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Research Article

Measuring inaccuracies in the proportional hazard rate model based on extropy using a length-biased weighted residual approach

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Abstract

In this paper, we consider the concept of the residual inaccuracy measure and extend it to its weighted version based on extropy. The properties of this measure are studied, and the discrimination principle is applied in the class of proportional hazard rate models. A characterization problem for the proposed weighted extropy-inaccuracy measure is studied. Some alternative expressions are provided as well as upper and lower limits and various inequalities related to the proposed measure. Non-parametric estimators based on the kernel density estimation method and empirical distribution function for the proposed measure are obtained, and the performance of the estimators are also discussed using some simulation studies. Finally, two real datasets are applied to illustrate our provided estimators. In general, our study highlights the potential of the weighted residual inaccuracy using extropy as a powerful tool to improve the quality and reliability of data analysis and modeling across various disciplines. Researchers and practitioners can benefit from incorporating this measure into their analytical toolkit to enhance the accuracy and effectiveness of their work.

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1. Introduction

The concept of entropy was first proposed by the physicist [20] to express the degree of chaos in the physical system. Later, Shannon [36] proposed and extended the entropy to the field of information as a measure of the uncertainty of the information. Shannon entropy represents the absolute limit of the best possible lossless compression of any communication. For additional details on this concept, see Cover and Thomas [6].

Suppose that we have two nonnegative continuous random variables X and Y, which represent the time to failure of two systems. These variables have probability density functions (PDF) $f(x)$ and $g(x)$, respectively. Furthermore, let $F(x)$ and $G(x)$ be the cumulative distribution functions (CDF) of X and Y, and let $\bar{F}(x)$ and $\bar{G}(x)$ be the

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survival functions (SF) of X and Y, respectively. The Shannon measure of uncertainty associated with the random variable X is defined as

$$H(X) = -E_f[\log f(X)] = -\int_{-\infty}^{+\infty} f(x)\log f(x)dx. \tag{1.1}$$

Similarly, Kerridge's measure of inaccuracy, as cited in Kerridge [17], is also denoted by Equation (1.2) as

$$H(X,Y) = -E_f[\log g(X)] = -\int_{-\infty}^{+\infty} f(x)\log g(x)dx, \tag{1.2}$$

where " log " represents the natural logarithm and following the convention that $0\log 0 = 0$, if we have $g(x) = f(x)$, then Equation (1.2) reduces to Equation (1.1).

In life testing and survival analysis, considering the current age of the system is important. Therefore, when calculating the uncertainty of a system or distinguishing between two systems, the measures referenced in Equations (1.1) and (1.2) may not be appropriate. Instead, when a system has survived up to time t, the corresponding dynamic measure of uncertainty of [9] and of discrimination of [15, 17] are denoted by Equations (1.3) and (1.4) as

$$H(X;t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)}\log\frac{f(x)}{\bar{F}(t)}dx, \tag{1.3}$$

and

$$H(X|Y;t) = \int_t^{\infty} \frac{f(x)}{\bar{F}(t)}\log\frac{f(x)\bar{G}(t)}{\bar{F}(t)g(x)}dx, \tag{1.4}$$

respectively. Obviously, when $t = 0$, then Equation (1.3) reduces to Equation (1.1).

According to [37], the dynamic measure of inaccuracy is defined as the measure associated with two residual lifetime distributions, denoted by F and G. This measure is known as Kerridge's measure of inaccuracy and is represented by Equation (1.5)

$$H(X,Y;t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)}\log\frac{g(x)}{\bar{G}(t)}dx. \tag{1.5}$$

Clearly, for $t = 0$, it reduces to Equation (1.2).

The relationship between information and inaccuracy can be quantified using the equation $H(X,Y) = H(X)+H(X|Y)$, where $H(X|Y)$ represents the Kullback-Leibler relative information measure of X about Y of [18], defined in Equation (1.6) as

$$H(X|Y) = E_f\left[\log\frac{f(X)}{g(X)}\right] = \int_0^{\infty} f(x)\log\frac{f(x)}{g(x)}dx. \tag{1.6}$$

It is clear that $H(X|Y;0) = H(X|Y)$.

The information measures mentioned in [7] do not consider the value of the random variable itself, but only its probability density function (PDF). They proposed a "length-biased" shift-dependent information measure that is related to the differential entropy. This measure assigns a higher weight to higher values of observed random variables.

The concept of a weighted distribution, as introduced by [31], is widely utilized in statistics and various other applications. Weighted distributions come into play when observations generated from a stochastic process are recorded with a certain weight function. In this context, let X represent a continuous non-negative random variable with PDF $f(x)$. Furthermore, let X_w be a weighted random variable associated with X, where the weight function $w(x)$ is positive for all values of $x \geq 0$. The corresponding PDF $f_w(x)$ of X_w can be determined in Equation (1.7) as

$$f^w(x) = \frac{w(x)f(x)}{E\left(w\left(X\right)\right)}, \quad x \geq 0. \tag{1.7}$$

When the function $w(x) = x$ is used, the random variable X_w is termed a random variable biased by length or biased by size. In this case, the PDF becomes

$$f_*(x) = \frac{xf(x)}{E(X)}, \quad x \geq 0. \tag{1.8}$$

For further details on this topic, see [10, 26]. If X is a random variable with a finite mean $E[X]$, the length biased CDF and SF are defined in Equations (1.9) and (1.10) as

$$F_*(t) = \int_0^t \frac{xf(x)}{E(X)} dx, \tag{1.9}$$

and

$$\bar{F}_*(t) = \int_t^\infty \frac{xf(x)}{E(X)} dx, \tag{1.10}$$

respectively. These functions describe weighted distributions that occur in sampling procedures where the probabilities of sampling are proportional to the values of the samples. Therefore, the measure of residual entropy in Equation (1.3) has been expanded to include the length biased weighted residual entropy, denoted by Equation (1.11) as

$$H_*(X,t) = -\int_t^\infty x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx. \tag{1.11}$$

The presence of the factor x in the integral on the right-hand side introduces a length biased shift-dependent information measure that assigns higher significance to larger values of the random variable X. Di Crescenzo and Longobardi [8] discussed weighted versions of the residual and past entropies. The weighted residual entropy is defined in Equation (1.12) as

$$H^w(X_t) = -\int_t^{+\infty} x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} \mathrm{d}x, \tag{1.12}$$

while the weighted past entropy is defined in Equation (1.13) as

$$H^w({}_tX) = -\int_0^t x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} \mathrm{d}x. \tag{1.13}$$

2. Extropy

Recently, Lad et al. [19] proposed an alternative measure of uncertainty of a random variable called extropy. The extropy is a measure of information introduced as a dual to entropy or as an antonym to entropy. The extropy of the random variable X is defined in Equation (2.1) as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx = -\frac{1}{2}\int_0^{+\infty} f(x)dF(x) = -\frac{1}{2}\int_0^1 f(F^{-1}(u))du. \tag{2.1}$$

For more details and applications of extropy, refer to [19]. The extropy may also be used to compare the uncertainties of two random variables. For two random variables Y and Z, $J(Y) \leq J(Z)$ implies that Y has more uncertainty than Z.

