xi, \cdot) = 0$. Obviously locally φ-symmetric spaces are locally flow-symmetric spaces such that ξ is the Reeb vector field. Boeckx, Bueken and Vanhecke proved that a warped product $\mathbb{E}^1 \times_f \overline{M}$ with $\xi = \partial/\partial_t$ is locally flow-symmetric with respect to ξ if and only if $\overline{M}$ is locally symmetric. Now we recall Example 5.5. Let $\overline{M}$ be a Hermitian symmetric space and equip a Kenmotsu structure (φ, ξ, η, g) on the warped product $M = \mathbb{E}^1 \times_{ce^t} \overline{M}$. Then M is locally flow-symmetric with respect to ξ. The hyperbolic $(2n+1)$-space $\mathbb{H}^{2n+1}$ is a typical example of locally flow-symmetric Kenmotsu manifold.

16.5. *Euclidean* 3*-space*

The possible dimension d of the connected isometry group G of $\mathbb{E}^3 = (\mathbb{R}^3(x, y, z), \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$ are 3, 4 or 6. More precisely there are three possibilities ([1, p. 186]):

1. $d = 6$.
2. $d = 4$ and $\dim(G \cap \mathrm{SO}(3)) = 1$.
3. $d = 3$ and $\dim(G \cap \mathrm{SO}(3)) = 0$.

16.5.1. The case $d = 6$ We start our investigation with the case $d = 6$.

As we saw in Example 4.8, the Cartesian 3-space $\mathbb{R}^3$ is isomorphic to $\mathfrak{so}(3)$ as a Lie algebra via the isomorphism $\iota : \mathbb{R}^3 \to \mathfrak{so}(3)$ given by (4.7). The Euclidean motion group $\mathrm{SE}(3) \ltimes \mathbb{R}^3$ has the Lie algebra

$$\mathfrak{se}(3) = \left\{ \left(\begin{array}{cccc} 0 & -w & v & x \\ w & 0 & -u & y \\ -v & u & 0 & z \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \; x, y, z, u, v, w \in \mathbb{R} \right\}.$$

The isotropy subgroup of $\mathrm{SE}(3)$ at the origin $o = (0, 0, 0)$ is isomorphic to the rotation group $\mathrm{SO}(3)$. The isotropy algebra $\mathfrak{h}$ is

$$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} 0 & -w & v & 0 \\ w & 0 & -u & 0 \\ -v & u & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \; u, v, w \in \mathbb{R} \right\} \cong \mathfrak{so}(3).$$

Take a basis $\{E_1, E_2, E_3, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of $\mathfrak{se}(3)$ by

$$E_1 = E_{32} - E_{23}, \quad E_2 = E_{13} - E_{31}, \quad E_3 = E_{21} - E_{12}, \quad \boldsymbol{e}_1 = E_{14}, \quad \boldsymbol{e}_2 = E_{24}, \quad \boldsymbol{e}_3 = E_{34}.$$

Then we have the commutation relations:

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2, \quad [E_i, \boldsymbol{e}_i] = 0, \quad [\boldsymbol{e}_i, \boldsymbol{e}_j] = 0,$$

$$[E_1, \boldsymbol{e}_2] = \boldsymbol{e}_3, \quad [E_1, \boldsymbol{e}_3] = -\boldsymbol{e}_2,$$

$$[E_2, \boldsymbol{e}_3] = \boldsymbol{e}_1, \quad [E_2, \boldsymbol{e}_1] = -\boldsymbol{e}_3,$$

$$[E_3, \boldsymbol{e}_1] = \boldsymbol{e}_2, \quad [E_3, \boldsymbol{e}_2] = -\boldsymbol{e}_1.$$

We identify $\mathbb{R}^3$ with $\mathbb{R}\boldsymbol{e}_1 \oplus \mathbb{R}\boldsymbol{e}_2 \oplus \mathbb{R}\boldsymbol{e}_3$ through the linear isomorphism $(x_1, x_2, x_3) \longmapsto x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$. The Lie algebra isomorphism $\iota : (\mathbb{R}^3, \times) \to \mathfrak{so}(3)$ is extended to a Lie algebra homomorphism $(x_1, x_2, x_3) \longmapsto x_1E_1 + x_2E_2 + x_3E_3$. Thus $\mathfrak{se}(3)$ is the direct sum $\mathbb{R}^3 \oplus \mathbb{R}^3$ (as a linear space) equipped with the above Lie bracket. More precisely, $\mathfrak{se}(3)$ is the direct sum of $(\mathbb{R}^3, \times)$ and abelian Lie algebra $\mathbb{R}^3$ as a Lie algebra. Here we compute the Lie algebra $\mathfrak{i}(\mathbb{E}^3)$ of Killing vector fields. Direct computation show that

$$\exp(tE_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \exp(tE_2) = \begin{pmatrix} \cos t & 0 & \sin t & 0 \\ 0 & 1 & 0 & 0 \\ -\sin t & 0 & \cos t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp(tE_3) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we get

$$E_1^\sharp = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad E_2^\sharp = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad E_3^\sharp = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

$$\boldsymbol{e}_1^\sharp = \frac{\partial}{\partial x}, \quad \boldsymbol{e}_2^\sharp = \frac{\partial}{\partial y}, \quad \boldsymbol{e}_3^\sharp = \frac{\partial}{\partial z}.$$

Introduce an inner product on $\mathfrak{se}(3) \oplus \mathbb{R}^3$ so that $\{E_1, E_2, E_3, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is orthonormal with respect to it. Then the orthogonal complement $\mathfrak{m} := \mathfrak{h}^\perp$ is $\mathfrak{m} = \mathbb{R}E_1 \oplus \mathbb{R}E_2 \oplus \mathbb{R}E_3$. The tangent space $\mathrm{T}_o\mathbb{E}^3$ is identified with $\mathfrak{m}$. Then one can see that $\mathrm{SE}(3)/\mathrm{SO}(3)$ with Lie subspace $\mathfrak{m}$ is a Riemannian symmetric space.

Let us look for Lie subspaces complementary to $\mathfrak{h}$. By the reductivity $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, the possible Lie subspaces are

$$\mathfrak{m}_3^\lambda = \mathbb{R}(\boldsymbol{e}_1 + \lambda E_1) \oplus \mathbb{R}(\boldsymbol{e}_2 + \lambda E_2) \oplus \mathbb{R}(\boldsymbol{e}_2 + \lambda E_3)$$

for some $\lambda \in \mathbb{R}$. Note that $\mathfrak{m}_3^0 = \mathfrak{m}$. The corresponding homogeneous Riemannian structure is

$$S_3^\lambda(X)Y = -\lambda\, \mathrm{d}V(X, Y).$$

One can see that $S_3^{\lambda_1}$ is isomorphic to $S_3^{\lambda_2}$ if and only if $\lambda_2 = \pm\lambda_1$. Thus we retrieve Theorem 4.7 in case of $\varepsilon = 0$. Since $\dim \mathrm{Aut}(\mathbb{E}^3) = 4$, the representation $\mathbb{E}^3 = \mathrm{SE}(3)/\mathrm{SO}(3)$ is not homogeneous coKähler.

16.5.2. The case $d = 3$ Next we consider the case $d = 3$. Theorem 16.2 implies that the isometry group G is a solvable Lie subgroup of $\mathrm{SE}(3)$. The solvable Lie algebra $\mathfrak{g}$ satisfies $\dim \mathfrak{g} = 3$ and $\mathfrak{g} \cap \mathfrak{so}(3) = \{0\}$. According to Abe [1], $\mathfrak{g}$ is isomorphic to

$$\mathfrak{g}_{2,3}^\lambda := \mathbb{R}\boldsymbol{e}_1 \oplus \mathbb{R}\boldsymbol{e}_2 \oplus \mathbb{R}(\boldsymbol{e}_3 + \lambda E_3) = \left\{ \begin{pmatrix} 0 & -\lambda z & 0 & x \\ \lambda z & 0 & 0 & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle|\; x, y, z \in \mathbb{R} \right\}$$

for some $\lambda \in \mathbb{R}$. The corresponding simply connected Lie group is

$$G_{2,3}^\lambda = \left\{ \begin{pmatrix} \cos(\lambda z) & -\sin(\lambda z) & 0 & x \\ \sin(\lambda z) & \cos(\lambda z) & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle|\; x, y, z \in \mathbb{R} \right\}.$$

The Lie group $G_{2,3}^\lambda$ is solvable. In particular, $G_{2,3}^0$ is abelian. Thus $\mathbb{E}^3$ is understood as $\mathbb{R}^3$ with multiplication

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2\cos(\lambda z_1) - y_2\sin(\lambda z_1), y_1 + x_2\sin(\lambda z_1) + y_2\cos(\lambda z_1), z_1 + z_2).$$

The Lie group $G^{\lambda}_{2,3}$ acts on $G^{\lambda}_{2,3}$ as left translations. The isotropy subgroup at $(x,y,z)=(0,0,0)$ is $\{E_4\}$. Thus we obtain the coset space representation $\mathbb{E}^3 = G^{\lambda}_{2,3}/\{E_4\}$. The tangent space at $o=E_4$ is the Lie algebra of $G^{\lambda}_{2,3}$. One can see that the coKähler structure of $\mathbb{E}^3$ is a left invariant coKähler structure on $G^{\lambda}_{2,3}$.

Indeed, take a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{g}^{\lambda}_{2,3}$, then we get

$$[e_1,e_2]=0, \quad [e_2,e_3]=\lambda\, e_1, \quad [e_3,e_1]=\lambda\, e_2.$$

Thus in case $\lambda \neq 0$, $\mathfrak{g}^{\lambda}_{2,3}$ is isomorphic to $\mathfrak{se}(2)$ as a Lie algebra. Thus $G^{\lambda}_{2,3}$ is isomorphic to $\widetilde{\mathrm{SE}}(2)$ when $\lambda \neq 0$. The left invariant metric on $G^{\lambda}_{2,3}$ determined by the condition that the basis $\{e_1,e_2,e_3\}$ is orthonormal is $\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$. Let us introduce a left invariant endomorphism field φ on $\mathfrak{g}^{\lambda}_{2,3}$ by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0$$

and set $\xi = e_3$ and $\eta = g(\xi,\cdot)$. Then (φ,ξ,η,g) is a flat left invariant coKähler structure on $G^{\lambda}_{2,3}$. One can see that

$$\varphi\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad \varphi\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}, \quad \varphi\frac{\partial}{\partial z} = 0.$$

Hence $G^{\lambda}_{2,3}$ is isomorphic to $\mathbb{E}^2\times\mathbb{E}^1$ as a coKähler manifold (*not* isomomorphic as a Lie group).

By using the coKähler structure (φ,ξ,η,g) of $\mathbb{E}^3=\mathbb{E}^2\times\mathbb{E}^1=G^{\lambda}_{2,3}$, the corresponding homogeneous Riemannian structure discovered by Abe [1] is described as

$$S^{\lambda}_{2,3}(X)Y = -\lambda\,\eta(X)\varphi Y.$$

On the coKähler $\mathbb{E}^3=\mathbb{E}^2\times\mathbb{E}^1$, the almost contact connections are given by

$$\nabla^r_X Y = \nabla_X Y - r\eta(X)\varphi Y.$$

Thus we get $S^{\lambda}_{2,3}(X)Y = A^{\lambda}$ and hence S^{λ} is a homogeneous almost contact Riemannian structure. Obviously $S^{\lambda_1}_{2,3}$ is isomorphic to $S^{\lambda_2}_{2,3}$ if and only if $\lambda_2=\pm\lambda_1$. Note that $S^0_{2,3}$ corresponds to the abelian Lie group $\mathbb{E}^3=\mathbb{E}^3/\{\mathbf{0}\}$. Moreover $\nabla + S^{\lambda}_{2,3}$ is the $(-)$-connection of $G^{\lambda}_{2,3}$. The non-trivial connection coefficients of the $(-)$-connection of $G^{\lambda}_{2,3}$ are given by the following formulas:

$$\nabla^{(-)}_{\partial_z}\partial_x = -\lambda\,\partial_y, \quad \nabla^{(-)}_{\partial_z}\partial_y = \lambda\,\partial_x. \tag{16.6}$$

Comparing (16.6) with (8.8) we notice that the $(-)$-connection of $\widetilde{\mathrm{SE}}(2)$ and that of $G^1_{2,3}$ coincide.

16.5.3. The case $d=4$ Here we observe the product manifold $\mathbb{E}^2\times\mathbb{E}^1$. More precisely we consider the Riemannian product of Euclidean plane $\mathbb{E}^2(x,y)=\mathrm{SE}(2)/\mathrm{SO}(2)$ and the Euclidean line $\mathbb{E}^1(z)=(\mathbb{R}(z),\mathrm{d}z^2)$. Here we regard $\mathbb{E}^2(x,y)$ as a Hermitian symmetric space and $\mathbb{E}^2(x,y)\times\mathbb{E}^1(z)$ as a coKähler manifold.

