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Some Data Dependence Results From Using $\mathscr{C}$-Class Functions in Partial Metric Spaces

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Abstract

This research paper examines the data dependence of fixed point sets for pseudo-contractive multifunctions in partial metric spaces using the notion of $\mathscr{C}$-class functions. By building upon previous findings from the literature, this work sheds more light on some new perspectives as well as generalizations on this issue. To illustrate how the $\mathscr{C}$-class function can be applied to study the data dependence of fixed point sets for a certain pseudo-contractive multifunction, an illustrative example is given.

1. Introduction

The fixed point theory is a powerful tool with numerous applications in various fields such as biology, chemistry, economics, engineering, game theory, computer science, and mathematical modeling [1, 2]. Recent developments in fixed point theory have focused on extending classical results to more general abstract spaces like b-metric spaces, partial metric spaces, and fuzzy metric spaces [3]. One notable advancement is the introduction of $\mathscr{C}$-class functions [4], which have been used to prove fixed point theorems in different abstract spaces, particularly in the context of partial metric spaces. $\mathscr{C}$-class functions provide a unified framework for studying various types of contractions and have applications in solving differential equations, integral equations, and variational inequalities [5, 6].
The study of partial metric spaces began in 1994, when Matthews introduced this generalization of traditional metric spaces in [7]. Since then, partial metric spaces have found applications across many fields because of their ability to represent asymmetric distance relationships [8]. During the same period, there was significant progress in multivalued mapping research, which greatly contributed to the development of generalized fixed point theory [9–11]. Multifunctions emerged as a natural approach to addressing problems that involved non-uniqueness or set-valued constraints. Initially, the focus was on establishing fixed point results for multifunctions defined on traditional metric and topological spaces.
However, as people became more interested in using these methods in real-life situations modeled by partial metrics, it became necessary to come up with new ways to think about data dependence for multifunctions in this new setting [12]. Data dependence properties play a crucial role in examining how perturbations in the domain affect or propagate to the range sets. In the case of single-valued mappings on metric spaces, classic results have established strong connections between input and output distances (see [13–17]).
However, for multifunctions whose domain and range reside in different partial metric spaces, new approaches were required. Researchers created the partial Hausdorff metric [18] to measure the distance between nonempty subsets using the basic partial metrics. This allowed generalizing key notions like continuity, contraction properties, and more.
In the beginning, researchers came up with the weak contraction and fixed point theorems for multifunctions that behave in certain ways when contracted with respect to the induced partial Hausdorff metric. The mapping had fixed points if the partial Hausdorff distance between images of any two points satisfied a Lipschitz-type condition based on their domain distance.
Today, data dependence results for multifunctions defined in partial and more exotic spaces remain an active area of research. Future directions include investigating new contraction conditions, establishing fixed point theorems for alternative structures, and discovering additional applications inspired by practical problems. Overall, the field has grown significantly since its inception, broadening the scope of generalized fixed point theory.

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The second part of this research paper provides a summary of partial metric spaces, $\mathscr{C}$-class functions, and existing results. The primary emphasis is on the significant contributions made by [12].
This section establishes a solid foundation for our main result, which is outlined in Section 3. Through $\mathscr{C}$-class functions, we aim to enhance our understanding of data dependence in partial metric spaces. The final section deals with the implications of our main result, thereby giving us a better understanding of data dependence in this context.

2. Preliminaries

To fully understand the complexities of data dependence in partial metric spaces, one needs a strong foundation. We propose the concept of partial metric spaces according to Matthews's 1994 study on partial metrics [7]. These metric spaces intriguingly extend to non-zero self-distances.
As a result, let us proceed to review the essential characteristics and definitions of partial metric spaces.

Definition 2.1. *[7] The function $p : X \times X \to \mathbb{R}^+$ defines a partial metric on a nonempty set X, where $\mathbb{R}^+$ includes all nonnegative real numbers. If the following four conditions are satisfied for every $x,y,z \in X$, we call the pair (X,p) a partial metric space:*

$\mathfrak{P}_1$*: $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$.*
$\mathfrak{P}_2$*: $p(x,x) \leqslant p(x,y)$.*
$\mathfrak{P}_3$*: $p(x,y) = p(y,x)$*
$\mathfrak{P}_4$*: $p(x,y) + p(z,z) \leqslant p(x,z) + p(z,y)$.*

The partial metric space represented by the pair (X,p).
We define the concept of closed p-balls, $\overline{\mathbb{B}_{p,r}}(x)$, and the open p-balls, $\mathbb{B}_{p,r}(x)$, to simplify our analysis. These sets are defined as

$$\overline{\mathbb{B}_{p,r}}(x) = \{y \in X | p(x,y) \leqslant p(x,x) + r\}, \qquad \mathbb{B}_{p,r}(x) = \{y \in X | p(x,y) < p(x,x) + r\}.$$

We denote the full space X as $\overline{\mathbb{B}_{p,+\infty}}(x)$ to keep things simple. By using this notation, we can express important ideas and claims about p-distance thresholds throughout the whole domain X in a more concise and accurate way.
The metric that is related to p, which is a partial metric on X, can be expressed as a new function $p^s : X \times X \to \mathbb{R}^+$. This formula may be used to get the metric p^s:

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

This metric p^s satisfies all the properties of a metric space: nonnegativity, symmetry, and triangle inequality. Therefore, while p is only a partial metric, the associated metric p^s transforms the partial metric space (X,p) into an actual metric space (X,p^s). We have laid the necessary groundwork to rigorously examine the concept of data dependence within partial metric spaces, a topic we will now explore through theoretical analysis. Let (X,p) be a partial metric space. The following properties hold:

- If $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$, then $\{x_n\}$ is said to converge to a point $x \in X$.
- If a sequence $\{x_n\}$ has a finite limit as n and m approach infinity, it is termed a Cauchy sequence.
- If each Cauchy sequences $\{x_n\}$ in X converge to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$, then the partial metric space (X,p) is complete.