Qiu and Jia [28] considered a random variable $X_t = [X - t|X > t]$, $t \geq 0$ and defined uncertainty of such a system based on extropy, given in Equation (2.2) as

$$J(X;t) = -\frac{1}{2}\int_t^\infty \left[\frac{f(x)}{\bar{F}(t)}\right]^2 dx. \tag{2.2}$$

Analogous to the weighted entropy, Balakrishnan et al. [3] introduced the concept of weighted extropy defined in Equation (2.3) as

$$J^w(X) = -\frac{1}{2}E[Xf(X)] = -\frac{1}{2}\int_0^{+\infty} xf^2(x)\mathrm{d}x, \tag{2.3}$$

which can also be rewritten in Equation (2.4) as

$$J^w(X) = -\frac{1}{2}\int_0^\infty f^2(x)\int_0^x \mathrm{d}y\ \mathrm{d}x = -\frac{1}{2}\int_0^\infty \mathrm{d}y\int_y^\infty f^2(x)\mathrm{d}x. \tag{2.4}$$

Also, they introduced the weighted residual extropy (WRJ) by Equation (2.5) as

$$J^w\left(X_t\right) = -\frac{1}{2\bar{F}^2(t)}\int_t^\infty x f^2(x)\mathrm{d}x. \tag{2.5}$$

Jahanshahi et al. [14] introduced an alternative measure of uncertainty of non-negative continuous random variable X which they called it cumulative residual extropy (CRJ) by Equation (2.6) as

$$\xi J(X) = -\frac{1}{2}\int_0^\infty \bar{F}^2(x)dx. \tag{2.6}$$

They studied some properties of the aforementioned information measure. The measure defined in Equation (2.6) is not applicable to a system that has survived for some unit of time. Hence, Sathar and Nair [35] proposed a dynamic version of CRJ (called dynamic survival extropy) to measure residual uncertainty of lifetime random variable X as follows

$$\xi J(X;t) = -\frac{1}{2\bar{F}^2(t)}\int_t^\infty \bar{F}^2(x)dx, \quad t \geq 0. \tag{2.7}$$

It is clear that $\xi J(X;0) = \xi J(X)$. Recently, Hashempour et al. [11] introduced a weighted cumulative residual extropy as an extended version of Equation (2.6) as follows

$$\xi J^w(X) = -\frac{1}{2}\int_0^\infty x\bar{F}^2(x)dx. \tag{2.8}$$

They studied the characterization problem, estimation, and testing for this measure. Also, Mohammadi and Hashempour [22] proposed a modified interval weighted cumulative residual and past extropies, respectively in Equations (2.9) and (2.10) as

$$WCRJ(X;t_1,t_2) = -\frac{1}{2}\int_{t_1}^{t_2}\varphi(x)\left(\frac{\bar{F}(x)}{\bar{F}(t_1)-\bar{F}(t_2)}\right)^2 dx, \tag{2.9}$$

and

$$WCPJ(X;t_1,t_2) = -\frac{1}{2}\int_{t_1}^{t_2}\varphi(x)\left(\frac{F(x)}{F(t_2)-F(t_1)}\right)^2 dx, \tag{2.10}$$

where φ is a weight function. They provided non-parametric estimators for these measures based on the kernel method. Also, For more details, concepts, generalizations, applications and estimations in the field of extropy, one can refer to the following references. Qiu [27] used the extropy for record values and ordered statistics. Qiu and Jia [29] considered the estimations of extropy measure. The extropy of a mixed system's lifetime was considered by [30]. Recently, Hashempour and Mohammadi [13] consider the extropy measure of inaccuracy for record statistics. Also, readers can refer to [11,12,16,23,24] and references therein. In this article, an attempt has been made to present new criteria that have a more general form compared to the criteria introduced in other articles. One of these criteria is the presentation of residual weight criteria based on extropy. In this regard, data available often do not have equal importance and value. In such cases, the presented criteria should be considered in a weighted form. Additionally, in many cases, there is a need to obtain information about future events, where we use the SF instead of the PDF. Considering the aforementioned points, the extropy-based residual weight criteria can meet the needs of researchers. In this paper we extend the concept of residual inaccuracy to length biased weighted residual inaccuracy (WRJI). In the rest of this paper, in Section 2, some concepts, definitions and generalizations related to extropy are given. In Section 3, we define and study WRJI and weighted residual discrimination information (WRDJ) by using some examples and remarks. In Section 4, we study that when F and G follow the PHR model then WRJI uniquely determines the survival function $\bar{F}$ and some alternative expressions related to WRJI are studied. In Section 5, we obtain some bounds and inequalities for our proposed measure. In Section 6, we give two non-parametric estimators for the proposed measure. Based on the results in the previous section, a simulation study is also obtained in Section 7 for comparing our proposed estimators. Finally, Section 8 investigates the behavior of the provided estimators for two real datasets. We conclude the paper in Section 9.

3. Weighted residual inaccuracy measure

In this section, for two non-negative continuous random variables with the same support, we introduce some measures of uncertainty based on extropy and some properties are studied.

Definition 3.1. Let X and Y be two non-negative continuous random variables with PDFs f and g, respectively. The weighted measure of discrimination of X about Y based on extropy is defined in Equation (3.1) as

$$J^w(X|Y) = \frac{1}{2}\int_0^\infty xf(x)\,[f(x)-g(x)]\,dx. \tag{3.1}$$

The weighted measure of inaccuracy between X and Y based on extropy (WJI) denoted by $J^w(X,Y)$ is defined as follows:

Definition 3.2. Let X and Y be non-negative continuous random variables with PDFs $f(x)$ and $g(x)$ and CDFs $F(x)$ and $G(x)$, respectively. Then WJI between the distributions X and Y is defined in Equation (3.2) as

$$J^w(X,Y) = -\frac{1}{2}\int_0^\infty xf(x)g(x)dx. \tag{3.2}$$

On WJI measure in Equation (3.2), $\bar{F}(.)$ is the actual SF corresponding to the observations and $\bar{G}(.)$ is the SF assigned by the experimenter. If $f(x) = g(x)$, then the WJI in Equation (3.2) reduces to Equation (2.3) introduced by [3]. WJI measures the value of misspecifying the correct model in which $f(x)$ is the actual PDF of observations and $g(x)$ is the PDF assigned by the experimenter such that the value of each observation is taken into account in its formula. In other words, by WJI, the discrimination measure between f and g can be affected by the strength of each observation.

In the provided example, we demonstrate the application of Equation (3.2) to compare statistical models.

Example 3.3. The statistical model for the random variable X is represented by SF $\bar{Q}(x) = 1-x$, where x is within the range $(0,1)$. Furthermore, two SFs, $\bar{F}(x) = 1-x^2$ and $\bar{S}(x) = 1-x^3$, $x \in (0,1)$, have been determined through nonparametric statistical tests to approximate the random variable X. Using Equation (3.2), we can calculate the WJI values. Specifically, we have $J^w(X,X) = J^w(X) = -0.25$, $J^w(X,Y) = -0.33$, and $J^w(X,Z) = -0.5$. Based on these WJI values, we can conclude that the WJI between X and Y, which follows the SF $\bar{F}(x)$, is closer to the WJI of X itself compared to the WJI between X and Z, which follows the SF $\bar{S}(x)$. Therefore, Y provides a better approximation of X than Z. In other words, the statistical model represented by the survival function $\bar{F}(x)$ is the closest approximation to the statistical model represented by $\bar{Q}(x)$ that generated the data. □

Balakrishnan et al. [3] examined a random variable denoted by $X_t = [X - t|X > t]$, where $t \geq 0$. They introduced the concept of uncertainty in this system using extropy in

Equation (3.3), denoted by

$$J^w(X;t) = -\frac{1}{2}\int_t^{\infty} x\left[\frac{f(x)}{\bar{F}(t)}\right]^2 dx. \tag{3.3}$$

$J^w(X;t)$ is a suitable metric for quantifying information in situations where uncertainty is linked to future events.

In the subsequent discussion, we introduce a weighted measure of inaccuracy that pertains to two residual lifetime distributions, denoted by $G(.)$ and $F(.)$, which are associated with the measure of inaccuracy.