We identify it with the Lie group

$$\mathbb{E}^2\times\mathbb{E}^1 = \left\{ \begin{pmatrix} x \\ y \\ 1 \\ e^z \end{pmatrix} \,\middle|\, x,y,z\in\mathbb{R} \right\}.$$

Then the product Lie group

$$\mathrm{SE}(2)\times\mathbb{R} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & u & 0 \\ \sin\theta & \cos\theta & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix} \,\middle|\, u,v,t\in\mathbb{R},\ 0\le\theta<2\pi \right\}$$

acts isometrically and transitively on $\mathbb{E}^2\times\mathbb{E}^1$ via the usual matrix multiplication action. More explicitly the action is described as

$$(e^{i\theta},(u,v),t)\cdot(x,y,z)=(x\cos\theta-y\sin\theta+u,x\sin\theta+y\cos\theta+v,z+t).$$

The Lie algebra $\mathfrak{se}(2)\oplus\mathbb{R}$ is

$$\left\{\left(\begin{array}{cccc}0&-w&u&0\\ w&0&v&0\\ 0&0&0&0\\ 0&0&0&t\end{array}\right)\,\middle|\,u,v,w,t\in\mathbb{R}\right\}.$$

Thus we can take a basis

$$E_3=E_{21}-E_{12},\quad \bar{\boldsymbol{e}}_1:=E_{13},\quad \bar{\boldsymbol{e}}_2:=E_{23},\quad \bar{\boldsymbol{e}}_3:=E_{44}.$$

The non-trivial commutation relations are

$$[E_3,\bar{\boldsymbol{e}}_1]=\bar{\boldsymbol{e}}_2,\quad [E_3,\bar{\boldsymbol{e}}_2]=-\bar{\boldsymbol{e}}_1.$$

The isotropy subgroup at the origin $o=((0,0),0)\in\mathbb{E}^2\times\mathbb{E}$ is

$$\left\{\left(\begin{array}{cccc}\cos\theta&-\sin\theta&0&0\\ \sin\theta&\cos\theta&0&0\\ 0&0&1&0\\ 0&0&0&1\end{array}\right)\,\middle|\,0\leq\theta<2\pi\right\}\cong\mathrm{SO}(2).$$

The isotropy algebra is $\mathfrak{h}=\mathbb{R}E_3\cong\mathfrak{so}(2)$. We introduce an inner product $\langle\cdot,\cdot\rangle$ so that $\{E_3,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is orthonormal. Then the orthogonal complement of $\mathfrak{h}$ is $\mathfrak{m}:=\mathbb{R}\bar{\boldsymbol{e}}_1\oplus\mathbb{R}\bar{\boldsymbol{e}}_2\oplus\mathbb{R}\bar{\boldsymbol{e}}_3$. Then $\mathfrak{se}(3)\oplus\mathbb{R}=\mathfrak{h}\oplus\mathfrak{m}$ is reductive. Abe [1] proved that other Lie subspaces are isomorphic to

$$\mathfrak{m}^\lambda=\mathbb{R}\bar{\boldsymbol{e}}_1\oplus\mathbb{R}\bar{\boldsymbol{e}}_2\oplus\mathbb{R}(\bar{\boldsymbol{e}}_3+\lambda E_3)=\left\{\left(\begin{array}{cccc}0&-w\lambda&u&0\\ w\lambda&0&v&0\\ 0&0&0&0\\ 0&0&0&w\end{array}\right)\,\middle|\,u,v,w\in\mathbb{R}\right\}.$$

One can see that $\mathfrak{m}^\lambda$ is a Lie subalgebra and $\mathfrak{m}^0=\mathfrak{m}$. The corresponding simply connected Lie group is

$$G_3^\lambda=\left\{\left(\begin{array}{cccc}\cos(\lambda z)&-\sin(\lambda z)&x&0\\ \sin(\lambda z)&\cos(\lambda z)&y&0\\ 0&0&1&0\\ 0&0&0&e^z\end{array}\right)\,\middle|\,x,y,z\in\mathbb{R}\right\}.$$

Obviously G_3^λ is isomorphic to $G_{2,3}^\lambda$ and hence this Lie groups is isomorphic to $\widetilde{\mathrm{SE}}(2)$ if $\lambda\neq0$. Thus the homogeneous Riemannian space $(\mathrm{SE}(2)\times\mathbb{R})/\mathrm{SO}(2)$ is isometric to $G_{2,3}^\lambda=G_{2,3}^\lambda/\{E_4\}$.

Now we obtain the following classification which is a reformulation of [1]:

Proposition 16.3. *Up to isomorphisms, the Euclidean 3-space $\mathbb{E}^3(x,y,z)$ admits two types of homogeneous Riemannian structures up to isomorphisms*:

1. *The homogeneous Riemannian structures S^λ of type $\mathcal{T}_3$*

$$S_3^\lambda(X)Y=-\lambda\mathrm{d}V(X,Y),\quad \lambda\geq0.$$

The corresponding coset space representation of $\mathbb{E}^3$ is $\mathrm{SE}(3)/\mathrm{SO}(3)$ with Lie subspace

$$\mathfrak{m}_3^\lambda=\left\{\left(\begin{array}{cccc}0&-\lambda z&\lambda y&x\\ \lambda z&0&-\lambda x&y\\ -\lambda y&\lambda x&0&z\\ 0&0&0&0\end{array}\right)\,\middle|\,x,y,z\in\mathbb{R}\right\}.$$

In particular, $S_3^0=0$ defines a Riemannian symmetric space $\mathrm{SE}(3)/\mathrm{SO}(3)$ with Lie subspace $\mathfrak{m}_3^0$. The homogeneous Riemannian structures S_3^λ are not homogeneous almost contact Riemannian structures.

2. *The homogeneous Riemannian structure $S^{\lambda}_{2,3}(X)Y = A^{\lambda}_X Y = -\lambda\eta(X)\varphi Y$ of type $\mathcal{T}_2 \oplus \mathcal{T}_3$ with $\lambda \geq 0$. These homogeneous Riemannian structures are homogeneous almost contact Riemannian structures. The coset space representations of $\mathbb{E}^3$ corresponding to each $S^{\lambda}_{2,3}$ is*

$$G^{\lambda}_{2,3}/\{E_4\},$$

where $G^{\lambda}_{2,3}$ is a solvable Lie group

$$G^{\lambda}_{2,3} = \left\{ \left. \begin{pmatrix} \cos(\lambda z) & -\sin(\lambda z) & 0 & x \\ \sin(\lambda z) & \cos(\lambda z) & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}.$$

In case $\lambda > 0$, $G^{\lambda}_{2,3}$ is isomorphic to $\widetilde{\mathrm{SE}}(2)$. In particular $G^{1}_{2,3}$ is isometric to $\widetilde{\mathrm{SE}}(2)$. Hence we have the coset space representation:

$$(\mathrm{SE}(2) \times \mathbb{R})/\mathrm{SO}(2) = G^{\lambda}_{2,3}/\{E_4\}.$$

Moreover, the Ambrose-Singer connection $\nabla + S^{\lambda}_{2,3}$ coincides with the $(-)$-connection of $G^{\lambda}_{2,3}$ when $\lambda > 0$.

Up to isomorphisms, all the homogeneous almost coKähler structures is parametrized as $\{S^{\lambda}_{2,3}\}_{\lambda \geq 0} = \{A^{\lambda}\}_{\lambda \geq 0}$.

Proof. The homogeneous structure $S^{\lambda}(X)Y = -\lambda\eta(X)\varphi Y$ satisfies $c_{12}(S^{\lambda}_{\flat})(Z) = 0$ and

$$\underset{X,Y,Z}{\mathfrak{S}} S^{\lambda}_{\flat}(X, Y, Z) = -\lambda\{\eta(X)g(\varphi Y, Z) + \eta(Y)g(\varphi Z, X) + \eta(Z)g(\varphi X, Y)\}.$$

Hence for $\lambda \neq 0$, $S^{\lambda} \in \mathcal{T}_2 \oplus \mathcal{T}_3$ but neither $S^{\lambda} \in \mathcal{T}_2$ nor $S^{\lambda} \in \mathcal{T}_3$.

Analogous to $\mathbb{H}^3$-case, S^{λ}_3 are *not* homogeneous almost contact Riemannian structures. On the other hand all $S^{\lambda}_{1,2,3}$ are so. $\square$

The trivial homogeneous Riemannian structure $S = 0$ corresponds to both the Riemannian symmetric space $\mathrm{SE}(3)/\mathrm{SO}(3)$ and the abelian Lie group $G^{0}_{2,3} = G^{0}_{2,3}/\{E_4\}$.

17. The product space $\mathbb{S}^2 \times \mathbb{E}^1$

The model space $\mathbb{S}^2(c^2) \times \mathbb{E}^1$ is a coKähler space form. In this section we study homogeneous coKähler structures of $\mathbb{S}^2(c^2) \times \mathbb{E}^1$. One can see that it suffices to study $\mathbb{S}^2 \times \mathbb{E}^1$. As we saw in Example 4.8, $\mathbb{S}^2$ has only trivial homogeneous Riemannian structure, we may restrict our attention to Hermitian symmetric space $\mathbb{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$. The connected component of the isometry group of $\mathbb{S}^2 \times \mathbb{E}^1$ is $\mathrm{SO}(3) \times \mathbb{R}$.

Here we point out the following fundamental fact. (see [222, §8.D]):

Proposition 17.1. *Let (M_1, g_1, S_1) and (M_2, g_2, S_2) be Riemannian manifolds equipped with homogeneous Riemannian structures. Then $S_1 + S_2$ gives a homogeneous Riemannian structure on a Riemannian product $(M_1 \times M_2, g_1 + g_2)$. The homogeneous Riemannian structure $S_1 + S_2$ is called the direct sum of S_1 and S_2.*

In case $M_1 = \mathbb{S}^2$ and $M_2 = \mathbb{E}^1$, both homogeneous Riemannian strutures are trivial. Thus the direct sum $S_1 + S_2$ is also trivial.

Let us realize the Riemannian product $\mathbb{S}^2 \times \mathbb{E}^1$ as a hyperquadric

$$\mathbb{S}^2 \times \mathbb{E}^1 = \{(\boldsymbol{x}, t) = (x_1, x_2, x_3, t) \in \mathbb{E}^4 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

of the Euclidean 4-space $\mathbb{E}^4$. Then $\mathrm{Iso}_{\circ}(\mathbb{S}^2 \times \mathbb{E}^1) = \mathrm{SO}(3) \times \mathbb{R}$ acts isometrically and transitively on $\mathbb{S}^2 \times \mathbb{E}^1$. To view this isometric action, we identify $\mathbb{S}^2 \times \mathbb{E}^1$ with

$$\left\{ \left. \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ e^t \end{pmatrix} \right| x_1, x_2, x_3, t \in \mathbb{R} \right\}.$$

Then $\mathrm{Iso}_\circ(\mathbb{S}^2\times\mathbb{E}^1)$ is identified with

$$\left\{(A,s)=\begin{pmatrix} A & 0\\ 0 & e^s\end{pmatrix}\;\middle|\; A\in \mathrm{SO}(3),\ s\in\mathbb{R}\right\}$$

acts isometrically and transitively on $\mathbb{S}^2\times\mathbb{E}^1$ via the usual matrix multiplication action. The action is simply described as

$$(A,s)\cdot(\boldsymbol{x},t)=(A\boldsymbol{x},t+s).$$

Since there are no 2-dimensional closed subgroups of $\mathrm{SO}(3)$, the only connected Lie subgroup acting transitively on $\mathbb{S}^2\times\mathbb{E}^1$ is $\mathrm{SO}(3)\times\mathbb{R}$. The Lie algebra of $\mathrm{SO}(3)\times\mathbb{R}$ is given by

$$\left\{\begin{pmatrix} 0 & -w & v & 0\\ w & 0 & -u & 0\\ -v & u & 0 & 0\\ 0 & 0 & 0 & s\end{pmatrix}\;\middle|\; u,v,w,s\in\mathbb{R}\right\}$$

With respect to the inner product induced from the direct product metric, we can take an orthonormal basis (*cf*. Example 4.8):

$$e_0=E_{21}-E_{12},\quad e_1=E_{32}-E_{23},\quad e_2=E_{13}-E_{31},\quad e_3=E_{44}.$$

The non-trivial commutation relations are

$$[e_0,e_1]=e_2,\quad [e_1,e_2]=e_0,\quad [e_2,e_0]=e_1.$$

The isotropy subgroup at the origin $(x_1,x_2,x_3,t)=(0,0,1,0)$ is $\mathrm{SO}(2)$ which is identified with

$$\left\{\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\;\middle|\; 0\leq\theta<2\pi\right\}.$$

The isotropy algebra is $\mathfrak{h}=\mathbb{R}e_0$.

The tangent space $\mathrm{T}_o(\mathbb{S}^2\times\mathbb{E}^1)$ of $\mathbb{S}^2$ at the origin o is identified with the Lie subspace $\mathfrak{m}=\mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}e_3$. The coKähler structure is determined by

$$\varphi e_1=e_2,\quad \varphi e_2=-e_1,\quad \xi=e_3,\quad \eta=g(\xi,\cdot).$$

Note that $\mathbb{S}^2\times\mathbb{E}^1$ is a coKähler space form and $\dim\mathrm{Aut}(\mathbb{S}^2\times\mathbb{E}^1)=4$. In particular the identity component of $\mathrm{Aut}(\mathbb{S}^2\times\mathbb{E}^1)=4$ is $\mathrm{SO}(3)\times\mathbb{R}$. One can see that

$$\mathfrak{m}=\mathfrak{h}^\perp,\quad [\mathfrak{h},\mathfrak{h}]=\{0\},\quad [\mathfrak{h},\mathfrak{m}]=\mathfrak{m},\quad [\mathfrak{m},\mathfrak{m}]=\mathfrak{h}.$$

Thus $(\mathrm{SO}(3)\times\mathbb{R})/\mathrm{SO}(2)$ is a Riemannian symmetric space. The other Lie subspaces are isomorphic to $\mathfrak{m}_\lambda=\mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}(e_3+\lambda e_0)$ for some $\lambda\in\mathbb{R}$.

The following classification is a reformulation of [174].

Theorem 17.1 ([174])**.** *Up to isomorphisms, all the homogeneous Riemannian structures on $\mathbb{S}^2\times\mathbb{E}^1$ are given by*

$$S^\lambda(X)Y=A^{\lambda/2}_XY=-\frac{\lambda}{2}\eta(X)\varphi Y,\quad \lambda\geq 0.$$

The corresponding Lie subspace $\mathfrak{m}_\lambda$ is

$$\mathfrak{m}_\lambda=\left\{\begin{pmatrix} 0 & \lambda w & v & 0\\ -\lambda w & 0 & -u & 0\\ -v & u & 0 & 0\\ 0 & 0 & 0 & w\end{pmatrix}\;\middle|\; u,v,w\in\mathbb{R}\right\}.$$

The corresponding coset space representations are $(\mathrm{SO}(3)\times\mathbb{R})/\mathrm{SO}(2)$ for any λ. Every S^λ is a homogeneous coKähler structure. The homogeneous Riemannian structure is of type $\mathcal{T}_2\oplus\mathcal{T}_3$. In particular, S^λ is of type $\mathcal{T}_3$ if and only if $\lambda=0$. In this case $\mathfrak{m}_0=\mathfrak{m}$. For $\lambda\neq 0$, S^λ is not of type $\mathcal{T}_2$.