Consider the collection $C^p(X)$, which represents all nonempty closed subsets of the partial metric space (X,p). In this framework, we introduce the following definitions for $x \in X$ and $A,B \in C^p(X)$:

$$\begin{aligned} H_p(A,B) &= \max\{\delta_p(A,B), \delta_p(B,A)\}, \\ &= \max\{\sup\{p(a,B) \mid a \in A\}, \sup\{p(b,A) \mid b \in B\}\}, \end{aligned}$$

such that

$$p(x,A) = \inf\{p(x,a) \mid a \in A\}.$$

Following the established conventions

$$p(x,\emptyset) = +\infty, \quad \delta_p(\emptyset,B) = 0. \tag{2.1}$$

Lemma 2.2. *[5, 19] In a partial metric space (X,p) with $A \subset X$, the equivalence relation $a \in \overline{A} \Leftrightarrow p(a,A) = p(a,a)$ holds. Additionally, $p(a,a) = 0$ and $a \in \overline{A} \Leftrightarrow p(a,A) = 0$, in which $\overline{A}$ represents the closure of A relative to the partial metric p.*

Lemma 2.3. *[20] Consider $x \in X$ and $A \in C^p(X)$ in a partial metric space (X,p). If $\mu > 0$ and $p(x,A) < \mu$, we can find that there is an element a in A such that $p(x,a) < \mu$.*

Furthermore, we introduce the intervals J and J' on the nonnegative real numbers, which include the value 0. These intervals can take the form of $[0,a[$, $[0,a]$, or $[0,+\infty[$, where a represents a nonnegative real number.
The following notations are used for a multivalued mapping $T : X \to 2^X$, where 2^X represents any nonempty subsets of X.

- $Fix(T) = \{x \in X | x \in T(x)\}$.
- $M_T(x,y) = \max\left\{p(x,y), p(x,T(x)), p(y,T(y)), \dfrac{p(x,T(y)) + p(y,T(x))}{2}\right\}$.

Definition 2.4. *[21] On the interval J, a (c)-comparison function or a Bianchini-Grandolfi gauge function is defined as a non-decreasing function $\varphi : J \to J$ that satisfies the condition:*

$$\mathfrak{s}(t) := \sum_{n=0}^{\infty} \varphi^n(t) \text{ is convergent, for all } t \in J,$$

where φ^n represents the n-th iteration of the function φ and $\varphi^0(t) = t$, i.e.,

$$\varphi^0(t) = t, \varphi^1(t) = \varphi(t), \varphi^2(t) = \varphi(\varphi(t)), \ldots, \varphi^n(t) = \varphi(\varphi^{n-1}(t)).$$

By utilizing Bianchini-Grandolfi gauge functions, we gain a more nuanced understanding of data dependence within partial metric spaces. This allows for a thorough examination of the interconnections between space elements. A theorem has been proven building on prior work, particularly the impactful results of [12]. The theorem elucidates the importance of these relationships identified through application of gauge functions, furthering the theoretical foundations of data dependence within this structure.

Theorem 2.5. *Consider a partial metric space (X, p), with $\bar{x} \in X$, $\lambda \in [0,1]$, and $r > 0$, satisfying the condition that the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Let T and F be multivalued mappings from $\overline{\mathbb{B}_{p,r}}(\bar{x})$ to $C^p(X)$. Additionally, let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and upper semicontinuous function, serving as a (c)-comparison function on the interval J. Under the assumption that there exists $\alpha \in J$ satisfying the following two conditions:*

(a) $p(z, F(z)) < \alpha$ *where* $\mathfrak{s}(\alpha) \leqslant (1-\lambda)r, \quad \forall z \in \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leqslant \varphi(M_F(x,y)), \quad \forall x, y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, we have

$$\delta_p(Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, Fix(F)) \leqslant \mathfrak{s}(\mathfrak{M}'),$$

where $\mathfrak{M}' := \sup_{x \in \overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x)).$

This theorem extends several results within the framework of partial metric spaces. Specifically, it expands upon the findings of:

- Azé et al., who presented Proposition 2.1 in their work [13].
- Lim, who presented Lemma 1 in their work [14].
- Geoffroy et al., who presented Proposition 4.5 in their work [22].
- Mansour et al., who presented Theorem 14 in their work [23].

Theorem 2.5 builds upon and generalizes prior work in the area of partial metric spaces.
Ansari's work in [4] proposed $\mathscr{C}$-class functions which have gone a long way in advancing our understanding and analysis of many mathematical phenomena. According to [6], the idea is useful for generalizing important results in fixed point theory. It is more comprehensive than the gauge function by Bianchini-Grandolfi.
Through the use of $\mathscr{C}$-class functions given by knowledge and structure, we can get deeper insights into, and navigate through the complexities of the issue at hand. Indeed, Ansari's contributions are invaluable as they continue shaping and inspiring further research on this subject area thereby leaving a lasting impact on the field of study.

Definition 2.6 ($\mathscr{C}$-class functions). *[4, 5] Assume that there is a continuous mapping $\mathfrak{F} : J \times J' \to \mathbb{R}$. If $\mathfrak{F}$ satisfies these requirements, we will classify it as a $\mathscr{C}$-class function.*

($\mathfrak{F}_1$) *For any $(s,t) \in J \times J'$, we have $s \geqslant \mathfrak{F}(s,t)$.*
($\mathfrak{F}_2$) *If $\mathfrak{F}(s,t) = s$, then the product $st = 0$.*

In addition, note that $\mathfrak{F}(0,0) = 0$ and that $\mathscr{C}$ is the set of all functions of the $\mathscr{C}$-class on $J \times J'$.

In the work [5], the authors introduced the following collections of $\mathscr{C}$-class functions:

Definition 2.7. *[5] The set of functions of the $\mathscr{C}$-class that satisfy these criteria is called $\mathscr{C}_I$:*

- *$\mathfrak{F}(s,t)$ is non-decreasing for both s and t when $(s,t) \in J \times J'$.*
- *For any fixed $t \in J'$, the series*

$$\widetilde{\mathfrak{w}}(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s,t)$$

converges for all $s \in J$. The function $\mathfrak{F}$ is defined as follows, and $\mathfrak{F}^n$ represents the n-th iteration of this function:

$$F^0(s,t) = s, \mathfrak{F}^1(s,t) = \mathfrak{F}(s,t), and \mathfrak{F}^{n+1}(s,t) = \mathfrak{F}(\mathfrak{F}^n(s,t), t).$$

Definition 2.8. *[5] $\mathscr{C}_{II}$ comprises a set of $\mathscr{C}$-class functions that adhere to the following specifications:*

- *$\mathfrak{F}(s,t)$ exhibits non-decreasing behavior in s and non-increasing behavior in t.*
- *For any given $t \in J'$, the series*

$$\widetilde{\mathfrak{w}}(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s,t)$$

converges for every $s \in J$, where the n-th iteration of the function $\mathfrak{F}$ with the following recurrence relation is represented as $\mathfrak{F}^n$:

$$\mathfrak{F}^0(s,t) = s, \mathfrak{F}^1(s,t) = \mathfrak{F}(s,t), and \mathfrak{F}^{n+1}(s,t) = \mathfrak{F}(\mathfrak{F}^n(s,t), \mathfrak{F}^n(s,t))$$

Here are some examples of functions belonging to both $\mathscr{C}_I$ and $\mathscr{C}_{II}$, as presented in Ansari et al. [5]. These examples illustrate the definitions provided in Definition 2.7 and Definition 2.8.