Definition 3.4. Consider two non-negative continuous random variables X and Y with probability density functions f and g respectively. WRJI measure between X and Y, utilizing extropy, can be defined in Equation (3.4) as follows

$$J^w(X,Y;t) = -\frac{1}{2}\int_t^{\infty} x\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{\bar{G}(t)}dx, \tag{3.4}$$

meanwhile, $\bar{F}(t)$ and $\bar{G}(t)$ can not be zero in (3.4).

Remark 3.5. From Equation (3.4), it is observed that $J^w(X,Y;t) = J^w(Y,X;t)$ and $J^w(X,Y;t) \leq 0$. Furthermore, by taking the limit as $t \to 0$ in Equation (3.4), the WRJI transforms into the inaccuracy measure in Equation (3.2). Furthermore, when two random variables X and Y have the same SFs, the WRJI simplifies to the WRJ as given in Equation (3.3).

Remark 3.6. The weight functions used in WRJI play a crucial role in determining the impact of different data points on the overall measure of the fit of the model. It is important to explain why these particular weight functions were chosen over others and how they align with the objectives of the study. Discussing the rationale behind the choice of weight functions and their potential effects on the results will improve the understanding of the methodology used in the study. Furthermore, sensitivity analysis on different weight functions could be beneficial in assessing the robustness of the results.

In addition, we introduce the concept of uncertainty-weighted discrimination information of variable X with respect to variable Y using the concept of extropy.

Definition 3.7. Assume that $\bar{F}(x)$ and $\bar{G}(x)$ denote the SFs of non-negative continuous random variables X and Y, respectively. The weighted residual discrimination information (WRDJ) between X and Y can be defined in Equation (3.5) as a quantity denoted by

$$J(X|Y;t) = \frac{1}{2}\int_t^{\infty} x\frac{f(x)}{\bar{F}(t)}\left[\frac{f(x)}{\bar{F}(t)} - \frac{g(x)}{\bar{G}(t)}\right]dx. \tag{3.5}$$

Clearly, when $t = 0$, then Equations (3.3)-(3.5) reduce to Equations (2.1), (3.2) and (3.1), respectively. By adding Equations (3.3) and (3.5), we obtain Equation (3.4), i.e. $J^w(X,Y;t) = J^w(X;t) + J^w(X|Y;t)$.

The WRJI measure based on extropy offers a valuable approach for evaluating predictive models in a dynamic data environment. By incorporating weights and extropy, this measure provides a more comprehensive assessment of model performance, considering the importance of different time periods and the unpredictability of the data. The practical value of WRJI lies in its ability to capture the nuances of data and provide a more accurate evaluation of predictive models. This measure can be applied in various fields where accurate predictions are crucial, such as finance, health care, weather forecasting, and supply chain management. By considering the weighted residuals and extropy concepts, decision makers can gain insights into the model's performance over time and make informed decisions based on the most relevant and reliable information. Furthermore, the potential impact of WRJI extends beyond model evaluation. It can aid in model selection, parameter tuning, and identifying areas for improvement in predictive models. By understanding the strengths and weaknesses of different models in dynamic scenarios, organizations can enhance their decision-making processes and optimize their operations. Together, the WRJI measure based on extropy offers a robust and practical solution for evaluating predictive models in dynamic environments. Its potential impact in various fields is significant, allowing for more accurate predictions, informed decision making, and improved performance of predictive models.

Example 3.8. Let X be a non-negative random variable with SF of $\bar{F}(x) = 1 - x^2$ for values of x between 0 and 1. Furthermore, Y is a random variable with a uniform distribution between 0 and 1, and its SF is denoted by $\bar{Q}(x) = 1 - x$, $x \in (0, 1)$. we obtain

$$J^w(X,Y;t) = \frac{t^2+t+1}{3t^2-3}, \quad t \neq 1,$$

$$J^w(X,t) = \frac{t^2+1}{2t^2-2},$$

and

$$J^w(X|Y;t) = -\frac{t-1}{6t+6}, \quad t \neq 1.$$

Example 3.9. Suppose that X and Y have exponential distributions with SFs as follows

$$\bar{F}(t) = e^{-\theta t}; \qquad t \geq 0,\ \theta > 0,$$

$$\bar{G}(t) = e^{-\lambda t}; \qquad t \geq 0,\ \lambda > 0.$$

From Equation (3.2), we given

$$J^w(X,Y) = -\frac{\theta\lambda e^{t(\theta+\lambda)}}{2\left(\lambda+\theta\right)^2},$$

$$J^w(X,Y;t) = -\frac{\theta\lambda \cdot (t\lambda + t\theta + 1)}{2\left(\lambda+\theta\right)^2 e^{-t(\theta+\lambda)}}.$$

Example 3.10. Let X and Y have Weibull distributions with same shape parameter 2 and SFs as

$$\bar{F}(t) = e^{-\theta t^2}; \qquad t \geq 0,\ \theta > 0,$$

$$\bar{G}(t) = e^{-\lambda t^2}; \qquad t \geq 0,\ \lambda > 0.$$

From Equation (3.2), we obtain

$$J^w(X,Y) = -\frac{\theta\lambda e^{t^2(\theta+\lambda)}}{\left(\lambda+\theta\right)^2},$$

also, from Equation (3.4), we have

$$J^w(X,Y;t) = -\frac{\theta\lambda\left(t^2(\lambda+\theta)+1\right)}{2\left(\lambda+\theta\right)^2 e^{-t^2(\theta+\lambda)}}.$$

In what follows, we prove another result to show the effect of monotone transformations on WJI defined in Equation (3.2). In this context, we prove the following theorem.

Theorem 3.11. Let X be a nonnegative absolutely continuous random variable with PDF $f(x)$ and CDF $F(x)$. Assume $Y = \phi(X)$, where ϕ is a strictly monotonically increasing and differentiable function. Let $G(y)$ and $g(y)$ denote the distribution and density functions of Y, respectively. Then,

$$J^w(Y) = J(X, \frac{\phi(X)}{\phi'(X)}X). \tag{3.6}$$

Proof. The PDF of $Y = \phi(X)$ is $g_Y(y) = |\frac{1}{\phi'(\phi^{-1}(y))}|f_X(\phi^{-1}(y))$. Therefore,

$$J^w(Y) = -\frac{1}{2}\int_0^\infty y g_Y^2(y)dy.$$

This gives

$$J^w(Y) = -\frac{1}{2}\int_0^\infty y \left[\frac{1}{\phi'(\phi^{-1}(y))}\right]^2 f_X^2(\phi^{-1}(y))dy$$

Substituting $x = \phi^{-1}(y)$, we get

$$\begin{aligned} J^w(Y) &= -\frac{1}{2}\int_0^\infty \phi(x)\left(\frac{1}{\phi'(x)}\right)^2 f_X^2(x)\phi'(x)dx \\ &= -\frac{1}{2}\int_0^\infty f_X^2(x)\frac{\phi(x)}{\phi'(x)}dx \\ &= -\frac{1}{2}\int_0^\infty f_X(x)\frac{\phi(x)}{\phi'(x)}f_X(x)dx \\ &= J(X, \frac{\phi(X)}{\phi'(X)}X), \end{aligned}$$

the proof is completed. □

Example 3.12. Suppose $\bar{F}(x) = \exp\{-\theta x\}$, $x \in (0, \infty)$, is the true statistical model for random variable X that generated some data. Also, suppose $\bar{G}(x) = \exp\{-2\theta x\}$ and $\bar{S}(x) = \exp\{-5\theta x\}$, $x \in (0, \infty)$, be two SFs determined through non-parametric statistical tests to approximate X. From Equation (3.2), we obtain $J^w(X, X) = J^w(X) = -1/8$, $J^w(X, Y) = -1/9$ and $J^w(X, Z) = -5/72$. Thus, WJI between X and random variable Y which follows the survival function $\bar{G}(x)$ is closer than that of between X and random variable Z which follows the survival function $\bar{S}(x)$. Therefore, Y provides a better approximation to X than Z i.e. the statistical model $\bar{F}(x)$ is the closest to the statistical model $\bar{G}(x)$ that generated data.