The set of all homogeneous coKähler structures is identified with $\{A^r\}_{r\geq 0}$.

Remark 17.1. The reductive decomposition $\mathfrak{g} \oplus \mathbb{R} = \mathfrak{h} + \mathfrak{m}_\lambda$ satisfies $[\mathfrak{m}_\lambda, \mathfrak{m}_\lambda] \subset \mathfrak{h}$ when and only when $\lambda = 0$.

Finally we investigate the projection $\pi : \mathbb{S}^2 \times \mathbb{E}^1 \to \mathbb{S}^2$.

Let $\overline{M} = (\overline{M}, \bar{g}, J)$ be a Kähler manifold. Take a Riemannian product $M = \overline{M} \times \mathbb{E}^1$ and equip a regular coKähler structure on M as we explained in Example 5.2. Assume that M admits a homogeneous coKähler structure S. Then the tensor field $\bar{S}$ defined by (15.5) induces a homogeneous Riemannian structure (see [36, Proposition 6.3.6], [42]).

Now let us investigate the homogeneous Riemannian structure $\bar{S}$ on $\mathbb{S}^2$. Since $S(X)Y = -(\lambda/2)\eta(X)\varphi(Y)$, we have $S(\overline{X}^{\mathsf{h}})\overline{Y}^{\mathsf{h}} = 0$. Thus $\bar{S} = 0$. This fact is consistent with the fact that $\mathbb{S}^2$ has only trivial homogeneous Riemannian structure.

18. The product space $\mathbb{H}^2 \times \mathbb{E}^1$

In this section we discuss the homogeneous Riemannian structures on $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$. The Riemannian product admits coKähler structure and almost Kenmotsu structures. There are two different classes of almost Kenmotsu structures. One is defined for $\mathbb{H}^2(-4)$ and the other is defined for $\mathbb{H}^2(-c^2)$ with $c^2 > 4$. On this reason we do not normalize the curvature of $\mathbb{H}^2(-c^2)$ to -1 in this section.

18.1. The product metric

Let us realize the homogeneous Riemannian space $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ as an open submanifold

$$\mathbb{H}^2(-c^2) \times \mathbb{E}^1 = (\{(x, y, t) \in \mathbb{R}^3 \; y > 0\}, g), \quad g = \bar{g} + \mathrm{d}t^2, \quad \bar{g} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{c^2 y^2}$$

of $\mathbb{R}^3$. We can take a global orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ of the form

$$e_1 = (cy)\frac{\partial}{\partial x}, \quad e_2 = (cy)\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial t}.$$

The coframe field $\Theta = (\vartheta^1, \vartheta^2, \vartheta^3)$ metrically dual to $\mathcal{E}$ is given by

$$\vartheta^1 = \frac{\mathrm{d}x}{cy}, \quad \vartheta^2 = \frac{\mathrm{d}y}{cy}, \quad \vartheta^3 = \mathrm{d}t,$$

Since

$$\mathrm{d}\vartheta^1 = \frac{1}{cy^2}\,\mathrm{d}x \wedge \mathrm{d}y, \quad \mathrm{d}\vartheta^2 = \mathrm{d}\vartheta^3 = 0,$$

we have

$$\omega = \begin{pmatrix} 0 & -c\,\vartheta^1 & 0 \\ c\,\vartheta^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & -c^2\,\vartheta^1 \wedge \vartheta^2 & 0 \\ c^2\,\vartheta^1 \wedge \vartheta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the Levi-Civita connection is given by

$$\nabla_{e_1} e_1 = ce_2, \quad \nabla_{e_1} e_2 = -ce_1, \quad \nabla_{e_1} e_3 = 0,$$

$$\nabla_{e_2} e_1 = \nabla_{e_2} e_2 = \nabla_{e_2} e_3 = 0,$$

$$\nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0.$$

The sectional curvatures are given by

$$K(e_1 \wedge e_2) = -c^2, \quad K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = 0.$$

The Ricci tensor field of $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ is given by

$$\mathrm{Ric} = -c^2 g + c^2\,\vartheta^3 \otimes \vartheta^3.$$

The product Lie group

$$\mathrm{SL}_2\mathbb{R} \times \mathbb{R} = \{((a_{ij}), s) \mid (a_{ij}) \in \mathrm{SL}_2\mathbb{R}, \; s \in \mathbb{R}\}$$

acts isometrically and transitively on $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ via the action:

$$((a_{ij}),s)\,(x+\sqrt{-1}y,t)=\left(\frac{a_{11}(x+\sqrt{-1}y)+a_{12}}{a_{21}(x+\sqrt{-1}y)+a_{22}},t+s\right).$$

The isotropy subgroup at the origin $o=(\sqrt{-1},0)$ is $\mathrm{SO}(2)\times\{0\}$. We identify this isotropy subgroup with $\mathrm{SO}(2)$. Hence we obtain a homogeneous Riemannian space representation $(\mathrm{SL}_2\mathbb{R}\times\mathbb{R})/\mathrm{SO}(2)$. We identify the product Lie group $\mathrm{SL}_2\mathbb{R}\times\mathbb{R}$ with

$$\left\{\left.\begin{pmatrix} e^s & 0 & 0\\ 0 & a_{11} & a_{12}\\ 0 & a_{21} & a_{22}\end{pmatrix}\right|\ a_{11},a_{12},a_{21},a_{22},s\in\mathbb{R},\ a_{11}a_{22}-a_{12}a_{21}=1\right\}.$$

Then the Lie algebra of $\mathrm{SL}_2\mathbb{R}\times\mathbb{R}$ is identified with

$$\mathfrak{sl}_2\mathbb{R}\oplus\mathbb{R}=\left\{\left.\begin{pmatrix} u_3 & 0 & 0\\ 0 & -u_2 & -u_4+u_1\\ 0 & u_4+u_1 & u_2\end{pmatrix}\right|\ u_1,u_2,u_3,u_4\in\mathbb{R}\right\}.$$

We can take a basis

$$e_1=\begin{pmatrix} 0 & 0\\ 0 & \boldsymbol{j}'\end{pmatrix}=\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad e_2=\begin{pmatrix} 0 & 0\\ 0 & \boldsymbol{k}'\end{pmatrix}=\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$

$$e_3=\begin{pmatrix} 1 & 0\\ 0 & \boldsymbol{0}\end{pmatrix}=\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad e_4=\begin{pmatrix} 0 & 0\\ 0 & \boldsymbol{i}\end{pmatrix}=\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}.$$

The isotropy algebra $\mathfrak{h}$ is spanned by e_4. The tangent space $\mathrm{T}_o(\mathbb{H}^2(-c^2)\times\mathbb{E}^1)$ is identified with

$$\mathfrak{m}_0=\mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}e_3.$$

One can see that $[\mathfrak{m}_0,\mathfrak{m}_0]\subset\mathfrak{h}$. Thus $(\mathrm{SL}_2\mathbb{R}\times\mathbb{R})/\mathrm{SO}(2)$ is a Riemannian symmetric space.

Let us equip an almost contact structure (φ_0,ξ_0,η_0) by

$$\xi_0=e_3,\quad \eta_0=g(e_3,\cdot)$$

and

$$\varphi_0e_1=e_2,\quad \varphi_0e_2=-e_1,\quad \varphi_0e_3=0.$$

Then (φ_0,ξ_0,η_0) is compatible to g and the resulting almost contact Riemannian manifold $(\mathbb{H}^2(-c^2)\times\mathbb{E}^1,\varphi_0,\xi_0,\eta_0,g)$ is coKähler.

18.2. The solvable Lie group model

As we saw in Example 16.4 and Example 16.5, $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ admits a solvable Lie group structure. On the other hand, we know the solvable Lie group model $\overline{\mathcal{S}}$ of $\mathbb{H}^2(-c^2)$. We use the model $\mathbb{H}^2(-c^2)=\overline{\mathcal{S}}$ and consider the product Lie group $\mathbb{R}\times\overline{\mathcal{S}}$. The product Lie group $\mathbb{R}\times\overline{\mathcal{S}}$ is identified with

$$\mathcal{S}=\left\{\left.\begin{pmatrix} e^t & 0 & 0\\ 0 & \sqrt{y} & x/\sqrt{y}\\ 0 & 0 & 1/\sqrt{y}\end{pmatrix}\right|\ x,y,t\in\mathbb{R},\ y>0\right\}.$$

The dressing action of $\mathrm{SL}_2\mathbb{R}$ on $\overline{\mathcal{S}}$ is naturally extended to $\mathcal{S}$. Here we replace $\overline{\mathcal{S}}$ by the solvable Lie group $\overline{\mathcal{M}}$ defined by (4.9). The solvable Lie group $\mathcal{S}$ is isomorphic to

$$\mathcal{M}=\left\{\left.\begin{pmatrix} e^t & 0 & 0\\ 0 & y & x\\ 0 & 0 & 1\end{pmatrix}\right|\ x,y,t\in\mathbb{R},\ y>0\right\}$$

via the Lie group isomorphism

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & \sqrt{y} & x/\sqrt{y} \\ 0 & 0 & 1/\sqrt{y} \end{pmatrix} \longmapsto \begin{pmatrix} e^t & 0 & 0 \\ 0 & y & x \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra of $\mathcal{M}$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} w & 0 & 0 \\ 0 & v & u \\ 0 & 0 & 0 \end{pmatrix} \middle| \, u, v, w \in \mathbb{R} \right\}.$$

The product metric $g = \bar{g} + \mathrm{d}t^2$ is a left invariant metric on $\mathcal{M}$. Moreover $\{e_1, e_2, e_3\}$ is a global left invariant orthonormal frame field on $\mathcal{M}$. At the origin, we have

$$e_1\big|_o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2\big|_o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3\big|_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Homogeneous geometry, especially Grassmann geometry of $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$, we refer to [123].

18.3. *Homogeneous Riemannian structures*

Now let us classify the homogeneous Riemannian structures on $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$. Let $S_\flat$ be a tensor field of type $(0, 3)$ satisfying (4.11) then $S_\flat$ is represented as (4.12). Let S be a tensor field of type $(1, 2)$ associated to $S_\flat$ and set $\tilde{\nabla} = \nabla + S$. Then the parallelism condition (4.14) is computed as

$$\begin{aligned}(\nabla_X \mathrm{Ric})(e_i, e_j) =& c^2\{\nabla_X(\vartheta^3 \otimes \vartheta^3)\}(e_i, e_j) = (\nabla_X \eta)(e_i)\eta(e_j) + \eta(e_i)(\nabla_X \eta)(e_j) \\ =& g(\nabla_X e_3, e_i)\eta(e_j) + \eta(e_i)g(\nabla_X e_3, e_j) = c^2(\omega_3^{\,i}(X)\eta(e_j) + \omega_3^{\,\text{æ}}(X)\eta(e_i)).\end{aligned}$$

On the other hand we have

$$\mathrm{Ric}(S(X)Y, Z) + \mathrm{Ric}(Y, S(X)Z) = c^2\{S_\flat(X, e_i, e_3)\eta(e_j) + S_\flat(X, e_j, e_3)\eta(e_i).\}$$

Hence we get

$$\omega_i^{\,3}(X)\eta(e_j) + \omega_j^{\,3}(X)\eta(e_i) = -S_\flat(X, e_i, e_3)\eta(e_j) - S_\flat(X, e_j, e_3)\eta(e_i).$$

From this system we get

$$S_\flat(X, e_3, e_1) = \omega_1^{\,3}(X) = 0, \quad S_\flat(X, e_3, e_3) = \omega_2^{\,3}(X) = 0.$$

Thus $S_\flat$ is expressed as

$$S_\flat = 2\sigma \otimes (\theta^1 \wedge \theta^2),$$

where the 1-form $\sigma = \sum_{i=1}^{3} \sigma_i \vartheta^i$ is defined by

$$\sigma(X) = S_\flat(X, e_1, e_2).$$

The connection 1-forms $\{\tilde{\omega}_i^{\,j}\}$ of $\tilde{\nabla}$ relative to $\{e_1, e_2, e_3\}$ are computed as

$$\tilde{\omega}_1^{\,2}(X) = \omega_1^{\,2}(X) + S_\flat(X, e_1, e_2) = c\vartheta^1(X) + \sigma(X), \quad \tilde{\omega}_1^{\,3}(X) = \tilde{\omega}_2^{\,3}(X) = 0.$$

Let us compute $\tilde{\nabla} S_\flat$.

First we get

$$\begin{aligned}\tilde{\nabla}_X \vartheta^1 =& \tilde{\omega}_1^{\,2}(X)\vartheta^2 = \left\{c\vartheta^1(X) + \sigma(X)\right\}\vartheta^2, \\ \tilde{\nabla}_X \vartheta^2 =& \tilde{\omega}_2^{\,1}(X)\vartheta^1 = -\left\{c\vartheta^1(X) + \sigma(X)\right\}\vartheta^1, \quad \tilde{\nabla}_X \vartheta^3 = 0.\end{aligned}$$

From these we get $\tilde{\nabla}_X(\vartheta^1 \wedge \vartheta^2) = 0$. Hence $\tilde{\nabla}_X S_\flat = 2(\nabla_X \sigma) \otimes (\vartheta^1 \wedge \vartheta^2)$. Thus $\tilde{\nabla} S_\flat = 0$ is equivalent to $\tilde{\nabla}\sigma = 0$. The covariant derivative $\tilde{\nabla}\sigma$ is computed as

$$\begin{aligned}\tilde{\nabla}_X \sigma = \quad & \left\{(\mathrm{d}\sigma_1)(X) - \left(c\vartheta^1(X) + \sigma(X)\right)\sigma_2\right\}\vartheta^1 \\ & + \left\{(\mathrm{d}\sigma_2)(X) + \left(c\vartheta^1(X) + \sigma(X)\right)\sigma_1\right\}\vartheta^2 \\ & + (\mathrm{d}\sigma_3)(X)\vartheta^3.\end{aligned}$$

Thus the parallelism of S with respect to $\tilde{\nabla}$ is equivalent to the system

$$\mathrm{d}\sigma_1 = \left(c\vartheta^1 + \sigma\right)\sigma_2, \quad \mathrm{d}\sigma_2 = -\left(c\vartheta^1 + \sigma\right)\sigma_1, \quad \mathrm{d}\sigma_3 = 0.$$

Thus σ_3 is a constant, say λ.