Example 2.9.
- *Given the functions* $\mathfrak{F}(s,t) = s-t$ *and* $\widetilde{\mathfrak{w}}(s,t) = 2s-t$, *it can be concluded that* $\mathfrak{F} \in \mathscr{C}_{II}$.
- *For* $\mathfrak{F}(s,t) = \lambda s$ *with* $\lambda \in [0,1)$, $\widetilde{\mathfrak{w}}(s,t) = \dfrac{s}{1-\lambda}$ *is derived. Hence,* $\mathfrak{F}$ *belongs to* $\mathscr{C}_I \cap \mathscr{C}_{II}$.
- *Since* $\widetilde{\mathfrak{w}}(s,t) = \mathfrak{s}(s)$ *and* φ *is a (c)-comparison function on* J, $\mathfrak{F}$ *belongs to* $\mathscr{C}_I \cap \mathscr{C}_{II}$.
- *Given* $\mathfrak{F}(s,t) = \dfrac{s^2}{2\sqrt{s^2+a^2}}$ *with* $a \geqslant 0$, *the corresponding transformation is* $\widetilde{\mathfrak{w}}(s,t) = s+\sqrt{s^2+a^2}-a$ *for* $s,t \geqslant 0$. *Hence,* $\mathfrak{F}$ *lies in* $\mathscr{C}_I \cap \mathscr{C}_{II}$.
- *When* $\mathfrak{F}(s,t) = st^k$ *with* $k > 1$, *the corresponding transformation is* $\widetilde{\mathfrak{w}}(s,t) = \dfrac{s}{1-t^k}$. *Here,* $\mathfrak{F}$ *is categorized under* $\mathscr{C}_I$.

Remark 2.10. *If* $\mathfrak{F}$ *is a* $\mathscr{C}$*-class function in either* $\mathscr{C}_I$ *or* $\mathscr{C}_{II}$, *then the following functional equations are satisfied by the functions* $\widetilde{\mathfrak{w}}$ *and* $\mathfrak{F}$:

- *For* $\mathfrak{F} \in \mathscr{C}_I$:
$$\widetilde{\mathfrak{w}}(\mathfrak{F}(s,t),t) = \widetilde{\mathfrak{w}}(s,t) - s.$$
- *For* $\mathfrak{F} \in \mathscr{C}_{II}$:
$$\widetilde{\mathfrak{w}}(\mathfrak{F}(s,t),\mathfrak{F}(s,t)) = \widetilde{\mathfrak{w}}(s,t) - s.$$

A class of functions Ξ that were mentioned in [5] are recalled in the following. These functions, represented as $\tau : X^2 \times (2^X)^2 \to J'$, satisfy a crucial condition. To be more precise, for any $x,y \in X$ and $A,C \in 2^X$, $\tau(x,y,A,C) = 0$ implies that $x=y$ or $p(x,y)=0$ is true. Additionally, we establish the nondecreasing property of $\tau \in \Xi$ within the (X,p) space, as indicated by the following inequality:

$$p(x,y) \leqslant p(a,b) \Rightarrow \tau(x,y,A_x,C_y) \leqslant \tau(a,b,A_a,C_b) \quad \forall A_x,A_a,C_y,C_b \in 2^X.$$

Example 2.11.
- $\tau(x,y,A,C) = \dfrac{p(x,y)}{1+\exp(-p(x,A)+p(y,C))}$,
- $\tau(x,y,A,C) = \log(1+p^s(x,y))$,
- $\tau(x,y,A,C) = p(x,y)^n$, *where* n *is a positive real number.*

3. Main results

The major finding of our research can be succinctly expressed as follows:

Theorem 3.1. *Consider a partial metric space* (X,p), *where* $\bar{x} \in X$, $\lambda \in [0,1]$, *and* $r > 0$ *such that the subspace* $\overline{\mathbb{B}_{p,r}}(\bar{x})$ *is complete. Let* $T,F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^p(X)$ *be multivalued mappings. Assuming that* $\tau \in \Xi$, $\alpha \in J$, *and* $\mathfrak{F} \in \mathscr{C}$, *which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:*

- $\mathfrak{F} \in \mathscr{C}_I$ *and* τ *is nondecreasing,*
- $\mathfrak{F} \in \mathscr{C}_{II}$ *and* $\tau(x,y,F(x),F(y)) \geqslant \alpha$ *for* $x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

We will establish our assumptions based on the satisfaction of the following two conditions:

(a) $p(z,F(z)) < \alpha$ *where* $\widetilde{\mathfrak{w}}(\alpha,\cdot) \leqslant r(1-\lambda)$, $\quad \forall z \in \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leqslant \mathfrak{F}(M_F(x,y), \tau(x,y,F(x),F(y)))$, $\quad \forall x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, *and for every* $y \in Fix(T)$ *and* $w \in F(y)$, *satisfying* $p(y,w) < \alpha$, *we can establish the following inequality:*

$$\delta_p(Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, Fix(F)) \leqslant \widetilde{\mathfrak{w}}(\mathfrak{M}, \tau(y,w,F(y),F(w))) \tag{3.1}$$

where $\mathfrak{M} := \sup_{x \in \overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x))$.