Example 3.13. According to Example 3.12 and Equation (3.4), we have

$$J^w(X, X; t) = J(X; t) = -\frac{2t\theta + 1}{8},$$

$$J^w(Y, Y; t) = J(Y; t) = -\frac{4t\theta + 1}{8},$$

$$J^w(Z, Z; t) = J(Z; t) = -\frac{10t\theta + 1}{8}.$$

Also, from Equation (3.4), we obtain the following equations:

$$J^w(Y, Z; t) = -\frac{35t\theta + 5}{49},$$

$$J^w(X, Z; t) = -\frac{30t\theta + 5}{72}.$$

It is seen that $J^w(Y, Z; t)$ is greater than $J^w(X, Z; t)$ for all $t, \theta > 0$.

Example 3.14. Let X and Y be two non-negative random variables with survival functions $\bar{F}(x) = (x + 1)e^{-x}$ and $\bar{G}(x) = e^{-2x}$, $x > 0$ respectively. We obtain

$$J^w(X, Y; t) = -\frac{9t^2 + 6t + 2}{27\,(t + 1)},$$

and

$$J^w(X|Y;t) = -\frac{36t^3 + 78t^2 - 34t - 49}{432t^2 + 864t + 432}.$$

Figure 1 provides the graphs of $J^w(X,Y;t)$ and $J^w(X|Y;t)$ for various values of t in the case where X and Y are random variables with the given SFs. From Figure 1, we can see that both $J^w(X,Y;t)$ and $J^w(X|Y;t)$ are decreasing functions of t.

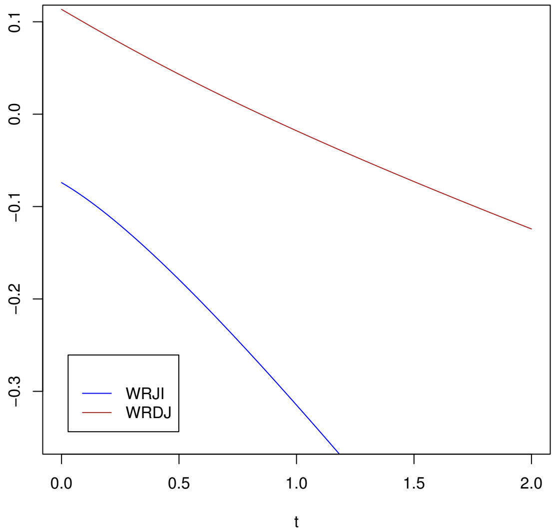

Figure 1. Graph of $J^w(X,Y;t)$ and $J^w(X|Y;t)$ as a function of t.

In the following, we investigate the relationship between WRIJ and WJI.

Corollary 3.15. Suppose that X and Y are continuous nonnegative random variables with SFs $\bar{F}(x)$ and $\bar{G}(x)$, respectively. Then, we given

$$J^w(X,Y;t) = a(t)\left[J^w(X,Y) + c(t)\right], \tag{3.7}$$

where $a(t) = [\bar{F}(t)\bar{G}(t)]^{-1}$ and $c(t) = \frac{1}{2}\int_0^t xf(x)g(x)dx$.

Proof.

$$\begin{aligned} J^w(X,Y;t) &= -\frac{1}{2}\left[\int_0^\infty x\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{\bar{G}(t)}dx - \int_0^t x\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{\bar{G}(t)}dx\right] \\ &= \frac{1}{\bar{F}(t)\bar{G}(t)}\left[J^w(X,Y) + \frac{1}{2}\int_0^t xf(x)g(x)dx\right]. \end{aligned}$$

This completes the proof. □

Corollary 3.16. Let X and Y be two non-negative continuous random variables with PDFs f and g, respectively. Then, we given

$$J^w(X,Y;t) = k_1 J^w(X,Y) - k_2 \bar{J}^w(X,Y;t), \tag{3.8}$$

where $k_1 = [\bar{F}(x)\bar{G}(x)]^{-1}$, $k_2 = \frac{F(x)G(x)}{\bar{F}(x)\bar{G}(x)}$ and $\bar{J}^w(X,Y;t)$ is the weighted past inaccuracy measure.

4. Some properties of WRJI measure

Recently, several authors studied the subject of characterizing underlying distribution of a sample based on the extropy. The general characterization problem is to determine when the residual measure characterizes the CDF uniquely. In this section, we study characterization problem for the proposed WRJI in Equation (3.4) under the PHR model. We study characterization problem for the WRJI under the assumption that the distribution functions of X and Y satisfy the PHR model. Under this model, the SFs of two random lifetime variables are related by

$$\bar{G}(x) = \left[\bar{F}(x)\right]^{\gamma}, \quad \gamma > 0. \tag{4.1}$$

Notice that, based on the PHR model in Equation (4.1), the hazard rate functions (HRFs) $\mu_F(x)$ and $\mu_G(x)$ satisfy the relation $\mu_G(x) = \gamma\mu_F(x)$. The PHR model assumes that the hazard rate for an individual at any given time is a constant multiple of the hazard rate for a reference individual. This assumption implies that the hazard ratios between different groups remain constant over time. It is important to discuss why this assumption is appropriate for the specific context of the study and how it aligns with the data being analyzed. Justifying the use of the PHR model will add credibility to the results obtained using this methodology. Since X and Y satisfy the PHR model, $J^w(X, Y)$, $J^w(X, Y; t)$ and $J^w(X|Y; t)$ can be rewritten as follows. Let X and Y be two nonnegative continuous random variables satisfying the PHR model. Then, we have

$$(I) \quad J^w(X, Y) = -\frac{\gamma}{2}\int_0^{\infty} x\mu_F^2(x)\bar{F}^{\gamma+1}(x)dx, \tag{4.2}$$

$$(II) \quad J^w(X, Y; t) = -\frac{\gamma}{2}\int_t^{\infty} x\mu_F^2(x)\left[\frac{\bar{F}(x)}{\bar{F}(t)}\right]^{\gamma+1} dx, \tag{4.3}$$

$$(III) \quad J^w(X|Y; t) = \frac{1}{2}\int_0^{\infty} x\left(\frac{f(x)}{\bar{F}(t)}\right)^2\left[1 - \gamma\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{\gamma+1}\right] dx. \tag{4.4}$$

When $\gamma = 1$, that is, $\bar{F}(x) = \bar{G}(x)$, then Equation (4.3) becomes Equation (3.3).

Some alternative expressions to Equations (4.2) and (4.3) of WJI and WRJI of a non-negative random variable X are provided hereafter.

Corollary 4.1. *Let X and Y be two nonnegative continuous random variables satisfying the PHR model. Then, we have*

$$(I) \quad J^w(X, Y) = -\frac{\gamma}{2}\int_0^{\infty} xf^2(x)\bar{F}^{\gamma-1}(x)dx, \tag{4.5}$$

$$(II) \quad J^w(X, Y; t) = -\frac{\gamma}{2}\int_t^{\infty} x\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{\gamma-1}\left(\frac{f(x)}{\bar{F}(t)}\right)^2 dx, \tag{4.6}$$

$$(III) \quad J^w(X, Y; t) = -\frac{\gamma}{2}\int_t^{\infty} xf^2(x)\left(\frac{\bar{F}^{\gamma-1}(x)}{\bar{F}^{\gamma+1}(t)}\right) dx. \tag{4.7}$$

In this following, we show that the WRJI measurement of inaccuracy can uniquely determine the underlying distribution. In addition, we study the following properties of the WRJI.