From the integrability condition we deduce that (see [174] for detail):

1. $\sigma_1 = \sigma_2 = 0$ or
2. $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = -c$.

In the former case, we have $\sigma = \lambda\vartheta^3$. Hence $S_\flat$ has the form

$$S_\flat^\lambda = 2\lambda\vartheta^3 \otimes (\vartheta^1 \wedge \vartheta^2) = \lambda\eta_0 \otimes \mathrm{d}v_{\bar{g}}, \quad \lambda \in \mathbb{R}.$$

where $\mathrm{d}v_{\bar{g}}$ is the area element of $\mathbb{H}^2(-c^2)$. Note that $S_\flat^{\lambda_1}$ and $S_\flat^{\lambda_2}$ are isomorphic each other if and only if $\lambda_2 = \pm\lambda_1$.

Let us compare the canonical connections $\nabla + S^\lambda$ with the almost contact connections ∇^r. Since the structure $(\varphi_0, \xi_0, \eta_0, g)$ is coKähler, we have $A^r(X)Y = -r\eta_0(X)\varphi_0 Y$. On the other hand S^λ is rewritten as $S^\lambda(X)Y = \lambda\eta_0(X)\varphi_0 Y$ Hence we get $S^\lambda = -A^\lambda = A^{-\lambda}$.

In the latter case, we have

$$S_\flat = -2c\,\vartheta^1 \otimes (\vartheta^1 \wedge \vartheta^2) = -c\,\vartheta^1 \otimes \mathrm{d}v_{\bar{g}}.$$

Obviously, we have

$$S(e_3)Y = 0, \quad S(X)e_3 = 0.$$

Thus we conclude that $S_\flat$ is a extension of the homogeneous Riemannian structure $\bar{S}_\flat$ of $\mathbb{H}^2(-c^2) = \overline{\mathcal{S}}$ to $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$. In other words, $(\mathbb{H}^2(-c^2) \times \mathbb{E}^1, g, S)$ is the direct sum of $(\mathbb{H}^2(-c^2), \bar{g}, \bar{S})$ and $(\mathbb{E}^1, \mathrm{d}t^2, 0)$. By using the coKähler structure, S is rewritten as $S(X)Y = -cg(X, e_1)\varphi_0 Y$. By using the solvable Lie group model $\mathcal{M}$, one can confirm that $\nabla + S$ coincides with the Cartan-Schouten's $(-)$-connection.

The following theorem is a reformulation of Ohno's result [174].

Theorem 18.1. *Up to isomorphisms, the model space $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ admits two types of homogeneous Riemannian structures*

1. *$S_{2,3}^\lambda(X)Y = A^{-\lambda}(X)Y = \lambda\,\eta(X)\varphi Y$, $\lambda \geq 0$. The corresponding coset space representation is $\mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2)$. The homogeneous structure $S_{2,3}^\lambda$ is of type $\mathcal{T}_2 \oplus \mathcal{T}_3$. The homogeneous Riemannian structure $S_{2,3}^\lambda$ is of type $\mathcal{T}_3$ if and only if $\lambda = 0$. In this case $\mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2)$ is a Riemannian symmetric space. For $\lambda \neq 0$, $S_{2,3}^\lambda \notin \mathcal{T}_2$. The homogeneous Riemannian structure $S_{2,3}^\lambda$ is a homogeneous coKähler structure for any λ. The homogeneous Kähler structure on $\mathbb{H}^2(-c^2)$ induced from $S_{2,3}^\lambda$ by reduction is the trivial one.*
2. *$S(X)Y = -cg(X, e_1)\varphi_0 Y$. The corresponding coset space representation is the solvable Lie group model $\mathcal{S} = \mathcal{S}/\{\mathsf{e}\}$. The homogeneous Riemannian structure S is the extension of the homogeneous Riemannian structure $\bar{S}$ of type $\mathcal{T}_1$ on $\mathbb{H}^2(-c^2)$ by the rule*

$$S(e_3)Y = S(X)e_3 = 0.$$

In other words, $(\mathbb{H}^2(-c^2) \times \mathbb{E}^1, g, S)$ is the direct sum of $(\mathbb{H}^2(-c^2), \bar{g}, \bar{S})$ and $(\mathbb{E}^1, \mathrm{d}t^2, 0)$. As a left invariant connection on the solvable Lie group $\mathcal{S} \cong \mathcal{M}$, the canonical connection $\nabla + S$ is the Cartan-Schouten's $(-)$-connection. The homogeneous Riemannian structure S is a homogeneous coKähler structure. The connection $\nabla + S$ coincides with almost contact connection $\nabla + A^0$, where A^0 is derived from the $(-c/2)$-almost Kenmotsu structure exhibited in Examples 16.4, 16.5.

Proof. The only task we need to check that $\tilde{\nabla}\varphi_0 = 0$ with respect to the homogeneous Riemannian structure $S = \bar{S} + 0$. Since $(\varphi_0, \xi_0, \eta_0, g)$ is coKäher, we have $\nabla\varphi = 0$. Thus the covariant derivative $\tilde{\nabla}\varphi_0$ with respect to $\tilde{\nabla} = \nabla + S$ is

$$(\tilde{\nabla}_X\varphi_0)Y = S(X)\varphi_0 Y - \varphi_0 S(X)Y = 0.$$

Thus S is a homogeneous coKäher structure. □

The model space $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ admits compatible coKähler structure as well as $(-c/2)$-almost Kenmotsu structure. From almost contact structure viewpoint, $S_{2,3}^\lambda$ fits with coKähler structure. Indeed we know that $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ is a coKähler space form and it has 4-dimensional automorphism group.

On the other hand, the homogeneous structure $S = \bar{S} + A^0$ fits with $(-c/2)$-almost Kenmotsu structure (see Theorem 16.5 and next subsection). The automorphism group of $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ with respect to the $(-c/2)$-almost Kenmotsu structure is 3-dimensional.

18.4. Almost Kenmotsu structures

Here we mention almost b-Kenmotsu structures on $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$.
Up to homothety we may restrict our attention to almost Kenmotsu structures on $\mathbb{H}^2(-c^2)$.
First we recall the following classification of locally symmetric almost Kenmotsu 3-manifolds.

Proposition 18.1 ([103]). *Let M be a non-normal almost Kenmotsu 3-manifold. Then M is locally symmetric if and only if M is locally isomorphic to one of the following spaces*

1. *If ξ is an eigenvector field of a Ricci operator, then M is Kenmotsu and of constant curvature -1 or locally isometric to $\mathbb{H}^2(-4)\times\mathbb{E}^1$.*
2. *If ξ is not an eigenvector field of a Ricci operator, then M is locally isometric to $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ of constant curvature $-c^2<-4$.*

Note that the third example $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ was discovered in [192, Theorem 1.2 Case (IV)].
In Example 16.4 and Example 16.5 we exhibited almost $(-c/2)$-Kenmotsu structures on $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$. We recall those structure here. In Example 16.4, $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ is realized as

$$G(0,c)=\left\{\left.\begin{pmatrix}1&0&0&w\\0&1&0&u\\0&0&e^{cw}&v\\0&0&0&1\end{pmatrix}\right|u,v,w\in\mathbb{R}\right\}\cong\left\{\left.\begin{pmatrix}1&0&u\\0&e^{cw}&v\\0&0&1\end{pmatrix}\right|u,v,w\in\mathbb{R}\right\}$$

with a left invariant metric $\mathrm{d}u^2+e^{-2cw}\mathrm{d}v^2+\mathrm{d}w^2$. Then $G(0,c)$ is isometric to $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ via the isometry:

$$x=cv,\quad y=e^{cw},\quad t=u.$$

The orthonormal frame field

$$\frac{\partial}{\partial u},\quad e^{cw}\frac{\partial}{\partial v},\quad \frac{\partial}{\partial w}$$

corresponds to

$$e_3=\frac{\partial}{\partial t},\quad e_1=(cy)\frac{\partial}{\partial x},\quad e_2=(cy)\frac{\partial}{\partial y}.$$

Let (φ_c,ξ_c,η_c) be the almost contact structure introduced in Section 16, then

$$\varphi_c e_3=e_1,\quad \varphi_c e_1=-e_2,\quad \varphi_c e_2=0,\quad \xi_c=e_2,\quad \eta_c=\vartheta^2=\frac{1}{cy}\,\mathrm{d}y.$$

Note ξ_c is an eigenvector field of the Ricci operator. Since we have $\nabla\xi_c=-c\vartheta^1\otimes e_1$, the almost contact connection $\nabla+A^{0,c}$ with respect to the structure $(\varphi_c,\xi_c,\eta_c,g)$ is given by

$$A^{0,c}_X Y=-\eta_c\nabla_X\xi_c+g(\nabla_X\xi_c,Y)\xi_c=-cg(X,e_1)\varphi_0 Y=S(X)Y.$$

Note that φ_{-2} is not parallel with respect to the connection $\nabla+S^{\lambda}_{2,3}$.

Corollary 18.1. *Let $\mathbb{H}^2(-4)\times\mathbb{E}^1$ be product manifold of the hyperbolic plane of curvature -4 and the real line equipped with a homogeneous almost Kenmotsu structure $(\varphi_{-2},\xi_{-2},\eta_{-2},g)$ such that ξ_{-2} is an eigenvector field of the Ricci operator, then the only homogeneous Riemannian structure is given by A^0. The homogeneous Riemannian structure A^0 is a homogeneous almost Kenmotsu structure and coincides with the Cartan-Schouten's $(-)$-connection. The corresponding coset space representation is the solvable Lie group model $G(0,-2)/\{\mathsf{e}\}$.*

Next we consider the almost Kenmotsu structure on $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ of constant curvature $-c^2<-4$. Set $c^2=4+\gamma^2$.
In Example 16.5, $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$ is realized as

$$G(c,0)=\left\{\left.\begin{pmatrix}1&0&0&w\\0&e^{cw}&0&u\\0&0&1&v\\0&0&0&1\end{pmatrix}\right|u,v,w\in\mathbb{R}\right\}\cong\left\{\left.\begin{pmatrix}e^{cw}&0&u\\0&1&v\\0&0&1\end{pmatrix}\right|u,v,w\in\mathbb{R}\right\}$$

with a left invariant metric $e^{-2cw}\mathrm{d}u^2 + \mathrm{d}v^2 + \mathrm{d}w^2$. Then $G(c, 0)$ is isometric to $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ via the isometry:

$$x = cu, \quad y = e^{cw}, \quad t = v.$$

The orthonormal frame field

$$e^{cw}\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial w}$$

corresponds to

$$e_1 = (cy)\frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial t}, \quad e_2 = (cy)\frac{\partial}{\partial y}.$$

Take a nonzero constant γ and set $c = -\sqrt{\gamma^2+4} < 0$. Then $G(-\sqrt{\gamma^2+4}, 0)$ is isomorphic to

$$\left\{ \begin{pmatrix} y & 0 & -x/\sqrt{\gamma^2+4} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \middle|\ x, y, t \in \mathbb{R},\ y > 0 \right\}.$$

Then the almost Kenmotsu structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g)$ on the model space $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{E}^1$ mentioned in the case (2) of Proposition 18.1 is given by (see [103]):

$$\tilde{\varphi}\tilde{e}_1 = \tilde{e}_2, \quad \tilde{\varphi}\tilde{e}_2 = -\tilde{e}_1, \quad \tilde{\varphi}\tilde{e}_3 = 0, \quad \tilde{\xi} = \tilde{e}_3, \quad \tilde{\eta} = g(\tilde{e}_3, \cdot),$$

where

$$\begin{aligned}
\tilde{e}_1 =& e_1 = -\sqrt{\gamma^2+4}\,y\,\frac{\partial}{\partial x}, \\
\tilde{e}_2 =& \frac{1}{\sqrt{\gamma^2+4}}(\gamma e_2 - 2e_3) = -\gamma y\,\frac{\partial}{\partial y} - \frac{2}{\sqrt{\gamma^2+4}}\frac{\partial}{\partial t}, \\
\tilde{e}_3 =& \frac{1}{\sqrt{\gamma^2+4}}(2e_2 + \gamma e_3) = -2y\,\frac{\partial}{\partial y} + \frac{\gamma}{\sqrt{\gamma^2+4}}\frac{\partial}{\partial t}.
\end{aligned}$$

In terms of the original orthonormal frame field, $\tilde{\varphi}$ is expressed as

$$\tilde{\varphi}e_1 = \frac{\gamma}{\sqrt{\gamma^2+4}}e_2 - \frac{2}{\sqrt{\gamma^2+4}}e_3, \quad \tilde{\varphi}e_2 = \frac{\gamma}{\sqrt{\gamma^2+4}}e_1, \quad \tilde{\varphi}e_3 = -\frac{2}{\sqrt{\gamma^2+4}}e_1.$$

One can see that $\tilde{\varphi}$ is not parallel with respect to $\nabla + S^{\lambda}_{2,3}$.