Proof. If the quantity $\mathfrak{M} \in \{0,+\infty\}$, there is nothing to prove; therefore, we may assume that $0 < \mathfrak{M} < +\infty$. Moreover, if $Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K = \emptyset$, then according to the convention (2.1), we are finished.
So we assume that $Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K \neq \emptyset$ and we take $x_0 \in Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$, i.e., $x_0 \in T(x_0) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$.
Fix $\varepsilon \geqslant \varepsilon' > 1$ such that $\delta_p(T(x_0) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, F(x_0)) \leqslant \mathfrak{M} < \varepsilon\mathfrak{M}$. Thus, using (a), we have

$$p(x_0,F(x_0)) < \min\{\alpha, \varepsilon\mathfrak{M}\}.$$

According to Lemma 2.3, there exists $x_1 \in F(x_0)$ such that

$$p(x_0,x_1) < \min\{\alpha, \varepsilon\mathfrak{M}\}.$$

Moreover, $x_1 \in \overline{\mathbb{B}_{p,r}}(\bar{x})$, indeed,

$$\begin{aligned} p(x_1,\bar{x}) &\leqslant p(x_1,x_0) + p(x_0,\bar{x}) - p(x_0,x_0) \\ &\leqslant \alpha + \lambda r + p(\bar{x},\bar{x}) \\ &\leqslant \widetilde{\mathfrak{w}}(\alpha,\cdot) + \lambda r + p(\bar{x},\bar{x}) \\ &\leqslant (1-\lambda)r + \lambda r + p(\bar{x},\bar{x}) \\ &\leqslant p(\bar{x},\bar{x}) + r. \end{aligned}$$

If $x_1 = x_0$, then $x_0 \in T(x_0) \cap F(x_0)$, and subsequently for any $x_0 \in Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$, we have

$$p(x_0, Fix(F)) \leqslant p(x_0, x_0) \leqslant \min\left\{\mathfrak{M}, \widetilde{\mathfrak{w}}\left(\mathfrak{M}, \tau(x_0, x_0, F(x_0), F(x_0))\right)\right\}.$$

This demonstrates that such x_0 satisfy inequality (3.1), since the distance between x_0 and the fixed point set $Fix(F)$ is positive, thereby fulfilling the requirement defined by inequality (3.1).
Moreover, for any $w \in F(x_1)$, we have

$$\begin{aligned} M_F(x_0,x_1) &= \max\left\{p(x_0,x_1), p(x_0,F(x_0)), p(x_1,F(x_1)), \frac{p(x_0,F(x_1)) + p(x_1,F(x_0))}{2}\right\} \\ &\leqslant \max\left\{p(x_0,x_1), p(x_1,w), \frac{p(x_0,w) + p(x_1,x_1)}{2}\right\} \\ &\leqslant \max\left\{p(x_0,x_1), p(x_1,w), \frac{p(x_0,x_1) + p(x_1,w)}{2}\right\} \\ &= \max\left\{p(x_0,x_1), p(x_1,w)\right\}. \end{aligned}$$

Assuming that $\max\{p(x_0,x_1), p(x_1,w)\} = p(x_1,w)$. Then, from condition (b), and the definition of $\mathscr{F}$, we derive a contradiction. Thus, we infer that $M_F(x_0,x_1) \leqslant p(x_0,x_1)$.
Given (b), we can deduce that

$$\begin{aligned} p(x_1,F(x_1)) &\leqslant \delta_p(F(x_0) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x_1)) \\ &\leqslant \mathfrak{F}\left(M_F(x_0,x_1), \tau(x_0,x_1,F(x_0),F(x_1))\right) \\ &\leqslant \mathfrak{F}\left(p(x_0,x_1), \tau(x_0,x_1,F(x_0),F(x_1))\right) \\ &< \min\{\mathfrak{F}(\alpha, \tau(x_0,x_1,F(x_0),F(x_1)), \mathfrak{F}(\varepsilon'\mathfrak{M}, \tau(x_0,x_1,F(x_0),F(x_1))\} \\ &= \min\{\mathfrak{F}(\alpha,\tau_0), \mathfrak{F}(\varepsilon'\mathfrak{M},\tau_0)\} \end{aligned}$$

where $\tau_0 = \tau(x_0,x_1,F(x_0),F(x_1))$, and subsequently, $\tau_k = \tau(x_k,x_{k+1},F(x_k),F(x_{k+1}))$.
This indicates the existence of $x_2 \in F(x_1) \cap \overline{\mathbb{B}_{p,r}}(\bar{x})$ such that

$$p(x_1,x_2) < \min\{\mathfrak{F}(\alpha,\tau_0), \mathfrak{F}(\varepsilon'\mathfrak{M},\tau_0)\} \in J.$$

Given that $n \in \mathbb{N}$ and a finite sequence $x_0,\ldots,x_n$ has been formed, let's assume that it satisfies:

$$\begin{cases} x_n \in F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), \\ M_F(x_{n-1},x_n) \leqslant p(x_{n-1},x_n) \in J \\ p(x_{n-1},x_n) < \min\{\mathfrak{F}^{n-1}(\alpha,\tau_0), \mathfrak{F}^{n-1}(\varepsilon'\mathfrak{M},\tau_0)\}. \end{cases}$$

If either $x_n = x_{n-1}$ or $x_{n-1} \in F(x_{n-1})$ for any $n \in \mathbb{N}^*$, then our task is complete. Hence, let us assume that for every $n \in \mathbb{N}^*$, it holds that $x_{n-1} \notin F(x_{n-1})$ and $x_{n-1} \neq x_n$, thereby implying that $p(x_{n-1},x_n) > 0$.
Now, let us consider the case of any $n \in \mathbb{N}^*$, and we can proceed as follows:

$$\begin{aligned} M_F(x_{n-1},x_n) &= \max\left\{p(x_{n-1},x_n), p(x_{n-1},F(x_{n-1})), p(x_n,F(x_n)), \frac{p(x_{n-1},F(x_n)) + p(x_n,F(x_{n-1}))}{2}\right\} \\ &= \max\left\{p(x_{n-1},x_n), p(x_n,F(x_n)), \frac{p(x_{n-1},F(x_n)) + p(x_n,x_n)}{2}\right\} \\ &\leqslant \max\left\{p(x_{n-1},x_n), p(x_n,F(x_n)), \frac{p(x_{n-1},x_n) + p(x_n,F(x_n))}{2}\right\} \\ &= \max\left\{p(x_{n-1},x_n), p(x_n,F(x_n))\right\}. \end{aligned}$$

In the event that $\max\{p(x_{n-1},x_n), p(x_n,F(x_n))\} = p(x_n,F(x_n))$, it leads to a contradiction based on condition (b) and the definitions of δ and $\mathfrak{F}$. Consequently, we can conclude that $M_F(x_{n-1},x_n) \leqslant p(x_{n-1},x_n) \in J$, reinforcing the validity of this inequality.
As $x_n \in F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x})$, we have

$$\begin{aligned} p(x_n,F(x_n)) &\leqslant \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x_n)) \\ &\leqslant \mathfrak{F}\left(M_F(x_{n-1},x_n), \tau_{n-1}\right) \\ &\leqslant M_F(x_{n-1},x_n). \end{aligned}$$