Theorem 4.2. Let X and Y be two non-negative continuous random variables that satisfy the PHR model, then $J^w(X, Y; t)$ uniquely determines the survival function $\bar{F}(x)$ of the random variable X.

Proof. Suppose X_1, Y_1 and X_2, Y_2 are two sets of random variables satisfying PHR model, that is, $\lambda_{G_1}(x) = \gamma\lambda_{F_1}(x)$, $\lambda_{G_2}(x) = \gamma\lambda_{F_2}(x)$ and

$$J^w(X_1, Y_1; t) = J^w(X_2, Y_2; t), \quad t \geq 0. \tag{4.8}$$

By differentiating from both side of Equation (4.8) with respect to t and using $\lambda_G(x) = \gamma\lambda_F(x)$, we have

$$\begin{aligned}\frac{d}{dt}J^w(X,Y;t) &= -\frac{\gamma}{2}\left[-t\lambda_F^2(t) + (\gamma+1)\frac{f(t)\int_t^\infty x\lambda_F^2(x)\bar{F}^{\gamma+1}(x)dx}{\bar{F}^{\gamma+2}(t)}\right] \qquad (4.9)\\ &= -\frac{\gamma}{2}\left[-t\lambda_F^2(t) + (\gamma+1)\lambda_F(t)\int_t^\infty x\lambda_F^2(x)\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{\gamma+1}dx\right]\\ &= \frac{\gamma}{2}t\lambda_F^2(t) + \lambda_F(t)J^w(X,Y;t) + \gamma\lambda_F(t)J^w(X,Y;t)\\ &= \lambda_F(t)\left[\frac{t\gamma\lambda_F(t)}{2} + (\gamma+1)J^w(X,Y;t)\right].\end{aligned}$$

From Equation (4.8) we given

$$\lambda_{F_1}(t)\left(\frac{\gamma t\lambda_{F_1}(t)}{2} + (\gamma+1)J^w(X_1,Y_1;t)\right) = \lambda_{F_2}(t)\left(\frac{\gamma t\lambda_{F_2}(t)}{2} + (\gamma+1)J^w(X_2,Y_2;t)\right) \tag{4.10}$$

Now to prove that Equation (4.8), under the assumption of PHR model in Equation (4.1), implies $\bar{F}_1(t) = \bar{F}_2(t)$, it is sufficient to prove that

$$\lambda_{F_1}(t) = \lambda_{F_2}(t), \quad \forall t > 0. \tag{4.11}$$

In the sequel, define a set $\Omega = \{t : t \geq 0, and \quad \lambda_{F_1}(t) \neq \lambda_{F_2}(t)\}$ and suppose the set Ω is not empty. Thus for some $t_0 \in \Omega$, $\lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0)$. Without loss of generality assume that $\lambda_{F_1}(t_0) > \lambda_{F_2}(t_0)$ and hence Equation (4.10) holds, when either

$$\frac{\gamma t\lambda_{F_1}(t)}{2} + (\gamma+1)J^w(X_1,Y_1;t_0) < \frac{\gamma t\lambda_{F_2}(t)}{2} + (\gamma+1)J^w(X_2,Y_2;t_0) \tag{4.12}$$

or

$$\frac{\gamma t\lambda_{F_1}(t)}{2} + (\gamma+1)J^w(X_1,Y_1;t_0) = \frac{\gamma t\lambda_{F_2}(t)}{2} + (\gamma+1)J^w(X_2,Y_2;t_0) = 0. \tag{4.13}$$

Let inequality (4.12) holds, then using Equation (4.8), inequality (4.12) reduces to $\lambda_{F_1}(t_0) < \lambda_{F_2}(t_0)$. If equality (4.13) holds, then using Equation (4.8), it reduces to $\lambda_{F_1}(t_0) = \lambda_{F_2}(t_0)$. Combining these two results, we get $\lambda_{F_1}(t_0) \leq \lambda_{F_1}(t_0)$. This contradicts our assumption and therefore set Ω is empty and this concludes the proof. □

According to Equation (3.4), the measure of inaccuracy WRJI has ordinary upper bound 0. We will establish a lower bound.

Remark 4.3. Let X and Y be two nonnegative random variables satisfying the PHR model. If $M = f(m) \leq \infty$, where $m = \sup\{x : f(x) \leq M\}$ is the mode of X, then

$$a_1 M^2 \leq J^w(X,Y;t) \leq 0, \tag{4.14}$$

where $a_1 = -\frac{\gamma}{2\bar{F}^{\gamma+1}(t)}\int_t^\infty x\bar{F}^{\gamma-1}(x)dx$.

Similarly, according to Equation (3.2), we have

$$a_2 M^2 \leq J^w(X,Y) \leq 0, \tag{4.15}$$

where $a_2 = -\frac{\gamma}{2}\int_0^\infty x\bar{F}^{\gamma-1}(x)dx$.

Proposition 4.4. Let X and Y be two nonnegative continuous random variables satisfying the PHR model. The maxima of weighted dynamic residual inaccuracy (WDRJI) measurer exist when F is exponential.

Proof.

$$\begin{aligned} J^w(X,Y;t) &= -\frac{\gamma}{2\bar{F}^{\gamma+1}(t)}\int_t^\infty xf^2(x)\bar{F}^{\gamma-1}(x)dx \\ &= \frac{\gamma}{2\bar{F}^{\gamma+1}(t)}\int_t^\infty xf^2(x)\left[1-1+\bar{F}^{\gamma-1}(x)\right]dx \\ &= \frac{\gamma}{2\bar{F}^{\gamma-1}(t)}J^w(X,t)+\frac{\gamma}{2\bar{F}^{\gamma+1}(t)}\int_t^\infty xf^2(x)\left[1-\bar{F}^{\gamma-1}(t)\right]dx. \quad (4.16) \end{aligned}$$

The maxima of $J^w(X;t)$ exist, when $f(x)=\theta exp\{-x\theta\}$ and $\max\{J^w(X;t)\}=-\frac{1}{4}\left[\theta t+\frac{1}{2}\right]$. Thus, from Equation (4.16) the maxima of $J^w(X,Y;t)$ under PHR model also exists only when $f(x)=\theta exp\{-x\theta\}$, and $\max\{J^w(X,Y;t)\}=-\frac{\gamma}{2(\gamma+1)}\left[\theta t+\frac{1}{\gamma+1}\right]$. □

In the following, we express WDRJI in terms of the mean residual lifetime (MRL). Let X be a nonnegative continuous random variable with survival function $\bar{F}$, such that $E(X)$ is finite. Then MRL of X is defined in Equation (4.17) as

$$m(t)=\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}dx, \quad t\geq 0. \tag{4.17}$$

The MRL function is of interest in many fields such as survival analysis, actuarial studies, economics, reliability, and so on.

Remark 4.5. Let X and Y be two non-negative random variables that satisfy the PHR model. Then, we have

$$J^w(X,Y;t)=-\frac{\gamma}{2}\int_t^\infty c^* x m(x)^{-2}dx, \tag{4.18}$$

where $c^*=\left[1+m'(x)\right]^2\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{\gamma+1}$.