On the other hand, $\tilde{\varphi}$ is parallel with respect to the Cartan-Schouten's $(-)$-connection. The difference tensor field $S = \nabla^{(-)} - \nabla$ is described with the almost Kenmotsu structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, g)$ by

$$S(X)Y = g(X, \tilde{e}_1)\left\{\gamma\tilde{\varphi}Y + 2(g(Y, \tilde{e}_1)\tilde{\xi} - \tilde{\eta}(Y)\tilde{e}_1)\right\}.$$

19. Further problems

19.1. Sasakian space forms

The set of homogeneous Riemannian structures on a 3-dimensional Sasakian space form is completely determined.

Problem 1. *Determine the homogeneous Riemannian structures on Sasakian space forms of dimension greater than* 3.

Five-dimensional Sasakian φ-symmetric spaces are classified in [155]. Full classification was carried out in [135].

Problem 2. *Determine the homogeneous Riemannian structures on Sasakian φ-symmetric spaces of dimension greater than* 3.

19.2. Three-dimensional Lie groups

In this article we concentrate our attention to eight model spaces of Thurston geometry and Berger spheres. Other homogeneous Riemannian 3-spaces are realized as 3-dimensional Lie groups equipped with left invariant metrics. Homogeneous Riemannian structures on those spaces are classified by Calviño-Louzao, Ferreiro-Subrido, García-Río and Váazquez-Lorenzo [39] (under the left invariance assumption). In Proposition 15.2 we give the formula for the homogeneous contact Riemannian structure of the non-unimodular Sasakian Lie group $\tilde{G}(c)$ by using the Sasakian structure. In [125] we improved the classification due to [39]. More precisely we carried out the classification without left invariance assumption. Homogeneous contact Riemannian structures on non-Sasakian 3-dimensional non-unimodular Lie groups are described by using the structure (φ, ξ, η) in [125].

On the other hand, Perrone classified homogeneous contact Riemannian 3-manifolds [190] (see also [99]), homogeneous almost coKähler 3-manifolds and homogeneous almost Kenmotsu 3-manifolds [191, 192, 193] (see also [102, 103]). Here we propose the following problem:

Problem 3. *Classify all the homogeneous almost coKähler structures and homogeneous almost Kenmotsu structures on 3-dimensional Riemannian manifolds.*

19.3. Kenmotsu hyperbolic space

A simply connected and complete Kenmotsu manifold M has maximal dimensional automorphism group if and only if it is isomorphic to the hyperbolic space $\mathbb{H}^{2n+1}$.

Let us discuss the Kenmotsu structure on the hyperbolic space $\mathbb{H}^{2n+1}$. As we have saw in Example 4.6, we have a homogeneous representation $\mathbb{H}^{2n+1} = \mathrm{SO}^+(1, 2n+1)/\mathrm{SO}(2n+1)$. We have the Iwasawa decomposition $\mathrm{SO}^+(1, 2n+1) = NAK$, where K is the isotropy subgroup at o. The hyperbolic space $\mathbb{H}^{2n+1}$ is identified with the solvable part $\mathcal{S} = NA$. The isotropy subgroup of $\mathrm{SO}^+(1, 2n+1)$ at $o = (1, 0, \ldots, 0)$ is

$$K = \left\{ \left(\begin{array}{cc} 1 & {}^t\mathbf{0} \\ \mathbf{0} & A^\circ \end{array} \right) \middle| \ A^\circ \in \mathrm{SO}(2n+1) \right\} \cong \mathrm{SO}(2n+1).$$

The abelian part is isomorphic to $\mathbb{R}$. Here we remark that

$$\dim \mathrm{SO}^+(1, 2n+1) = 2n^2 + 3n + 1, \quad \dim \mathrm{SO}(2n) = 2n^2 + n, \quad \dim A = 1, \quad \dim N = 2n.$$

The maximal tori of $K = \mathrm{SO}(2n+1)$ are n-dimensional.

The hyperbolic space $\mathbb{H}^{2n+1}$ is represented as $\mathbb{H}^{2n+1} = \mathrm{SO}^+(1, n)/\mathrm{SO}(2n)$ as a Riemannian symmetric space. If $n > 1$, this is the unique naturally reductive representation.

On the other hand, for any Lie subgroup $\overline{K}$ of K, $\mathbb{H}^{2n+1}$ has a homogeneous space representation $\mathbb{H}^{2n+1} = (\mathcal{S} \times \overline{K})/\overline{K}$.

The maximum dimension of $\mathrm{Aut}(\mathbb{H}^{2n+1}) = (n+1)^2$. Thus as a Kenmotsu manifold with maximum automorphism group, $\mathbb{H}^{2n+1}$ is represented as

$$\mathbb{H}^{2n+1} = (\mathcal{S} \times \overline{K})/\overline{K}, \quad \dim \overline{K} = n^2.$$

For instance The subgroup

$$\mathrm{SO}(n+1) \times \mathrm{SO}(n) = \left\{ \left(\begin{array}{cc} A_1 & O \\ O & A_2 \end{array} \right) \middle| \ A_1 \in \mathrm{SO}(n+1), \ A_2 \in \mathrm{SO}(n) \right\}$$

is n^2-dimensional. When $n = 1$, $\overline{K} = \mathrm{SO}(2)$ and hence

$$\mathbb{H}^3 = (\mathcal{S} \times \mathrm{SO}(2))/\mathrm{SO}(2) = (\mathrm{B}_2^+\mathbb{C} \times \mathrm{U}(1))/\mathrm{U}(1).$$

Problem 4. *Determine the automorphism group of $\mathbb{H}^{2n+1}$ and give its nice expressions.*

If a connected Lie group G acts transitively and isometrically on $\mathbb{H}^n$ then $G\backslash\mathbb{H}^n$ is a point. It follows that $G\backslash\mathrm{SO}^+(n, 1)$ is an orbit space of $\mathrm{O}(n)$, thus G is a non-discrete co-compact subgroup of $\mathrm{SO}^+(n, 1)$. By Witte's structure theorem for co-compact Lie groups [229], the following result was obtained (see [40, 41]).

Theorem 19.1. *The connected Lie groups acting transitively on $\mathbb{H}^n$ are $\mathrm{SO}^+(n, 1)$ and $G = NF_r$, where N is the nilpotent part of the Iwasawa decomposition $\mathrm{SO}^+(n, 1) = N \cdot A \cdot \mathrm{SO}(n)$ and F_r is a connected subgroup of $A \cdot \mathrm{SO}(n-1)$ with nontrivial projection to the abelian part A.*

Remark 19.1. In the case of $\mathbb{H}^3$, $N \cdot F_r = N \cdot A \cdot \mathrm{SO}(2)$ which is isomorphic to $\mathrm{B}_2^+\mathbb{C} \times \mathrm{U}(1)$ (see [41, p. 567]).

19.4. Generalized symmetric Sasakian manifolds

The unit tangent sphere bundle $\mathrm{U}\mathbb{S}^n(1)$ equipped with the standard contact Riemannian structure is a φ-symmetric space fibered over the Grassmannian manifold $\widetilde{\mathrm{Gr}}_2(\mathbb{E}^4)$ of oriented 2-planes in $\mathbb{E}^4$.

In particular, $\mathrm{U}\mathbb{S}^3(1)$ is a Riemannian 4-symmetric space (see [104]). The only locally symmetric Sasakian manifold is $\mathbb{S}^{2n+1}$. But the Riemannian symmetric space $\mathrm{SO}(2n+2)/\mathrm{SO}(2n+1)$ is not homogeneous contact. Thus there are no Sasakian symmetric space. However $\mathrm{U}\mathbb{S}^3(1)$ is Sasakian 4-symmetric. According to Jiménez [134] Riemannian 4-symmetric spaces are fiber bundles over Riemannian symmetric spaces whose standard fibres are Riemannian symmetric. This fact motives us to study the following problem:

Problem 5. *Are there compatible 4-symmetric structures on φ-symmetric spaces ?*

For more information on Riemannian 4-symmetric spaces, we refer to [133, 156, 157].

19.5. Generalized φ-symmetric spaces

The notion of φ-geodesic symmetry in Definition 9.1 can be generalized to the following one:

Definition 19.1 ([128]). Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. Assume that ξ is a Killing vector field. A local diffeomorphism s_p is said to be an *axial symmetry* (or *ξ-preserving symmetry*) with base point $p \in M$ if

1. s_p fixes every point of the characteristic flow,
2. for each point q on the characteristic flow, s_p sends any φ-geodesic through q to a φ-geodesic through q,
3. in a small neighborhood, of p, the points on the characteristic flow are the only fixed points of s_p.

If there exists a least integer $k \geq 2$ such that $(s_p)^k$ is the identity map, then s_p is said to be of *order* k.

A φ-geodesic symmetry in the sense of Takahashi is an axial symmetry of order 2 if M is a contact Riemannian manifold (K-contact). The terminology ξ-preserving symmetry was suggested by Tsukada and used in [128].

Analogous to the generalized Riemannian symmetric spaces in the sense of Kowalski [149], one can introduce the notion of *locally generalized φ-symmetric space* in the following manner (due to Masami Sekizawa):

Definition 19.2. An almost contact Riemannian manifold M is said to be a *locally generalized φ-symmetric space* if ξ is a Killing vector field and at each point $p \in M$, there exists a globally defined axial symmetry s_p and the order of s_p is the common value, say $k \geq 2$ and called the *order* of M.

We know that globally φ-symmetric spaces are naturally reductive homogeneous spaces. What about generalized φ-symmetric spaces ?

Problem 6. *Construct explicit examples of generalized φ-symmetric spaces of order $k > 2$.*

Let M be an almost contact Riemannian manifold with Killing ξ, then at each point $p \in M$, we can take a sufficiently small neighborhood $\mathcal{U}$ of p on which ξ is regular. We obtain a local fibering $\pi : \mathcal{U} \to \mathcal{U}/\xi$. The K-contact structure induces an almost Hermitian structure $(\bar{g}, J)$ on the factor space $\overline{\mathcal{U}} = \mathcal{U}/\xi$. In particular, if M is contact Riemannian, then $(\bar{g}, J)$ is almost Kähler.

Now let us assume that M is locally generalized φ-symmetric, then s_p induces a local symmetry on $\mathcal{U}/\xi$ around $\pi(p)$. If s_p is of order k, then the induced symmetry is also of order k. Hence $\mathcal{U}/\xi$ is an almost Hermitian and locally k-symmetric.

Conversely, we know the following local construction.

Proposition 19.1 ([128, 127]). *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold with Killing ξ, p a point of M, $\mathcal{U}$ a sufficiently small normal neighborhood of p on which ξ is regular, $\pi : \mathcal{U} \to \overline{\mathcal{U}} = \mathcal{U}/\xi$ the local fibering and $(J, \bar{g})$ the almost Hermitian structure of $\overline{\mathcal{U}}$ induced from $\mathcal{U}$. If there exits a family $\{\bar{s}_{\bar{p}}\}$ of isometric symmetries on $\overline{\mathcal{U}}$ with base point $\bar{p} = \pi(p)$, then there exits a family of axial symmetries on $\mathcal{U}$. Moreover if $\bar{s}_{\bar{p}}$ is of order k, then so is s_p.*

Thus we are interested in almost contact Riemannian manifolds with Killing ξ which is fibered over almost Hermitian k-symmetric spaces with $k > 2$.

Let M be a homogeneous Sasakian manifold, then M is regular, hence we have the Boothby-Wang fibering $\pi : M \to \overline{M} = M/\xi$. The factor space is a homogeneous Kähler manifold (see [29, Theorem 8.3.5]). Moreover we know the following fundamental fact ([29, Theorem 8.3.6]).

Theorem 19.2. *Let (M, η) be a compact homogeneous contact manifold. Then*

1. *M admits a homogeneous Sasakian structure,*
2. *M is a non-trivial circle bundle overe a generalized flag manifold, and*
3. *M has finite fundamental group, and the universal covering $\widetilde{M}$ of M is compact with a homogeneous Sasakian structure.*

It is well known that every Kähler C-space, *i.e.*, simply connected compact homogeneous Kähler manifold is a generalized flag manifold. Jiménez constructed Hermitian k-symmetric spaces [132]. Moreover, Kähler C-spaces admit compatible k-symmetric structures (see [34, 221]). These facts motivates the following problem.

Problem 7. *Can we classify Sasakian generalized φ-symmetric spaces of order $k > 2$?*

We expect that compact Sasakian generalized φ-symmetric spaces are homogeneous Sasakian manifolds fibered over Kähler C-spaces equipped with k-symmetric structures.

Let us turn our attention to the case $k = 3$. Assume that the factor space $\overline{\mathcal{U}}$ of the local fibering is a Riemannian 3-symmetric space. As is well known, Riemannian 3-symmetric spaces admit compatible nearly Kähler structure (almost Tachibana structure) [82]. An almost Hermitian manifold $(\overline{M}, \bar{g}, J)$ is said to be *nearly Kähler* if it satisfies

$$(\nabla_{\overline{X}}J)\overline{X} = 0$$

for all vector field $\overline{X}$ on $\overline{M}$. On the other hand, an almost contact manifold M is said to be *nearly coKähler* (or *nearly cosymplectic*) if it satisfies

$$(\nabla_X\varphi)X = 0$$

for all vector field X on M [15, 19]. Typical example of strictly nearly cosymplectic manifold is the totally geodesic unit 5-sphere of the nearly Kähler 6-sphere [15]. One can see that the Riemannian product $M = \overline{M} \times \mathbb{E}^1$ or $M = \overline{M} \times \mathbb{S}^1$ of a nearly Kähler manifold $\overline{M}$ and the Euclidean line or the circle is nearly coKähler and it is generalized φ-symmetric of order 3.

Blair and Showers studied nearly cosymplectic manifolds satisfying $d\eta = 0$. They showed that nearly cosymplectic 5-manifolds satisfying $d\eta = 0$ are coKähler (cosymplectic). Note that the totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$ does not satisfy $d\eta = 0$.

In [56] De Nicola, Dileo and Yudin proved that a strictly nearly Sasakian manifold M of dimension $2n + 1 > 5$ is s locally isometric to one of the following Riemannian products: $\overline{M}^{2n} \times \mathbb{E}^1$ or $M_1^5 \times M_2^{2n-4}$, where $\overline{M}^{2n}$ is a strictly nearly Kähler manifold, M_2^{2n-4} is a nearly Kähler manifold and M_1^5 is a strictly nearly cosymplectic 5-manifold.