If we make the assumption that $M_F(x_{n-1},x_n) \leqslant p(x_n,F(x_n))$ or $\tau_{n-1} = 0$ for a certain value of $n \in \mathbb{N}^*$, we can deduce that

$$\mathfrak{F}\left(M_F(x_{n-1},x_n), \tau_{n-1}\right) = M_F(x_{n-1},x_n)$$

which implies that $M_F(x_{n-1},x_n)\tau_{n-1} = 0$. Consequently, we arrive at the contradiction that $x_{n-1} = x_n$ or $p(x_{n-1},x_n) = 0$. So we assume that $p(x_n,F(x_n)) < M_F(x_{n-1},x_n)$ and $\tau_{n-1} \neq 0$ for all $n \in \mathbb{N}^*$ and then there exists $x_{n+1} \in F(x_n)$ such that

$$p(x_n,x_{n+1}) < M_F(x_{n-1},x_n) \leqslant p(x_{n-1},x_n).$$

Furthermore, if τ is nondecreasing and $F \in \mathscr{C}_I$, then

$$\begin{aligned} p(x_n, x_{n+1}) &\leqslant \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x_n)) \\ &\leqslant \mathfrak{F}(M_F(x_{n-1}, x_n), \tau_{n-1}) \\ &\leqslant \mathfrak{F}(p(x_{n-1}, x_n), \tau_0) \\ &\leqslant \mathfrak{F}\left(\mathfrak{F}^{n-1}(\alpha, \tau_{n-1}), \tau_0\right) \\ &\leqslant \mathfrak{F}\left(\mathfrak{F}^{n-1}(\alpha, \tau_0), \tau_0\right) \\ &\leqslant \mathfrak{F}^n(\alpha, \tau_0) \end{aligned}$$

else if $F \in \mathscr{C}_{II}$ and $\tau_{n-1} \geqslant \alpha$

$$\begin{aligned} p(x_n, x_{n+1}) &\leqslant \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x_n)) \\ &\leqslant \mathfrak{F}(M_F(x_{n-1}, x_n), \tau_{n-1}) \\ &\leqslant \mathfrak{F}(p(x_{n-1}, x_n), \alpha) \\ &\leqslant \mathfrak{F}(\mathfrak{F}^{n-1}(\alpha, \tau_0), \mathfrak{F}^{n-1}(\alpha, \tau_0)) \\ &\leqslant \mathfrak{F}^n(\alpha, \tau_0). \end{aligned}$$

So, using induction, we can find $x_{n+1} \in F(x_n)$ with

$$p(x_n, x_{n+1}) < \min\{\mathfrak{F}^n(\alpha, \tau_0), \mathfrak{F}^n(\varepsilon' \mathfrak{M}, \tau_0)\},$$

and

$$\begin{aligned} p(x_{n+1}, \bar{x}) &\leqslant p(\bar{x}, x_0) + \sum_{j=0}^{n} p(x_{j+1}, x_j) - \sum_{j=0}^{n} p(x_j, x_j) \\ &< p(\bar{x}, x_0) + \sum_{j=0}^{\infty} \mathfrak{F}^j(\alpha, \tau_0) \\ &\leqslant p(\bar{x}, \bar{x}) + \lambda r + \widetilde{\mathfrak{w}}(\alpha, \tau_0) \\ &\leqslant p(\bar{x}, \bar{x}) + r. \end{aligned}$$

Therefore, it follows that $x_{n+1} \in \overline{\mathbb{B}_{p,r}}(\bar{x})$, and consequently, the sequence $\{x_n\}$ is a Cauchy sequence within $\overline{\mathbb{B}_{p,r}}(\bar{x})$. This observation is further reinforced by the fact that for any integers n and m satisfying $n > m$, we have the following:

$$\begin{aligned} p(x_n, x_m) &\leqslant \sum_{k=m}^{n-1} p(x_{k+1}, x_k) - \sum_{k=m+1}^{n-1} p(x_k, x_k) \\ &< \sum_{k=m}^{n-1} \mathfrak{F}^k(\alpha, \tau_0) \\ &\leqslant \widetilde{\mathfrak{w}}(\alpha, \tau_0). \end{aligned}$$

Consequently, we have

$$p^s(x_n, x_m) \leqslant 2p(x_n, x_m) < 2\widetilde{\mathfrak{w}}(\alpha, \tau_0).$$

Consequently, we can deduce that the sequence $\{x_n\}$ is actually a Cauchy sequence within the metric space (X, p^s). This conclusion is supported by the fact that $\widetilde{\mathfrak{w}}(\alpha, \tau_0)$ converges for every $\tau_0 \in J'$. Furthermore, since $(X \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), p)$ is a complete metric space, it follows that $(X \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), p^s)$ is also complete. As a result, the sequence $\{x_n\}$ converges to a point x^* with respect to p^s and satisfies the condition:

$$p(x^*, x^*) = \lim_{n \to +\infty} p(x_n, x^*) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0.$$

Now we assert that $x^* \in F(x^*)$. With the application of the partial metric's property ($\mathfrak{P}_4$), one can obtain

$$\begin{aligned} p(x^*, F(x^*)) &\leqslant p(x^*, x_n) + p(x_n, F(x^*)) - p(x_n, x_n) \\ &\leqslant p(x^*, x_n) + \delta_p(F(x_{n-1}) \cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(x^*)) \\ &\leqslant p(x^*, x_n) + \mathfrak{F}(M_F(x_{n-1}, x^*), \tau(x_{n-1}, x^*, F(x_{n-1}), F(x^*))) \\ &\leqslant p(x^*, x_n) + \mathfrak{F}(p(x_{n-1}, x^*), \tau(x_{n-1}, x^*, F(x_{n-1}), F(x^*))). \end{aligned}$$

Exploiting the upper semicontinuity property of the function $\mathfrak{F}$ concerning the first variable and employing the limit superior as n approaches infinity, we arrive at $p(x^*, F(x^*)) = 0 = p(x^*, x^*)$. This leads to the conclusion that $x^* \in F(x^*)$ as per Lemma 2.2.