Example 4.6. Let X and Y follow exponential and Lindley distributions, respectively, with SFs given by

$$\bar{F}(t)=e^{-\theta t};\quad f(t)=\theta e^{-\theta t},\quad \theta>0,\ t\geq 0,$$
$$\bar{G}(t)=(1+\frac{\lambda}{\lambda+1}t)e^{-\lambda t};\quad f(t)=\frac{\lambda^2}{\lambda+1}(1+t)e^{-\lambda t},\quad \lambda>0,\ t\geq 0.$$

After some algebraic manipulations, we have

$$J^w(X,Y;t)=-\frac{\theta\lambda^2\cdot\left(\left(t^2+t\right)\lambda^2+\left(\left(2t^2+2t\right)\theta+2t+1\right)\lambda+\left(t^2+t\right)\theta^2+\left(2t+1\right)\theta+2\right)}{2\left(\lambda+\theta\right)^3\left(\left(t+1\right)\lambda+1\right)},$$

$$J^w(X,Y)=-\frac{\theta\lambda^2\cdot(\lambda+\theta+2)}{2\left(\lambda+1\right)\left(\lambda^3+3\theta\lambda^2+3\theta^2\lambda+\theta^3\right)},$$

$$J^w(Y;t)=-\frac{\left(4t^3+8t^2+4t\right)\lambda^3+\left(6t^2+8t+2\right)\lambda^2+\left(6t+4\right)\lambda+3}{16\left(\left(t+1\right)\lambda+1\right)^2},$$

$$J^w(X;t)=-\frac{2t\theta+1}{8}.$$

$J^w(X,Y;t)$, $J^w(Y,t)$ and $J^w(X,t)$ are shown in Figure 2 for some selected values of λ and θ. Figure 2 shows that the WRJI inaccuracy measure and the residual extropy of both X and Y are decreasing over time t.

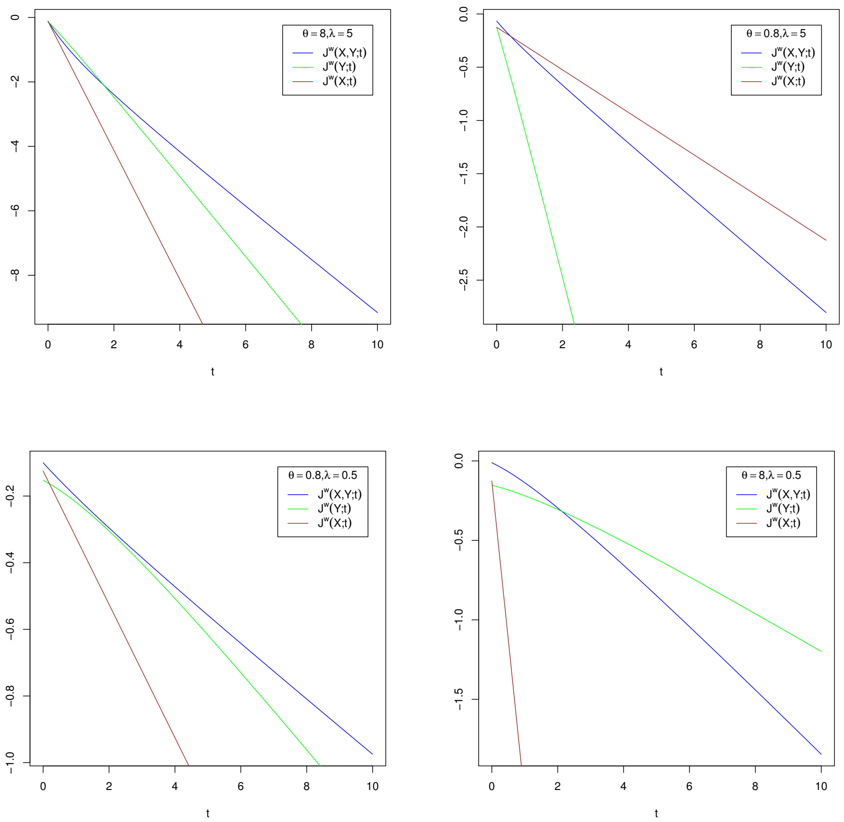

Figure 2. Graph of $J^w(X,Y;t)$, $J^w(X;t)$ and $J^w(Y;t)$ for some selected values of λ and θ.

Example 4.7. Let a non-negative random variable X be uniformly distributed over (c,d), such that $c<d$, with survival and density functions, respectively given by

$$\bar{F}(x)=\frac{d-x}{d-c},\ x\in(c,d),$$

and

$$f(x)=\frac{1}{d-c}.$$

If the random variables X and Y satisfy the PHR model, then the SF of the random variable Y is

$$\bar{G}(x)=\bar{F}^{\gamma}(x)=\left[\frac{d-x}{d-c}\right]^{\gamma},\ x\in(c,d),\ \gamma\in(0,\infty).$$

Substituting these in Equation (3.2) and simplifying, we obtain WRJI measure as

$$\begin{aligned} J^w(X,Y) &= -\frac{\gamma}{2(d-c)^{\gamma+1}}\int_c^d x(d-x)^{\gamma-1}dx \\ &= -\frac{(d-c)^{-\delta-1}\,(c\delta+d)\,\mathrm{e}^{\ln(d-c)\,\delta}}{2\,(\delta+1)} \\ &= -\frac{(c\delta+d)\delta}{2(d-c)^{\delta}(\delta+1)}. \end{aligned}$$

In the sequel, we characterize the uniform distribution in terms of the WRJI under the assumption that X and Y satisfy the PHR model. Differentiating of Equation (3.4) with respect to t and using Equation (4.1), we obtain

$$\frac{d}{dt}J^w(X,Y;t) = \frac{\gamma t}{2}\lambda_F^2(t) + (\gamma+1)\lambda_F(t)J^w(X,Y;t). \tag{4.19}$$

This gives

$$\frac{d}{dt}J^w(X,Y;t) - \frac{\gamma}{2}t\lambda_F^2(t) - (\gamma+1)\lambda_F(t)J^w(X,Y;t) = 0.$$

Hence for a fixed $t > 0$, $\lambda_F(t)$ is a solution of $g^w(x) = 0$, where

$$g^w(x) = \frac{d}{dt}J^w(X,Y;t) - \frac{\gamma}{2}x^2 t - (\gamma+1)xJ^w(X,Y;t). \tag{4.20}$$

Differentiating both side of Equation (4.20) with respect to x, we get

$$\frac{d}{dx}g^w(x) = -\gamma t x - (\gamma+1)\,J^w(X,Y;t). \tag{4.21}$$

Thus, $\frac{d}{dx}g^w(x) = 0$ gives,

$$x = -\frac{\gamma+1}{\gamma t}J^w(X,Y;t) = x_0. \tag{4.22}$$

In the following, we give a theorem which characterizes uniform distribution in terms of WRJI.

Theorem 4.8. Suppose non-negative continuous random variables X and Y satisfy the PHR model in Equation (4.1). Then random variable X over (c,d) such that $c < d$ has uniform distribution if and only if

$$J^w(X,Y;t) = \frac{\gamma t + d}{2\,(\gamma+1)\,(t-d)}. \tag{4.23}$$

Proof. The only if part of the theorem is straightforward since, in the case of a uniform distribution of the random variable X over (c,d)

$$f(x) = \frac{1}{d-c}, \quad \bar{F}(x) = \frac{d-x}{d-c}.$$

Hence, under PHR model, $G(x) = \left(\frac{d-x}{d-c}\right)^{\gamma}$ which gives $g(x) = \frac{\gamma}{(d-c)^{\gamma}}(d-x)^{\gamma-1}.$ Substituting these in Equation (3.4) and simplifying, we get

$$J^w(X,Y;t) = \frac{\gamma t + d}{2\,(\gamma+1)\,(t-d)}. \tag{4.24}$$

To prove the if part, let Equation (4.20) be valid. Then from Equation (4.23), we have $g^w(0) = \frac{d}{dt}J^w(X,Y;t) < 0$. Also we can show that $g^w(x)$ is a concave function with maximum occurring at $x = x_0$. Thus, $g^w(x) = 0$ has a unique solution if $g^w(x_0) = 0$. We have $x_0 = -\left(\frac{\gamma+1}{\gamma}\right)\frac{J^w(X,Y;t)}{t}$. Using Equation (4.24), we get $x_0 = (d-t)^{-1}$, $t < d$ and

$$g^w(x) = \frac{d}{dt}J^w(X,Y;t) - \frac{\gamma}{2}tx^2 - (\gamma+1)xJ^w(X,Y;t) = 0. \tag{4.25}$$

Thus, $g^w(x) = 0$ has the unique solution given by $x = x_0$. But $\lambda_F(t)$ is a solution to Equation (4.20). Hence $\lambda_F(t) = x_0 = 1/(d-t),\ t < d$ is the unique solution to $g^w(x) = 0$. So, the distribution is uniform. To illustrate the characterization results obtained above, we consider the following example.