Problem 8. *Are there irreducible strictly nearly cosymplectic generalized φ-symmetric spaces ?*

Remark 19.2. The model space $\mathrm{F}^4 = (\mathrm{SL}_2\mathbb{R} \ltimes \mathbb{R}^2)/\mathrm{SO}(2)$ of four dimensional geometry is an almost Kähler 3-symmetric space discovered by Kowalski [149] (see also [64]). We may expect that the circle bundle $\mathrm{SL}_2\mathbb{R} \ltimes \mathbb{R}^2$ admits a non-Sasakian K-contact generalized φ-symmetric structure of order 3. Calvaruso and Fino proved that the Lie algebra $\mathfrak{sl}_2\mathbb{R} + \mathbb{R}^2$ of $\mathrm{SL}_2\mathbb{R} \ltimes \mathbb{R}^2$ does not admit any K-contact structure [37, Proposition 4.5]. Foreman [72] also studied K-contact Lie groups.

19.6. Standard contact metric structure of the unit tangent sphere bundle

Let $M^n(\varepsilon c^2)$ be an n-dimensional Riemannian space form of curvature εc^2. Then its unit tangent bundle $\mathrm{U}M^n(\varepsilon c^2)$ equipped with standard contact Riemannian structure is a contact (κ, μ)-space with

$$\kappa = \varepsilon c^2(2 - \varepsilon c^2), \quad \mu = -2\varepsilon c^2.$$

In particular, $\mathrm{U}\mathbb{H}^n(-1)$ is a contact $(-3, 2)$-space.

Problem 9. *Classify homogeneous contact Riemannian structure on $\mathrm{U}\mathbb{H}^2(-c^2)$. Are there any homogeneous contact Riemannian structure other than Boeckx's homogeneous Riemannian structure ?*

19.7. Grassmann geometry

In our works [109, 120, 121, 122, 123, 158], we studied Grassmann geometry of surfaces of orbit type in 3-dimensional homogeneous Riemannian spaces.

Problem 10. *Determine Grassmann geometry of submanifolds of orbit type in homogeneous contact manifolds.*

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Competing interests

The authors declare that they have no competing interests.

References

[1] Abe, K.: *The classification of homogeneous structures on 3-dimensional space forms*, Math. J. Okayama Univ. **28** (1), 173-189 (1986).
[2] Agricola, T, Friedrich, F.: *On the holonomy of connections with skew-symmetric torsion*, Math. Ann. **328**,711-748 (2004).
[3] Alekseevsky, D., Gorodski, C.: *Semisimple symmetric contact spaces*, Indag. Math. New Ser. **31** (6), 1110-1133 (2020).
[4] Ambrose, W., Palais, R.S., Singer, I. M.: *Sprays*, Ann. Acad. Brasil. Ciĕncias **32**, 163-178 (1960).
[5] Ambrose, W., Singer, I. M.: *On homogeneous Riemannian manifolds*, Duke Math. J. **25**, 647-669 (1958).
[6] Auslander, L., Green, L., Hahn, F.: *Flows on homogeneous spaces*, with the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg, Ann. of Math. Stud. **53** Princeton University Press, Princeton, NJ, 1963.
[7] Azencott, R., Wilson, E. N.: *Homogeneous manifolds with negative curvature*. I, Trans. Am. Math. Soc. **215**, 323-362 (1976).
[8] Azencott, R., Wilson, E. N.: *Homogeneous manifolds with negative curvature*. II, Mem. Am. Math. Soc. **178** (1976).
[9] Bazdar, A., Teleman, A.: *Infinitesimal homogeneity and bundles*, Ann Glob Anal Geom **59**, 197-231 (2021).
[10] Belkhelfa, M., Dillen, F., Inoguchi, J.: *Surfaces with parallel second fundamental form in Bianchi-Cartan-Vranceanu spaces*, PDE's, Submanifolds and Affine Differential Geometry (Warsaw, 2000), Banach Center Publ., **57**, 67-87, (2002).
[11] Bérad-Bergery, L., Bourguignon, J. P.: *Laplacians and Riemannian submersions with totally geodesic fibers*, Illinois J. Math. **26**, 181-200 (1982).
[12] Berger, M.: *Les Varietes Riemanniennes homogenes normlales simplement connexes a courbure strictement positive*, Annli Scuola Norm. Sup. Pisa **15**, 179-246 (1961).
[13] Bianchi, L.: *Sugli sazi a tre dimensioni che ammettono un gruppo continuo di movimenti*, Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Serie Tereza, Tomo XI, 267-352 (1898); English translation: *On the three-dimensional spacse which admit a continuous group of motions*, General Relativity and Gravitation **33** (12), 2171-2252 (2001).
[14] Bieliavsky, P., Falbel, E., Gorodski, C.: *The classification of simply-connected contact sub-Riemannian symmetric spaces*, Pacific J. Math. **188** (1), 65-82 (1999).
[15] Blair, D. E.: *Almost contact manifolds with Killing structure tensors*, Pacific J. Math. **39** (2), 285-292 (1971)
[16] Blair, D. E.: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, Springer-Verlag, Berlin-Heidelberg-New-York, (1976).
[17] Blair, D. E.: *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Math. **203**, Birkhäuser, Boston, Basel, Berlin, (2002).
[18] Blair, D.E., Koufogiorgos, T., Papantoniou, B.J.: *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91**, 189-214 (1995).
[19] Blair, D.E., Showers, D.K.: *Almost contact manifolds with Killing structure tensors*. II, J. Differ. Geom. **9**, 577-582 (1974).
[20] Blair, D. E., Vanhecke, L.: *Symmetries and φ-symmetric spaces*, Tôhoku Math. J. (2) **39** (3), 373-383 (1987).
[21] Blair, D. E., Vanhecke, L.: *New characterizations of φ-symmetric spaces*, Kodai Math. J. **10** (1), 102-107 (1987).
[22] Blair, D. E., Vanhecke, L.: *Volume-preserving φ-geodesic symmetries*, C. R. Math. Acad. Sci. Soc. R. Can. **9**, 31-36 (1987).
[23] Boeckx, E.: *A class of locally φ-symmetric contact metric spaces*, Arch. Math. **72** (6), 466-472 (1999).
[24] Boeckx, E.: *A full classification of contact metric (κ, μ)-spaces*, Illinois J. Math. **44**, 212-219 (2000).
[25] Boeckx, E., Bueken, P., Vanhecke, L.: *Flow-symmetric Riemannian manifolds*, Beitr. Algebra Geom. **40**, (2), 459-474 (1999).
[26] Boeckx, E., Cho, J. T.: *Locally symmetric contact metric manifolds*, Monatsh. Math. **148** (4), 269-281 (2006).
[27] Boeckx, E., Cho, J. T.: *Pseudo-Hermitian symmetries*, Israel J. Math. **166**, 125-145 (2008).
[28] Boothby, W. M., Wang, H. C.: *On contact manifolds*, Ann. Math. (2) **68**, 721-734 (1958).
[29] Boyer, C. P., Galicki, K.: *Sasakian Geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.
[30] Bueken, P., Vanhecke, L.: *Geometry and symmetry on Sasakian manifolds*, Tsukuba J. Math. **12** (2), 403-422 (1988).
[31] Bueken, P., Vanhecke, L.: *Harmonic reflections on Sasakian manifolds*, Math. J. Okayama Univ. **30**, 187-197 (1988).
[32] Bueken, P., Vanhecke, L.: *Reflections in K-contact geometry*, Math. Rep. Toyama Univ. **12**, 41-49 (1989).
[33] Bueken, P., Vanhecke, L.: *Rotations and harmonicity in contact geometry*, Rend. Mat. Appl., VII. **12** (1), 127-141 (1992).
[34] Burstall, F. E., Rawnsley, J. H.: *Twistor Theory for Riemannian Symmetric Spaces With Applications to Harmonic Maps of Riemann Surfaces*, Lecture Notes in Math. **1424**, Springer Verlag, (1990).
[35] Cabrerizo, J. L. Fernández, M., Gómez, J. S.: *On the existence of almost contact structure and the contact magnetic field*, Acta Math. Hung. **125** (1-2), 191-199 (2009).
[36] Calvaruso, C., Castrillón López, M.: *Pseudo-Riemannian Homogeneous Structures*, Springer Verlag, (2019).
[37] Calvaruso, G., Fino, A.: *Five-dimensional K-contact Lie algebras*, Monatsh. Math. **167** (1), 35-59 (2012).
[38] Cartan, E.: *Leçon sur la geometrie des espaces de Riemann*, Second Edition, Gauthier-Villards, Paris, (1946).

[39] Calviño-Louzao, E., Ferreiro-Subrido, M., García-Río, E., Vázquez-Lorenzo, R.: *Homogeneous Riemannian structures in dimension three*, Rev. Real. Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **117**, Article number 70 (2023).
[40] Castrillón López, M., M. Gadea, P. M., Swann, A.: *Homogeneous structures on real and complex hyperbolic spaces*, Illinois J. Math. **53** (2), 561-574 (2009).
[41] Castrillón López, M., M. Gadea, P. M., Swann, A.: *The homogeneous geometries of real hyperbolic space*, Mediter. J. Math. **10** (2), 1011-1022 (2013).
[42] Castrillón López, M., Luján, I.: *Reduction of homogeneous Riemannian structures*, Proc. Edinb. Math. Soc. (2) **58** (1), 81-106 (2015).
[43] Chen, B. Y., Vanhecke, L.: *Isometric, holomorphic and symplectic reflections*, Geom. Dedicata **29** (3), 259-277 (1989).
[44] Chinea, D., Gonzalez, C.: *An example of an almost cosymplectic homogeneous manifolds*, Differential Geometry Peñiscola 1985, Lecture Notes in Math. **1209**, 133-142 (1986).
[45] Chinea, D., Gonzalez, C.: *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl. (4) **156**, 15-36 (1990).
[46] Chinea, D., Gonzalez, C.: *Classification of almost contact metric structures*, Rev. Roum. Math. Pures Appl. **37** (3), 199–211 (1992).
[47] Cho, J. T.: *On some classes of almost contact metric manifolds*, Tsukuba J. Math. **19** (1), 201-217 (1995).
[48] Cho, J. T., Inoguchi, J.: *Pseudo-symmetric contact 3-manifolds*, J. Korean Math. Soc. **42** (5), 913-932 (2005).
[49] Cho, J. T., Inoguchi, J.: *Pseudo-symmetric contact 3-manifolds*. II. *When is the tangent sphere bundle over a surface pseudo-symmetric ?*, Note Mat. **27** (1), 119-129 (2007).
[50] Cho, J. T., Inoguchi, J.: *Curvatures and symmetries of tangent sphere bundles*, Houston J. Math. **37** (4), 1125-1142.
[51] Cho, J. T., Inoguchi, J., Lee, J.-E.: *Pseudo-symmetric contact 3-manifolds*. III, Colloq. Math. **114** (1), 77–98 (2009)
[52] Cleyton, R., Moroianu, A., Semmelmann, U.: *Metric connections with parallel skew-symmetric torsion*, Adv. Math. **378**, 107519 (2021).
[53] D'Atri, J. E., Nickerson, H. K.: *Geodesic symmetries in spaces with special curvature tensor*, J. Diff. Geom. **9**, 251-262 (1974).
[54] D'Atri, J. E., Nickerson, H. K.: *Geodesic spheres and symmetries in naturally reductive homogeneous spaces*, Michigan Math. J. **22**, 71-76 (1975).
[55] D'Atri, J. E., Ziller, W.: *Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups*, Mem. Am. Math. Soc. **215** (1979).
[56] De Nicola, A., Dileo, G., Yudin, I.: *On nearly Sasakian and nearly cosymplectic manifolds*, Ann. Mat. Pura Appl. (4) **197** (1), 127-138 (2018).
[57] Dileo, G., Lotta, A.: *A classification of spherical symmetric CR manifolds*, Bull. Aust. Math. Soc. **80**, 251-274 (2009).
[58] Dombrowski, P.: *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210**, 73-88 (1962).
[59] Dorfmeister, J. F., Inoguchi, J., Kobayashi, S.-P.: *Constant mean curvature surfaces in hyperbolic 3-space via loop groups*, J. Reine Angew. Math. **686** (1), 1-36 (2014).
[60] Dorfmeister, J. F., Inoguchi, J., Kobayashi, S.-P.: *A loop group method for affine harmonic maps into Lie groups*, Adv. Math. **298**, 207-253 (2016).
[61] Dorfmeister, J. F., Inoguchi, J., Kobayashi, S.-P.: *A loop group method for minimal surfaces in the three-dimensional Heisenberg group*, Asian J. Math. **20** (3), 409-448 (2016).
[62] Draper, C., Garvín, A., Palomo, F. J.: *Invariant affine connections on odd-dimensional spheres*, Ann. Glob. Anal. Geom. **49**, 213-251 (2016).
[63] Eliashberg, Ya.: *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. l'institut Fourier **42** (1-2), 165-192 (1992).
[64] Erjavec, Z., Inoguchi, J.: *Geodesics and magnetic curves in the 4-dim almost Kähler model space* F^4, Complex Manifolds **11**, Article ID 20240001, 33 p. (2024).
[65] Falbel, E., Gorodski, C.: *On contact sub-Riemannian symmetric spaces*, Ann. Sci. Éc.Norm. Supér. (4) **28** (5), 571–589 (1995).
[66] Falbel, E., Gorodski, C.: *Sub-Riemannian homogeneous spaces in dimensions 3 and 4*, Geom. Dedicata **62** (3), 227-252 (1996).
[67] Falbel, E., Gorodski, C., Veloso, J. M.: *Conformal sub-Riemannian geometry in dimension 3*, Mat. Contemp. **9**, 61-73 (1995).
[68] Ferus, D., Pinkall, U.: *Constant curvature 2-spheres in the 4-sphere*, Math. Z. **200** (2), 265-271 (1989).
[69] Fino, A.: *Almost contact homogeneous manifolds*, Riv. Mat. Univ. Parma (5) **3**, 321-332 (1994).
[70] Fino, A.: *Almost contact homogeneous structures*, Boll. Un. Mat. Ital. A (7) **9**, 299–311 (1995).
[71] Fino, A., Grantcharov, G.: *Properties of manifolds with skew-symmetric torsion and special holonomy*, Adv. Math. **189** (2), 439-450 (2004).
[72] Foreman, B. J.: *K-contact Lie groups of dimension five or greater*, Kodai Math. J. **34** (1), 79-84 (2011).
[73] Friedrich, T., Ivanov, S.: *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2), 303-335 (2002),.
[74] Friedrich, T., Ivanov, S.: *Almost contact manifolds, connections with torsion, and parallel spinors*, J. Reine Angew. Math. **559**, 217-236 (2003).
[75] Fujioka, A., Inoguchi, J.: *Spacelike surfaces with harmonic inverse mean curvature*, J. Math. Sci. Univ. Tokyo **7**, 657-698 (2000).
[76] Gadea, P. M., González-Dávila, J. C., Oubiña, J. A.: *Cyclic metric Lie groups*, Monatsh. Math. **176** (2), 219-239 (2015).
[77] Gadea, P. M., González-Dávila, J. C., Oubiña, J. A.: *Cyclic homogeneous Riemannian manifolds*, Ann. Mat. Pura Appl. (4) **195** (5), 1619-1637 (2016).
[78] Gadea, P. M., González-Dávila, J. C., Oubiña, J. A.: *Homogeneous spin Riemannian manifolds with the simplest Dirac operator*, Adv. Geom. **18** (3), 289-302 (2018).
[79] Gadea, P. M., Oubiña, J. A.: *Homogeneous Riemannian structures on Berger 3-spheres*, Proc. Edinb. Math. Soc. (2) **48**, (2), 375-387 (2005).
[80] Gadea, P. M., Oubiña, J. A.: *Homogeneous Kähler and Sasakian structures related to complex hyperbolic spaces*, Proc. Edinb. Math. Soc. (2) **53** (2), 393-413 (2010).
[81] Geiges, G.: *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics **109**, (2008).
[82] Gray, A.: *Riemannian manifolds with geodesic symmetries of order 3*, J. Differ. Geom. **7**, 343-369 (1972).
[83] Gonzalez, C., Chinea, D.: *Quasi-Sasakian homogeneous structures on the generalized Heisenberg group $H(p, 1)$*, Proc. Am. Math. Soc. **105**, 173-184 (1989).
[84] Gordeeva, I.A., Pan'zhenskii, V.I., Stepanov, S.E.: *Riemann–Cartan manifolds*. J. Math. Sci. **169**, 342-361 (2010).
[85] Gordon, C. S.: *Naturally reductive homogeneous Riemannian manifolds*, Canadian J. Math. **37** (3), 467-487 (1985).
[86] Ha, K.Y., Lee, J.B.: *Left invariant metrics and curvatures on simply connected three-dimensional Lie groups*, Math. Nachr. **282** (6), 868-898 (2009).
[87] Halverscheid, S., Iannuzzi, A.: *On naturally reductive left-invariant metrics of* $\mathrm{SL}(2, \mathbb{R})$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **5** (2), 171-187 (2006).
[88] Hasegawa, Y.: *C-flows on a Lie group for Euler equations*, Nagoya Math. J. **40**, 67-84 (1970)
[89] Hassani M., Ahmadi, P.: *Isometric actions on the four dimensional Minkowski spacetime*, Differential Geom. Appl. **88**, Article number 102007, (2023).
[90] Hatakeyama, Y.: *Some notes on differentiable manifolds with almost contact structures*, Tôhoku Math. J. (2) **15**, 176-181 (1963).
[91] Heintze, E.: *On homogeneous manifolds of negative curvature*, Math. Ann. **211**, 23-34 (1974).