Through meticulous computations, we obtain the following results:

$$
\begin{aligned}
p(x_0,x^*) &\leqslant \sum_{j=0}^{\infty} p(x_{j+1},x_j) - \sum_{j=1}^{\infty} p(x_j,x_j) \\
&\leqslant \sum_{j=0}^{\infty} \min\{\mathfrak{F}^j(\alpha,\tau_0),\mathfrak{F}^j(\varepsilon'\mathfrak{M},\tau_0)\} \\
&\leqslant \widetilde{\mathfrak{w}}(\varepsilon'\mathfrak{M},\tau_0).
\end{aligned}
$$

Given that $p(x_0,Fix(F)) \leqslant p(x_0,x^*)$, we can deduce that

$$p(x_0,Fix(F)) \leqslant \widetilde{\mathfrak{w}}(\varepsilon\mathfrak{M},\tau_0).$$

This inequality holds for any $y := x_0 \in Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K$. Consequently, we obtain

$$\delta_p(Fix(T) \cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) \cap K, Fix(F)) \leqslant \widetilde{\mathfrak{w}}(\varepsilon\mathfrak{M},\tau_0).$$

By allowing ε to approach 1, we successfully complete the proof. □

You can see how to apply Theorem 3.1 in the following example.

Example 3.2. *Let $X = \mathbb{R}^+ = [0,+\infty[$ be equipped with the partial metric defined as follows:*

$$p(x,y) = \begin{cases} 0, & \text{if } x = y \in \left[0,\frac{125}{216}\right]; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Now, let $\mathfrak{F}(s,t)$ be defined as follows:

$$\mathfrak{F}(s,t) = \begin{cases} st^2, & \text{if} (s\times t) \in \left[0,\frac{5}{6}\right] \times \left[0,\frac{\sqrt{11}}{4}\right]; \\ 6s-4t, & \text{otherwise.} \end{cases}$$

The function $\mathfrak{F}$ belongs to the set $\mathscr{C}$ over the interval $J \times J' = \left[0,\frac{5}{6}\right] \times \left[0,\frac{\sqrt{11}}{4}\right]$. Specifically, $\mathfrak{F}$ is an element of $\mathscr{C}_I$ and $\widetilde{\mathfrak{w}}(s,t) = \frac{s}{1-t^2}$. Define $F : [0,1] \to C^p(X)$ as follows:

$$F(x) = \begin{cases} \{x^3\}, & \text{if } x \in \left[0,\frac{5}{6}\right]; \\ [1,+\infty[, & \text{if } x \in \left]\frac{5}{6},1\right]. \end{cases}$$

Utilizing the parameters specified as follows, we proceed to apply Theorem 3.1:

$$\bar{x} = \frac{1}{6},\ \ r = 1,\ \ \lambda = \frac{1}{5},\ \ \alpha = \frac{1}{4} \in J,\ \ \overline{\mathbb{B}_{p,r}}(\bar{x}) = [0,1]$$

and

$$\tau(x,y,F(x),F(y)) = p(x,y),$$

which is a nondecreasing.

First, observe that for every z in the closed interval, $\overline{\mathbb{B}_{p,\lambda r}}(\bar{x}) = \left[0,\frac{1}{5}\right]$, we can express the function as follows:

$$p(z,F(z)) = p(z,\{z^3\}) = \max\{z,z^3\} = z \leqslant \frac{1}{5} < \frac{1}{4} = \alpha.$$

Additionally, it holds that $\widetilde{\mathfrak{w}}(\alpha,t) = \frac{\alpha}{1-t^2} \leqslant \frac{1}{4}\cdot\frac{16}{5} = \frac{4}{5} = r(1-\lambda)$. Therefore, condition (a) of Theorem 3.1 is satisfied. To establish the validity of condition (b) in Theorem 3.1, it is enough to examine the following scenarios:

1. If $x = y \in \left[0,\frac{5}{6}\right]$ then

$$
\begin{aligned}
\delta_p(F(x)\cap[0,1],F(y)) = \delta_p(\{x^3\},\{x^3\}) = 0 &\leqslant \begin{cases} p(x,x)\cdot p(x,x)^2, & x \in \left[0,\frac{\sqrt{11}}{4}\right]; \\ 6p(x,x)-4p(x,x), & x \in \left]\frac{\sqrt{11}}{4},\frac{5}{6}\right]. \end{cases} \\
&\leqslant \mathfrak{F}(p(x,y),p(x,y))) \\
&\leqslant \mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).
\end{aligned}
$$

2. If $x,y \in \left[0, \frac{5}{6}\right]$ and $x \neq y$ then

$$\delta_p(F(x)\cap[0,1],F(y)) = \delta_p(\{x^3\},\{y^3\}) \leqslant \begin{cases} \max\{x,y\}\cdot(\max\{x,y\})^2, & x,y \in \left[0, \frac{\sqrt{11}}{4}\right]; \\ 6\max\{x,y\} - 4\max\{x,y\}, & x,y \in \left]\frac{\sqrt{11}}{4}, \frac{5}{6}\right]. \end{cases}$$
$$\leqslant \mathfrak{F}(p(x,y),p(x,y)))$$
$$\leqslant \mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).$$

3. If $x,y \in \left]\frac{5}{6}, 1\right]$, then

$$\begin{aligned}\delta_p(F(x)\cap[0,1],F(y)) &= \delta_p(\{1\},[1,+\infty[) = 1 \\ &\leqslant 6\max\{x,y\} - 4\max\{x,y\} \\ &\leqslant \mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).\end{aligned}$$

4. If $x \in \left[0, \frac{5}{6}\right]$ and $y \in \left]\frac{5}{6}, 1\right]$ then

$$\delta_p(F(x)\cap[0,1],F(y)) = \delta_p(\{x^3\},[1,+\infty[) = 1 \leqslant 2p(x,y) = \mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y)))$$

and

$$\delta_p(F(y)\cap[0,1],F(x)) = \delta_p(\{1\},\{x^3\}) = 1 \leqslant 2p(x,y) = \mathfrak{F}(M_F(x,y),\tau(x,y,F(x),F(y))).$$