Example 4.9. Let random variables $X_1, X_2, \ldots, X_k$ have an exponential distribution with PDF $f(x) = \theta\exp\{-\theta x\}$ and CDF $F(x) = 1 - \exp\{-\theta x\},\ x > 0, \theta > 0$ representing the lifetime of components, in a series system of components k, then the lifetime of the system is given by $Y = \min(X_1, X_2, \cdots, X_k)$. If G is the CDF for Y, then under the PHR model the CDF of Y and its PDF are given $\bar{G}(x) = \bar{F}^k(x)$ and $g(x) = kf(x)\bar{F}^{k-1}(x)$, respectively. From Equation (3.4) under PHR model, we given

$$J^w(X,Y;t) = -\frac{k\theta^2}{2e^{-t\theta(k+1)}} \int_t^\infty xe^{-x\theta(k+1)}dx = -\frac{k\left((k+1)\,t\theta+1\right)}{2\left(k^2+2k+1\right)}. \tag{4.26}$$

Taking *limit* as $k \to \infty$, we obtain

$$\lim_{k\to\infty} J^w(X,Y;t) = -\frac{t\theta}{2}. \tag{4.27}$$

Figure 3 shows that when k, the number of components increases in a series system, then the magnitude of the WRJI decreases. In addition, it is observed that the WRJI inaccuracy measure is decreasing in time t and parameter θ.

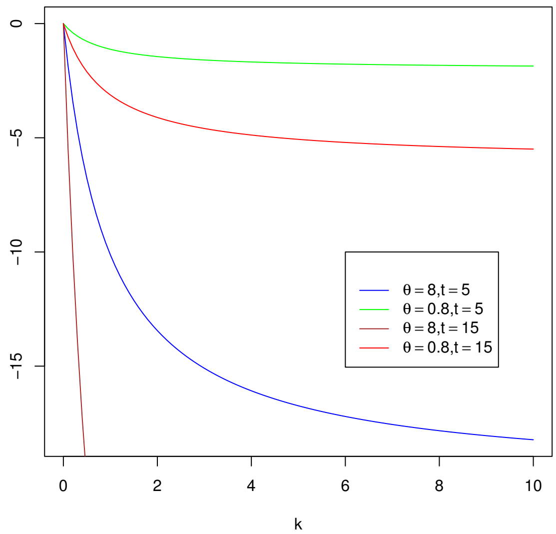

Figure 3. Graph of $J^w(X,Y;t)$ for some selected values of θ and t as a function of k.

5. Some bounds and inequalities for WRJI

In this section, the upper and lower bounds and some inequalities related to WRJI are determined. In the sequel, we express some lower bounds for WRJI in terms of the HRF.

Corollary 5.1. *Suppose X and Y are two non-negative random variables satisfying the PHR model. Then, we have*

$$J^w(X,Y;t) \geq -\frac{\gamma}{2}\int_t^\infty x\lambda_F^2(x)dx. \tag{5.1}$$

Proof. We knew that $t < x$ then $F(t) < F(x)$. This implies $\bar{F}(x)/\bar{F}(t) < 1$. Thus, $[\bar{F}(x)/\bar{F}(t)]^{\gamma+1} < \bar{F}(x)/\bar{F}(t)$. After some calculations and using Equation (4.3), the proof is complete. □

Remark 5.2. Suppose X and Y are two nonnegative random variables satisfying the PHRM. Then, we given

$$J^w(X,Y;t) \geq -\frac{\gamma}{2}\int_t^\infty x\left[-\log\bar{F}(x)\right]^2 dx. \tag{5.2}$$

Proposition 5.3. Let X and Y be two nonnegative random variables. Then, we have

$$J^w(X,Y;t) \geq \left[\bar{F}(t)\bar{G}(t)\right] J^w(X,Y). \tag{5.3}$$

In the following, we consider another example where $F(x)$ and $G(x)$ do not satisfy the PHR model.

Example 5.4. Suppose that X and Y are two non-negative random variables having distribution functions, respectively.

$$F_X(x) = \begin{cases} \frac{x^2}{2} & , \ 0 \leq x < 1 \\ \frac{x^2+2}{6} & , \ 1 \leq x < 2 \\ 1 & , \ x \geq 2 \ , \end{cases}$$

and

$$G_Y(x) = \begin{cases} \frac{x^2+x}{4} & , \ 0 \leq x < 1 \\ \frac{x}{2} & , \ 1 \leq x < 2 \\ 1 & , \ x \geq 2 \ , \end{cases}$$

The WDRJI and WRJI measures are given by

$$J^w(X,Y;t) = \begin{cases} -\dfrac{4-4(4t^3-3t^2)}{3(2-t^2)(4-t^2-t)} - \dfrac{3}{2(4-t^2)(2-t)} & , \ 0 \leq t < 1 \\ -\dfrac{4-t^2}{2(4-t^2)(2-t)} & , \ 1 \leq t < 2 \\ 0 & , \ t \geq 2 \ , \end{cases}$$

and

$$J^w(X,Y) = \begin{cases} -\frac{17}{48} & , \ 0 \leq x < 1 \\ -\frac{1}{4} & , \ 1 \leq x < 2 \\ 0 & , \ x \geq 2 \ . \end{cases}$$

Figure 4 provides the graphs of $J^w(X,Y;t)$ as a function of t. Notice that $J^w(X,Y;t)$ is a decreasing and continuous function in terms of t. □

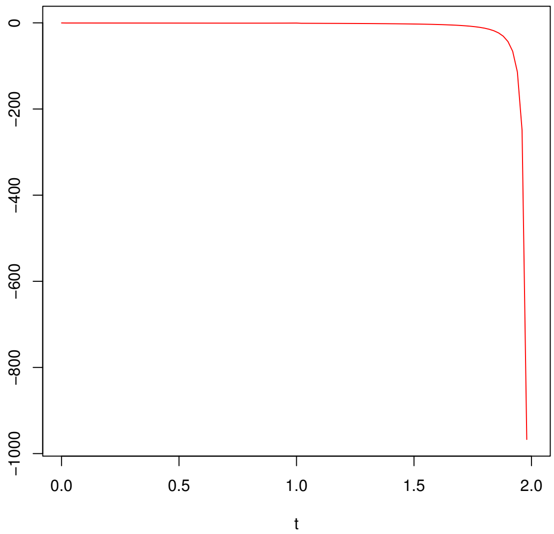

Figure 4. Graph of $J^w(X,Y;t)$ in Example 5.4.

In the sequel, we observe the following relation between three inaccuracy measures considered in this paper.