[92] Helgason, S.: *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics **34**, American Mathematical Society (2001).
[93] Hsu, C.: *On some structures which are similar to the quaternion structure*, Tôhoku Math. J. (2) **12**, 403–428 (1960).
[94] Inoguchi, J.: *Timelike surfaces of constant mean curvature in Minkowski* 3*-space*, Tokyo J. Math. **21**, 141-152 (1998).
[95] Inoguchi, J.: *Minimal surfaces in* 3*-dimensional solvable Lie groups*, Chinese Ann. Math. B. **24**, 73-84 (2003).
[96] Inoguchi, J.: *Invariant minimal surfaces in real special linear group of degree* 2, Ital. J. Pure Appl. **16**, 61-80 (2004).
[97] Inoguchi, J.: *Minimal surfaces in* 3*-dimensional solvable Lie groups* II, Bull. Austral. Math. Soc. **73**, 365-374 (2006).
[98] Inoguchi, J.: *Pseudo-symmetric Lie groups of dimension* 3, Bull. Fac. Edu. Utsunomiya Univ. Sect. 2 **57**, 1-5 (2007).
http://hdl.handle.net/10241/00004810
[99] Inoguchi, J.: *On homogeneous contact* 3*-manifolds*, Bull. Fac. Edu. Utsunomiya Univ. Sect. 2 **59**, 1-12 (2009).
http://hdl.handle.net/10241/00004788
[100] Inoguchi, J.: *A note on almost contact Riemannian* 3*-manifolds*, Bull. Yamagata Univ. Nat. Sci. **17** (1), 1-6 (2010).
[101] Inoguchi, J.: *A note on almost contact Riemannian* 3*-manifolds.* II, Bull. Korean Math. Soc. **54** (1), 85-97 (2017).
[102] Inoguchi, J.: *Characteristic Jacobi operator on almost cosymplectic* 3*-manifolds*, Internat. Elect. J. Geom. **12** (2), 276-299 (2019).
[103] Inoguchi, J.: *Characteristic Jacobi operator on almost Kenmotsu* 3*-manifolds*, Internat. Elect. J. Geom. **16** (2), 464–525 (2023).
[104] Inoguchi, J.: *Differential geometry of the unit tangent sphere bundle over the* 3*-sphere*, in preparation
[105] Inoguchi, J., Kumamoto, T., Ohsugi, N., Suyama, Y.: *Differential Geometry of curves and surfaces in* 3*-dimensional homogeneous spaces* I, Fukuoka Univ. Sci. Rep., **29** (2), 155-182 (1999).
[106] Inoguchi, J., Kumamoto, T., Ohsugi, N., Suyama, Y.: *Differential Geometry of curves and surfaces in* 3*-dimensional homogeneous spaces* II, Fukuoka Univ. Sci. Rep., **30** (1), 17-48 (2000).
[107] Inoguchi, J., Kumamoto, T., Ohsugi, N., Suyama, Y.: *Differential Geometry of curves and surfaces in* 3*-dimensional homogeneous spaces* III, Fukuoka Univ. Sci. Rep., **30** (20), 130-160 (2000).
[108] Inoguchi, J., Kumamoto, T., Ohsugi, N., Suyama, Y.: *Differential Geometry of curves and surfaces in* 3*-dimensional homogeneous spaces* IV, Fukuoka Univ. Sci. Rep., **30** (20), 161-168 (2000).
[109] Inoguchi, J., Kuwabara, K., Naitoh, H.: *Grassmann geometry on the* 3*-dimensional Heisenberg group*, Hokkaido Math. J. **34** (2), 375-391 (2005).
[110] Inoguchi, J., Lee, J,-E.: *Affine biharmonic curves in* 3*-dimensional homogeneous geometries*, Mediterr. J. Math. **10** (2013), 571-592 (2013)
[111] Inoguchi, J., Lee, S.: *A Weierstrass representation for minimal surfaces in* Sol., Proc. Am. Math. Soc. **136**, 2209-2216 (2008).
[112] Inoguchi, J., Munteanu, M. I.: *Periodic magnetic curves in Berger spheres*, Tôhoku Math. J. (2) **69** (1), 113–128 (2017).
[113] Inoguchi, J., Munteanu, M. I.: *Magnetic curves in tangent sphere bundles.* II, J. Math. Anal. Appl. **466** (2), 1570-1581.
[114] Inoguchi, J., Munteanu, M. I.: *Magnetic curves in the real special linear group*, Adv. Theor. Math. Phys. **23** (8), 2161-2205 (2019).
[115] Inoguchi, J., Munteanu, M. I.: *Magnetic curves in tangent sphere bundles*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM **113** (3), 2087-2112 (2019).
[116] Inoguchi, J., Munteanu, M. I.: *Homogeneity of magnetic trajectories in the real special linear group*, Proc. Am. Math. Soc. **152** (3), 1287–1300, (2024).
[117] Inoguchi, J., Munteanu, M. I.: *Magnetic curves in tangent sphere bundles.* III, submitted.
[118] Inoguchi, J., Munteanu, M. I.: *Homogeneity of magnetic trajectories in the Berger sphere*, submitted.
[119] Inoguchi, J., Munteanu, M. I.: *The oscillator group and homogeneous magnetic trajectories in the Heisenberg group*, submitted.
[120] Inoguchi, J., Naitoh, H.: *Grassmann geometry on the* 3*-dimensional unimodular Lie groups* I, Hokkaido Math. J. **38** (3), 427-496 (2009).
[121] Inoguchi, J., Naitoh, H.: *Grassmann geometry on the* 3*-dimensional unimodular Lie groups* II, Hokkaido Math. J. **40** (3), 411-429 (2011).
[122] Inoguchi, J., Naitoh, H.: *Grassmann geometry on the* 3*-dimensional non-unimodular Lie groups*, Hokkaido Math. J. **48** (2), 385-406 (2019).
[123] Inoguchi, J., Naitoh, H.: *Grassmann geometry on* $H^2 \times \mathbb{R}$, submitted.
[124] Inoguchi, J., Ohno, Y.: *Homogeneous structures of* 3*-dimensional Sasakian space forms*, Tsukuba J. Math., to appear.
[125] Inoguchi, J., Ohno, Y.: *Homogeneous structures of* 3*-dimensional Lie groups*, submitted.
[126] Inoguti, J.: *Rotations and* φ*-symmteric spaces*, Math. J. Toyama Univ. **15**, 123-130 (1992).
[127] Inoguti, J.: *Nearly cosymplectic manifolds with rotations of order* 3, preprint (1993).
[128] Inoguti, J., Sekizawa, M.: *Symmetries which preserve the characteristic vector fields of* K*-contact manifolds*, Note Mat. **13** (2), 229-236 (1993).
[129] Ise, M., Takeuchi, M.: *Lie groups I. Lie groups II.* (Translated by Nomizu), Translations of Mathematical Monographs. **85**, American Mathematical Society (1991).
[130] Itoh, M.: *Invariant connections and Yang–Mills solutions*, Trans. Am. Math. Soc. **267** (1981), 229-236 (1981).
[131] Janssense, D., Vanhecke, L.: *Almost contact structures and curvature tensors*, Kodai Math. J. **4**, 1-27 (1981).
[132] Jiménez, J. A.: *Existence of Hermitian* n*-symmetric spaces of non-commutative naturally reductive spaces*, Math. Z. **196**, 133-139 (1987). Addendum: Math. Z. **197**, (3), 455-456 (1988).
[133] Jiménez, J. A.: *The focal locus of a Riemannian* 4*-symmetric space*, Can. Math. Bull. **31** (2), 175-181 (1988).
[134] Jiménez, J. A.: *Riemannian* 4*-symmetric spaces*, Trans. Am. Math. Soc. **306** (2), 715-734 (1988).
[135] Jiménez, J. A., Kowalski, O.: *The classification of* φ*-symmetric Sasakian manifolds*, Monatsh. Math. **115** (1-2), 83-98 (1993).
[136] Kato, T., Motomiya, K.: *A study on certain homogeneous spaces*, Tohôku Math. J. (2) **21**, 1-20 (1969).
[137] Katsuda, A.: *A pinching problem for locally homogeneous spaces*, J. Math. Soc. Japan **41** (1), 57-74 (1989).
[138] Kaup, W., Zaitsev, D.: *On symmetric Cauchy–Riemann manifolds*, Adv. Math. **149**, 145-181 (2000).
[139] Kenmotsu, K.: *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. (2) **24** (1), 93-103 (1972).
[140] Kim, I. B., Takahashi, T.: *Isoparametric hypersurfaces in a space form and metric connections*, Tsukuba J. Math. **21** (1), 15-28 (1997).
[141] Klingenberg, W., Sasaki, S.: *The tangent sphere bundle of a* 2*-sphere*, Tôhoku Math. J. **27** (1), 45-57 (1975).
[142] Kričhenko, V. F.: *On homogeneous Riemannian spaces with invariant tensor structures*, Sov. Math. Dokl. **21**, 734-737 (1980).
[143] Kobayashi, S.: *Principal fibre bundles with the 1-dimensional toroidal group*, Tôhoku Math. J. (2) **8**, 29-45 (1956).
[144] Kobayashi, S.: *Homogeneous Riemannian manifolds of negative curvature*, Tôhoku Math. J. (2) **14**, 413-415 (1962).
[145] Kobayashi, S., Nomizu, K.: *Foundations of differential geometry*. II, Interscience Tracts in Pure and Applied Mathematics 15, New York-London: Interscience Publishers, a division of John Wiley and Sons. (1969).
[146] Koda, T., Watanabe, Y.: *Homogeneous almost contact Riemannian manifolds and infinitesimal models*, Boll. Un. Mat. Ital. (7) **11-B**, suppl., 11-24 (1997).