Hence, the condition (b) of Theorem 3.1 is satisfied, and $x^* \in Fix(F) = \{0,1\} \subset \overline{\mathbb{B}_{p,r}}(\bar{x})$ are the required points.
Hence, for arbitrary multivalued mapping $T:[0,1] \to C^p(X)$, all conditions of Theorem 3.1 are satisfied and then, for any $K \subseteq [0,1]$, and for every $y \in Fix(T)$ and $w \in F(y)$, satisfying $p(y,w) < \alpha$, we have

$$\begin{aligned}\delta_p(Fix(T)\cap[0,\tfrac{1}{5}]\cap K, Fix(F)) &\leqslant \sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\tfrac{1}{5}]\cap K, F(x)) \cdot \frac{1}{1-p(y,w)^2} \\ &\leqslant \widetilde{\mathfrak{w}}\left(\sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\tfrac{1}{5}]\cap K, F(x)), \tau(y,w,F(y),F(w))\right).\end{aligned}$$

Consider, as an illustrative example, the mapping $T:[0,1] \to C^p(X)$ defined by

$$T(x) = \begin{cases} \left\{\frac{1}{2}x + \frac{1}{12}\right\}, & x \in \left[0, \frac{143}{225}\right]; \\ [2,+\infty[, & x \in \left]\frac{143}{225}, 1\right]. \end{cases}$$

Furthermore, we observe that $Fix(T) = \{\frac{1}{6}\}$. Let us consider $y = \frac{1}{6} \in Fix(T)$ and any $w \in F(\frac{1}{6})$, specifically $w = \frac{1}{216}$. In this case, we can calculate:

$$\begin{aligned}p(y,w) &= p(\tfrac{1}{6},\tfrac{1}{216}) \\ &= \max\left\{\frac{1}{6},\frac{1}{216}\right\} \\ &= \frac{1}{6} < \frac{1}{4} = \alpha.\end{aligned}$$

Let $K \subseteq [0,1]$, then we have

Case 1. $Fix(T)\cap[0,\frac{1}{5}]\cap K = \emptyset$, then

$$\delta_p(Fix(T)\cap[0,\tfrac{1}{5}]\cap K, Fix(F)) = 0 \leqslant \widetilde{\mathfrak{w}}\left(\sup_{x\in[0,1]} \delta_p(T(x)\cap[0,\tfrac{1}{5}]\cap K, F(x)), \tau(y,w,F(y),F(w))\right).$$

Case 2. $Fix(T)\cap[0,\frac{1}{5}]\cap K \neq \emptyset$, i.e., $\frac{1}{6} \in K$, then

$$\delta_p(Fix(T)\cap[0,\tfrac{1}{5}]\cap K, Fix(F)) = \delta_p(\{\tfrac{1}{6}\},\{0,1\}) = \min\left\{p(\tfrac{1}{6},0), p(\tfrac{1}{6},1)\right\} = \frac{1}{6}.$$

Let $x \in [0,1]$, we consider the following cases:

- If $x > \frac{7}{30}$, we have $T(x) > \frac{1}{5}$ and then
$$\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) = 0.$$
- If $x \leqslant \frac{7}{30}$ and $T(x)\cap K = \emptyset$ then
$$\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) = 0.$$
- If $x \leqslant \frac{7}{30}$ and $T(x)\cap K \neq \emptyset$ then
$$\begin{aligned}\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) &= \delta_p(\{\frac{1}{2}x+\frac{1}{12}\},\{x^3\})\\ &= p(\frac{1}{2}x+\frac{1}{12},x^3)\\ &= \max\left\{\frac{1}{2}x+\frac{1}{12},x^3\right\}\\ &= \frac{1}{2}x+\frac{1}{12}.\end{aligned}$$

  Then
$$\sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K, F(x)) = \sup_{x\in[0,1]}\left(\frac{1}{2}x+\frac{1}{12}\right) = \frac{7}{12}.$$

Hence,
$$\begin{aligned}\widetilde{\mathfrak{w}}\left(\sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K,F(x)),\tau(y,w,F(y),F(w))\right) &= \sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K,F(x))\cdot\frac{1}{1-p(y,w)^2}\\ &= \sup_{x\in[0,1]}\delta_p(T(x)\cap[0,\frac{1}{5}]\cap K,F(x))\cdot\frac{36}{35}\\ &= \frac{36}{35}\cdot\frac{7}{12} = \frac{3}{5} \geqslant \frac{1}{6}\\ &\geqslant \delta_p(Fix(T)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K, Fix(F)).\end{aligned}$$

4. Some Consequences

This section deals with the ramifications of the theorem 3.1. Through these corollaries, we wish to clarify other insights and implications that have resulted from the study. The corollaries that we have derived from the main theorem are not universal but rather depend on the specific choices of the parameters $\mathfrak{F}$, τ, λ, and r. Different values of these parameters may lead to different outcomes or even invalidate some of the corollaries. Therefore, we need to be careful when applying the corollaries to concrete situations and always check the assumptions and conditions that are required for their validity.
Consider, for instance, $\mathfrak{F}(s,t) = \varphi(s)$ belonging to both $\mathscr{C}_I$ and $\mathscr{C}_{II}$, where $\varphi(s)$ is an arbitrary Bianchini-Grandolfi gauge on J and $\widetilde{\mathfrak{w}}(s,t) = \mathfrak{s}(s)$. Thus, under these conditions, Theorem 2.5 directly follows from Theorem 3.1.

Corollary 4.1. *Consider a partial metric space (X,p), where $\bar{x}\in X$, $\lambda\in[0,1]$, and $r>0$ such that the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Let $T,F:\overline{\mathbb{B}_{p,r}}(\bar{x})\to C^p(X)$ be multivalued mappings. Assuming that φ be an increasing and upper semicontinuous function, serving as Bianchini-Grandolfi gauge on J. Under the assumption that there exists $\alpha\in J$ satisfying the following two conditions:*

(a) $p(z,F(z)) < \alpha$ *where* $\mathfrak{s}(\alpha)\leqslant r(1-\lambda)$, $\quad\forall z\in\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x)\cap\overline{\mathbb{B}_{p,r}}(\bar{x}),F(y))\leqslant\varphi(M_F(x,y))$, $\quad\forall x,y\in\overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K\subseteq\overline{\mathbb{B}_{p,r}}(\bar{x})$, we can establish the following inequality:
$$\delta_p(Fix(T)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K,Fix(F))\leqslant\mathfrak{s}(\mathfrak{M})$$
where $\mathfrak{M} := \sup_{x\in\overline{\mathbb{B}_{p,r}}(\bar{x})}\delta_p(T(x)\cap\overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K,F(x))$.