[147] Kodama, H., Takahara, A., Tamaru, H.: *The space of left-invariant metrics on a Lie group up to isometry and scaling*, Manuscr. Math. **135** (1-2), 229-243 (2011).
[148] Kostant, B.: *A characterization of invariant affine connections*, Nagoya Math. J. **16**, 35-50 (1960).
[149] Kowalski, O.: *Generalized Symmetric Spaces*, Lecture Notes in Math. **805**, Springer Verlag (1980).
[150] Kowalski, O.: *On strictly locally homogeneous Riemannian manifolds*, Differential Geom. Appl. **7** (2), 131-137 (1997).
[151] Kowalski, O., Szenthe, J.: *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*, Geom. Dedicata **81**, 209-214 (2000).
[152] Kowalski, O., Tricerri, F.: *Riemannian manifolds of dimension $n \leq 4$ admitting a homogeneous structure of class $\mathcal{T}_2$*, Conf. Semin. Mat. Univ. Bari **222**, 24 pages. (1987).
[153] Kowalski, O., Vanhecke, L.: *A generalization of a theorem on naturally reductive homogeneous spaces*, Proc. Am. Math. Soc. 91(3), 433-435 (1984).
[154] Kowalski, O., Vanhecke, L.: *Riemannian manifolds with homogeneous geodesics*, Boll. Unione Mat. Ital., VII. Ser., **B 5** (1), 189-246 (1991).
[155] Kowalski, O., Węgrzynowski, S.: *A classification of five-dimensional ϕ-symmetric spaces*, Tensor (N.S.) **46**, 379-386 (1987).
[156] Kurihara, H., Tojo, K.: *Involutions of compact Riemannian 4-symmetric spaces*, Osaka J. Math. **45** (3), 643-689 (2008).
[157] Kurihara, H., Tojo, K.: *Involutions on a compact 4-symmetric space of exceptional type*, Osaka J. Math. 5**52** (4), 1101-1125 (2015).
[158] Kuwabara, K.: *Grassmann geometry on the groups of rigid motions on the Euclidean and the Minkowski planes*, Tsukuba J. Math. **30** (1), 49-59 (2006).
[159] Maeda, S., Tanabe, H.: *Redefinition of Berger spheres from the viewpoint of submanifold geometry*, Tôhoku Math. J., to appear.
[160] Martín Cabrera, F.: *On the classification of almost contact metric manifolds*, Differ. Geom. Appl. **64**, 13-28 (2019).
[161] Matsuzoe, H., Inoguchi, J.: *Statistical structures on tangent bundles*, Appl. Sci. **5** (1), 55-57 (2003).
[162] Meeks, W.H., III. and Pérez, J., *Constant mean curvature surfaces in metric Lie groups*, Geometric Analysis: Partial Differential Equations and Surfaces, Contemp. Math. **570**, 25-110 (2012)
[163] Milnor, J.: *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21**, 293-329 (1976).
[164] Morimoto, A.: *On normal almost contact structures*, J. Math. Soc. Japan **15**, 420-436 (1963).
[165] Motomiya, K.: *A study on almost contact manifolds*, Tôhoku Math. J. (2) **20**, 73-90 (1968).
[166] Ni, L., Zheng, F.: *On Hermitian manifolds whose Chern connection is Ambrose-Singer*, Trans. Am. Math. Soc. **376** (9), 6681-6707 (2023).
[167] Nicolodi, L.; Vanhecke, L.: *Rotations and Hermitian symmetric spaces*, Monatsh. Math. **109** (4), 279-291 (1990).
[168] Nicolodi, L.; Vanhecke, L.: *Rotations on a Riemannian manifold*, Recent topics in differential geometry, Proc. Workshop/Puerto de la Cruz/Spain 1990, Ser. Inf. **32**, 89-101 (1991).
[169] Nicolodi, L.; Vanhecke, L.: *Harmonic and isometric rotations around a curve*, Illinois J. Math. **37** (1), 85-100.
[170] Nomizu, K.: *Invariant affine connections on homogeneous spaces*, Am. J. Math. **76**, 33-56 (1954).
[171] Ogiue, K.: *On fiberings of almost contact manifolds*, Kodai Math. Sem. Rep. **17** (1), 53-62 (1965).
[172] Ohnita, Y.: *Canonical connections of a Sasakian manifold and invariant submanifolds with parallel second fundamental form*, Proceedings of The 23rd International Differential Geometry Workshop on Submanifolds in Homogeneous Spaces and Related Topics **23**, 31-40 (2021).
[173] Ohnita, Y.: *Parallel Kähler submanifolds and R-spaces*, Differential Geometry and Global Analysis in Honor of Tadashi Nagano, Contemp. Math. **777**, 163-184 (2022).
[174] Ohno, Y.: *Homogeneous structures on $\mathbf{S}^2 \times \mathbf{R}$ and $\mathbf{H}^2 \times \mathbf{R}$*, Tsukuba J. Math. **47** (2) 239-246 (2023).
[175] Okumura, M.: *Some remarks on space with a certain contact structures*, Tôhoku Math. J. (2) **14**, 135-145 (1962).
[176] Olmos, C., Reggiani, S.: *The skew-torsion holonomy theorem and naturally reductive spaces*, J. Reine Angew. Math. **664**, 29-53 (2012).
[177] Olmos, C., Reggiani, S.: *A note on uniqueness of the canonical connection of a naturally reductive space*, Monats. Math. **172** (3-4), 379–386 (2013).
[178] Olmos, C., Sánchez, C.: *A geometric characterization of the orbits of s-representations*, J. Reine Angew. Math. **420**, 195-202 (1991).
[179] Olszak, Z.: *Normal almost contact manifolds of dimension three*, Ann. Pol. Math. **47**, 42-50 (1986).
[180] Omori, H.: *Infinite dimensional Lie Transformation Groups*, Lecture Notes in Math. **427**, Springer-Verlag, Berlin, (1974).
[181] Omori, H.: *Infinite-dimensional Lie Groups*, Translations of Mathematical Monographs **158**, American Mathematical Society, Providence, RI, (1997).
[182] O'Neill, B.: *Semi Riemannian Geometry with Application to Relativity*, Academic Press, (1983).
[183] Pastore, A. M.: *On the homogeneous Riemannian structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$*, Geom. Dedicata **30**, 235-246 (1989).
[184] Pastore, A. M.: *Reducibility of homogeneous Riemannian structures of the class $\mathcal{T}_3$ in low dimension*, Geom. Dedicata **38** (2), 121-136 (1991).
[185] Pastore, A. M.: *Canonical connections with an algebraic curvature tensor field on naturally reductive spaces*, Geom. Dedicata **43** (3), 351-361 (1992).
[186] Pastore, A. M.: *Homogeneous representations of the hyperbolic spaces related to homogeneous structures of class $\mathcal{T}_1 \oplus \mathcal{T}_3$*, Rend. Mat. Appl. (7) **12**, (2), 445–453 (1992).
[187] Pastore, A. M., Verroca, F.: *Some results on the homogeneous Riemannian structures of class $\mathcal{T}_1 \oplus \mathcal{T}_2$*, Rend. Mat. Appl. (7) **11** (1), 105-121 (1991).
[188] Patrangenaru, V.: *Classifying 3- and 4-dimensional homogeneous Riemannian manifolds by Cartan triples*, Pacific J. Math. **173**, 511-532 (1996).
[189] Patrangenaru, V.: *5 dimensional strictly locally homogeneous Riemannian manifolds* Period. Math. Hungar. **45** (1-2), 123-129(2002).
[190] Perrone, D.: *Homogeneous contact Riemannian three-manifolds*, Illinois J. Math. **42**, 243-256 (1998)
[191] Perrone, D.: *Classification of homogeneous almost cosymplectic three-manifolds*, Differ. Geom. Appl. **30** (1), 49-58 (2012).
[192] Perrone, D.: *Classification of homogeneous almost α-coKähler three-manifolds*, Differ. Geom. Appl. **59**, 66-90 (2018).
[193] Perrone, D.: *Left-invariant almost α-coKähler structures on 3D semidirect product Lie groups*, Int. J. Geom. Methods Mod. Phys. **16** (1), Article ID 1950011, 18 pp. (2019).
[194] Prüfer, F.: *On compact Riemannian manifolds with volume-preserving symmetries*, Ann. Glob. Anal. Geom. **7**, 133-140 (1989).
[195] Rastrepina, A. O., Surina, O. P.: *Invariant almost contact structures and connections on the Lobachevsky space* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. (2023), (2), 47–56 (2023). English translation: Russian Math. (Iz. VUZ) **67** (2), 43-51 (2023).
[196] Salvai, M.: *Spectra of unit tangent bundles of compact hyperbolic Riemann surfaces*, Ann. Global Anal. Geom. **16**, 357-370 (1998).
[197] Salvai, M.: *Density of periodic geodesics in the unit tangent bundle of a compact hyperbolic surface*, Rev. Uni. Mat. Argentina **41**, 99-105 (1999).
[198] Salvai, M.: *On the geometry at inifinity of the universal covering of $Sl(2, \mathbb{R})$*, Rend. Sem. Mat. Univ. Padova **104**, 91-108 (2000).
[199] Sasaki, S.: *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. **10**, (3), 338-354 (1958).
[200] Sasaki, S.: *On the differential geometry of tangent bundles of Riemannian manifolds* II, Tôhoku Math. J. **14**, (2), 146–155 (1962).

[201] Sasaki, S.: *Notes on my mathematical works*, Shigeo Sasaki. Selected Papers (S. Tachibana ed), Kinokuniya, 3–30, (1985).
[202] Sasaki, S., Hatakeyama, Y.: *On differential manifolds with certain structures which are closely related to almost contact structure* II, Tôhoku Math. J. (2) **13**, 282-294 (1961).
[203] Schlarb, M.: *Covariant derivatives on homogeneous spaces. Horizontal lifts and parallel transport*, J. Geom. Anal. **34** (5), article number 150, 43 p., (2024).
[204] Sekigawa, K.: *Notes on some curvature homogeneous spaces*, Tensor, New Ser. **29**, 255-258 (1975).
[205] Sekigawa, K.: 3-*dimensional homogeneous Riemannian manifolds*. I, Sci. Rep. Niigata Univ., Ser. A **14**, 5-14 (1977).
[206] Sekigawa, K.: 3-*dimensional homogeneous Riemannian manifolds*. II, Sci. Rep. Niigata Univ., Ser. A **15**, 71-78 (1978).
[207] Sekigawa, K.: *Notes on homogeneous almost Hermitian manifolds*, Hokkaido Math. J. **7**, 206-213 (1978).
[208] Strichartz, R. S.:, *Sub-Riemannian geometry*, J. Differ. Geom. **24**, 221-263 (1986). Correction ibid. **30** (2), 595-596 (1989).
[209] Takahashi, Toshio: *Sasakian ϕ-symmetric spaces*, Tôhoku Math. J. (2) **29**, 91-113 (1977).
[210] Takahashi, Tsunoro: *An isometric immersion of a homogeneous Riemannian manifold of dimension* 3 *in the hyperbolic space*, J. Math. Soc. Japan **23**, 649–661 (1971).
[211] Tamaru, H.: *Riemannian g. o. spaces fibered over irreducible symmetric spaces*, Osaka J. Math. **36**, 835-851 (1999).
[212] Tanaka, N.: *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds*, Lecture in Math. Kyoto Univ. **9**, Kinokuniya Book Store (1975).
[213] Tanaka, N.: *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math. **2**, 131–190 (1976).
[214] Tanno, S.: *Some transformations on manifolds with almost contact and contact metric structures* I, II, Tôhoku Math. J. (2) **15**, 140-147, 322-331 (1963).
[215] Tanno, S.: *A theorem on regular vector fields and its applications to almost contact structures*, Tôhoku Math. J. (2) **17** (3), 235-238 (1965) .
[216] Tanno, S.: *Locally symmetric K-contact Riemannian manifolds*, Proc. Japan Acad. **43**, 581-583 (1967).
[217] Tanno, S.: *The automorphism groups of almost contact Riemannian manifolds*, Tôhoku Math. J. (2) **21**, 21-38 (1969).
[218] Tanno, S.: *Sasakian manifolds with constant φ-holomorphic sectional curvature*, Tôhoku Math. J. (2) **21**, 501-507 (1969).
[219] Tanno, S.: *Variational problems on contact Riemannian manifolds*, Trans, Am. Math. Soc. **314** (1), 349-379 (1989).
[220] Thurston, W. M.: *Three-dimensional Geometry and Topology* I (S. Levy ed.), Princeton Math. Series. **35**, (1997).
[221] Tojo, K.: *Kähler C-spaces and k-symmetric spaces*, Osaka J. Math. **34** (4), 803-820 (1997).
[222] Tricerri, F., Vanhecke, L.: *Homogeneous Structures on Riemannian Manifolds*, Lecture Notes Series, London Math. Soc. **52**, Cambridge Univ. Press (1983).
[223] Tricerri, F., Vanhecke, L.: *Naturally reductive homogeneous spaces and generalized Heisenberg groups*, Compositio Math. **52**, 389-408 (1984).
[224] Vanhecke, L., Willmore, T.J.: *Interaction of tubes and spheres*, Math. Ann. **263**, 31-42 (1983).
[225] Vezzoni, L.: *Connections on contact manifolds and contact twistor space*, Israel J. Math. **178**, 253-267 (2010).
[226] Vranceanu, G.: *Lecons de Geometrie Differentielle* I, Ed. Acad. Rep. Roum, Bucarest, (1947).
[227] Watanabe, Y., Tricerri, F.: *Characterizations of ϕ-symmetric spaces in terms of the canonical connection*, C. R. Math. Rep. Acad. Sci. Canada **15** (2-3), 61-66 (1993).
[228] Webster, S. M.: *Pseudohermitian structures on a real hypersurface*, J. Differential Geom. **13**, 25-41 (1978).
[229] Witte, D.: *Cocompact subgroups of semisimple Lie groups*, Lie algebra and related topics (Madison, WI, 1988), 309–313, Contemp. Math. **110**, Am. Math. Soc., Providence, RI, (1990).

Affiliations

J. INOGUCHI
ADDRESS: Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
E-MAIL: inoguchi@math.sci.hokudai.ac.jp
ORCID ID:0000-0002-6584-5739