Proof. Given a $\mathscr{C}$-class function $\mathfrak{F}(s,t)=\varphi(s)$ that is independent of the second variable t, it is possible to select any $\tau\in\Xi$ such that τ is either nondecreasing or greater than α. Subsequently, Theorem 3.1 can be applied. □

Let $\mathfrak{F}(s,t) = ks-t$ and $k\in]0,1]$ be an element of $\mathscr{C}_{II}$ for $J\times J' = \mathbb{R}^2_+$. Then, with $\widetilde{\mathfrak{w}}(s,t) = \frac{1}{2-k}(2s-t)$, we can state this corollary:

Corollary 4.2. *Suppose (X,p) is a partial metric space and $\bar{x}$ is in X. Let λ, k, and r be no-negative numbers with $0\leqslant\lambda\leqslant 1$, $k\leqslant 1$, and $r>0$. Then the subspace $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is complete. Suppose that T and F are multivalued mappings from $\overline{\mathbb{B}_{p,r}}(\bar{x})$ to $C^p(X)$. Let τ be an element of Ξ and α be a positive real number such that $\tau(x,y,F(x),F(y))\geqslant\alpha$ for all x and y in $\overline{\mathbb{B}_{p,r}}(\bar{x})$. We will establish our assumptions based on the satisfaction of the following two conditions:*

(a) $p(z,F(z)) < \alpha$ *where* $2\alpha \leqslant r(1-\lambda)(2-k)$, $\quad \forall z \in \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})$.
(b) $\delta_p(F(x)\cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leqslant kM_F(x,y) - \tau(x,y,F(x),F(y)), \quad \forall x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, *and for every* $y \in Fix(T)$ *and* $w \in F(y)$, *satisfying* $p(y,w) < \alpha$, *we can establish the following inequality:*

$$\delta_p(Fix(T)\cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K, Fix(F)) \leqslant \frac{1}{2-k}\left(2\mathfrak{M} - \tau(y,w,F(y),F(w))\right)$$

where $\mathfrak{M} := \sup\limits_{x\in \overline{\mathbb{B}_{p,r}}(\bar{x})} \delta_p(T(x)\cap \overline{\mathbb{B}_{p,\lambda r}}(\bar{x})\cap K, F(x))$.

When λ is zero, we get a simpler version of theorem 3.1:

Corollary 4.3. *Let* (X,p) *be a partial metric space and let* $T,F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^p(X)$ *be multivalued mappings such that* $\bar{x}$ *is a fixed point of* T *and* $\overline{\mathbb{B}_{p,r}}(\bar{x})$ *is a complete subspace for some positive* r*. Assuming that* $\tau \in \Xi$, $\alpha \in J$, *and* $\mathfrak{F} \in \mathscr{C}$, *which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:*

- $\mathfrak{F} \in \mathscr{C}_I$ *and* τ *is nondecreasing,*
- $\mathfrak{F} \in \mathscr{C}_{II}$ *and* $\tau(x,y,F(x),F(y)) \geqslant \alpha$ *for* $x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Our assumptions require two conditions:

(a) $p(\bar{x},F(\bar{x})) < \alpha$ *where* $\widetilde{\mathfrak{w}}(\alpha,\cdot) \leqslant r$.
(b) $\delta_p(F(x)\cap \overline{\mathbb{B}_{p,r}}(\bar{x}), F(y)) \leqslant \mathfrak{F}(M_F(x,y), \tau(x,y,F(x),F(y))), \quad \forall x,y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

Then, for any given $K \subseteq \overline{\mathbb{B}_{p,r}}(\bar{x})$, *and for every* $w \in F(\bar{x})$, *satisfying* $p(\bar{x},w) < \alpha$, *we can establish the following inequality:*

$$p(\bar{x}, Fix(F)) \leqslant \widetilde{\mathfrak{w}}\left(\mathfrak{M}, \tau(\bar{x},w,F(\bar{x}),F(w))\right)$$

where $\mathfrak{M} := \sup\limits_{x\in \overline{\mathbb{B}_{p,r}}(\bar{x})} p(\bar{x},F(x))$.

Let λ be a nonzero number, and let r go to infinity. Then $\overline{\mathbb{B}_{p,+\infty}}(x)$ is equal to X, and we have the following corollary:

Corollary 4.4. *Consider the complete partial metric space* (X,p)*. Let* $T,F : X \to C^p(X)$ *be multivalued mappings. Assuming that* $\tau \in \Xi$, $\alpha \in J$, *and* $\mathfrak{F} \in \mathscr{C}$, *which is upper semicontinuous with respect to the first variable, satisfy either of the following conditions:*

- $\mathfrak{F} \in \mathscr{C}_I$ *and* τ *is nondecreasing,*
- $\mathfrak{F} \in \mathscr{C}_{II}$ *and* $\tau(x,y,F(x),F(y)) \geqslant \alpha$ *for* $x,y \in X$.

Our assumptions require two conditions:

(a) $p(z,F(z)) < \alpha$ *where* $\widetilde{\mathfrak{w}}(\alpha,\cdot) < +\infty, \quad \forall z \in X$.
(b) $\delta_p(F(x),F(y)) \leqslant \mathfrak{F}(M_F(x,y), \tau(x,y,F(x),F(y))), \quad \forall x,y \in X$.

Then, for any given $K \subseteq X$, *and for every* $y \in Fix(T)$ *and* $w \in F(y)$, *satisfying* $p(y,w) < \alpha$, *we can establish the following inequality:*

$$\delta_p(Fix(T)\cap K, Fix(F)) \leqslant \widetilde{\mathfrak{w}}\left(\mathfrak{M}, \tau(y,w,F(y),F(w))\right)$$

where $\mathfrak{M} := \sup\limits_{x\in X} \delta_p(T(x)\cap K, F(x))$.

5. Conclusion

In this work, we have extended the results of [12] about the data dependence of fixed point sets for pseudo-contractive multifunctions in the context of partial metric spaces. By utilizing $\mathscr{C}$-class functions, we established new theorems on the data dependence of fixed point sets and implied corollaries. The practical examples given show our key findings. The present investigation adds to the literature on fixed point theory in partial metric spaces and offers tools that are useful in investigating the data dependence for various classes of multifunctions on their fixed points. This paper therefore opens up the possibility of studying other generalized metric structures and nonlinear operators based on techniques developed here that form an active area of research too.

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