Remark 5.5. Let X and Y be two non-negative continuous random variables with PDFs respectively $f(x)$ and $g(x)$. Suppose $F(x)$ and $G(x)$ are their CDFs, respectively. The relation between the three inaccuracy measures given by

$$J^w(X,Y) = F(t)G(t)\bar{J}^w(X,Y;t) + \bar{F}(t)\bar{G}(t)\bar{J}^w(X,Y;t). \tag{5.4}$$

Proof. From Equation (3.2), we have

$$\begin{aligned} J^w(X,Y) &= -\frac{1}{2}\int_0^t xf(x)g(x)dx - \frac{1}{2}\int_t^\infty xf(x)g(x)dx \\ &= -\frac{1}{2}F(t)G(t)\int_0^t x\frac{f(x)g(x)}{F(t)G(t)}dx - \frac{1}{2}\bar{F}(t)\bar{G}(t)\int_t^\infty x\frac{f(x)g(x)}{\bar{F}(t)\bar{G}(t)}dx \\ &= F(t)G(t)\bar{J}^w(X,Y,t) + \bar{F}(t)\bar{G}(t)J^w(X,Y,t), \end{aligned}$$

where $\bar{J}^w(X,Y,t)$ is weighted dynamic past inaccuracy measure.

In the next remarks, the relationship between $J^w(X,Y;t)$ and $J^w(X,Y)$ is presented.

Remark 5.6. Let X and Y be two nonnegative random variables satisfying the PHR model. Then, we have

$$J^w(X,Y;t) = c_3(t)J^w(X,Y) + c_4(t), \tag{5.5}$$

where $c_3(t) = \bar{F}^{-(\gamma+1)}(t)$ and $c_4 = \frac{\gamma}{2}\int_0^t x\frac{f^2(x)\bar{F}^{\gamma-1}(x)}{\bar{F}^{\gamma+1}(t)}dx$.

Remark 5.7. Suppose X and Y are two nonnegative random variables satisfying the PHR model. Then, we given

$$J^w(X,Y;t) = b_1(t)[J^w(X,Y) + b_2(t)], \tag{5.6}$$

where $b_1(t) = \bar{F}^{-(\gamma+1)}(t)$ and $b_2 = \frac{\gamma}{2}\int_0^t x\mu^2(x)\bar{F}^{\gamma+1}dx$.

In order to provide a lower bound for the WRJI measure of a random variable X, we study the following conditional mean value (Vitality function)

$$V(X;t) := E(X \mid X > t) = \frac{1}{\bar{F}(t)}\int_t^\infty xf(x)dx, \tag{5.7}$$

a result which finds applications in insurance and economics. For more details, refer to [10].

Theorem 5.8. If the hazard rate function $\lambda_G(x)$ is decreasing in x, then

$$J^w(X,Y;t) \geq -\frac{1}{2}\,\lambda_G(x)V(X;t).$$

Proof. $J^w(X,Y;t)$ can be rewritten as

$$J^w(X,Y;t) = -\frac{1}{2}\int_t^\infty x\frac{f(x)}{\bar{F}(t)}\lambda_G(x)\frac{\bar{G}(x)}{\bar{G}(t)}dx$$

Since $\frac{\bar{G}(x)}{\bar{G}(t)} \leq 1$, for $x \geq t$, and also by the assumption that $\lambda_G(x)$ is a decreasing function, the proof is completed. □

Example 5.9. If the true distribution function $F(x)$ and the reference distribution function $G(x)$ are exponentially distributed with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively, then the WRJI measure is derived as follows. We obtain

$$J^w(X,Y;t) = -\frac{\lambda_1\lambda_2\cdot((\lambda_2+\lambda_1)\,t+1)}{2\,(\lambda_2+\lambda_1)^2}. \tag{5.8}$$

Note that the hazard rate function is constant for an exponential distribution, that is, $\lambda(t) = \lambda$, and the conditional mean value $E(X \mid X > t) = t + \frac{1}{\lambda}$.

In this part, we obtain some lower and upper bounds for the measure of inaccuracy between X and Y. First, we express an upper (a lower) bound for WRJI in terms of the extropy.

Theorem 5.10. *Let X and Y be two nonnegative random variables satisfying the PHR model. Then, we have*

(i) *For $\gamma > 1$, $J^w(X,Y;t) \geq \gamma J^w(X;t)$,*
(ii) *For $0 < \gamma \leq 1$, $J^w(X,Y;t) \leq \gamma J^w(X;t)$.*

Proof. Since $t < x$ then $\bar{F}(x) < \bar{F}(t)$. Also, $\bar{F}(x)/\bar{F}(t) < 1$. Therefore, for $0 < \gamma \leq 1$, $[\bar{F}(x)/\bar{F}(t)]^{\gamma-1} > 1$ and for $\gamma > 1$, $[\bar{F}(x)/\bar{F}(t)]^{\gamma-1} < 1$. After some algebraic manipulations and using Definition 4, the proof is complete. □

In the following, we express a lower bound for $J^w(X,Y;t)$ in terms of the weighted extropy inaccuracy.

Remark 5.11. A lower bound for the WDRIJ between the distributions X and Y is obtained

$$J^w(X,Y;t) \geq a(t)J^w(X,Y), \tag{5.9}$$

where $a(t) = [\bar{F}(t)\bar{G}(t)]^{-1}$.

Proof.

$$\begin{aligned} J^w(X,Y;t) &= -\frac{1}{2}\int_t^\infty x\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{\bar{G}(t)}dx \\ &\geq -\frac{1}{2}\int_0^\infty x\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{\bar{G}(t)}dx \\ &= [\bar{F}(t)\bar{G}(t)]^{-1}J^w(X,Y). \end{aligned}$$

□

In the following, we express some lower bounds for WRJI in terms of the HRF.

Corollary 5.12. Let that X and Y be two non-negative random variables satisfying the PHR model. Then, we have

$$J^w(X,Y;t) \geq c_1\int_t^\infty \bar{F}(x)\mu_F^2(x)dx, \tag{5.10}$$

where $c_1 = -\frac{\gamma}{2\bar{F}(t)}$

Proof. We know that for $t < x$ then $F(t) < F(x)$. This implies $\bar{F}(x)/\bar{F}(t) < 1$. Thus, $[\bar{F}(x)/\bar{F}(x)]^{\gamma+1} < \bar{F}(x)/\bar{F}(x)$. After some calculations and using Equation (4.3), the proof is completed. □

Proposition 5.13. Let X and Y be two non-negative random variables that satisfy the PHR model. Then, we have

$$J^w(X,Y;t) \geq -\frac{\gamma}{2}\int_t^\infty x\mu_F^2(x)dx. \tag{5.11}$$

Proof. We know that $0 \leq \bar{F}(x) \leq 1$ and since $t < x$ then $[\bar{F}(x)/\bar{F}(t)]^{\gamma+1} < 1$. After some algebraic manipulations and using Definition 4, the proof is completed. □

Remark 5.14. Let X and Y be two nonnegative random variables satisfying the PHR model and decreasing PDFs such that $f(0) \leq 1$. Then, we have

$$J^w(X,Y;t) \geq c_2\int_t^\infty x\bar{F}^{\gamma-1}(x)dx, \tag{5.12}$$

where $c_2 = -\frac{\gamma}{2\bar{F}^{\gamma+1}(t)}$.

Remark 5.15. Let X and Y be two non-negative continuous random variables with PDFs f and g, respectively. Then, we have

(i) $J^w(X,Y;t) \geq J^w(X;t)$,
(ii) $J^w(X|Y;t) \leq -J^w(X;t)$.

Proposition 5.16. Let X and Y be two nonnegative random variables satisfying the PHR model. Then, we have

$$J^w(X,Y;t) \geq d_1 \int_t^\infty x\mu_F(x)f(x)dx, \tag{5.13}$$

where $d_1 = -\frac{\gamma}{2\bar{F}(t)}$

Proposition 5.17. Suppose X and Y are two non-negative random variables satisfying the PHR model. We have

(i) $J^w(X,Y;t) \geq d_2 \int_t^\infty x\mu_F^2(x)dx$,
(ii) $J^w(X,Y;t) \geq d_2 \int_t^\infty x\mu_F(x)f(x)dx$,

where $d_2 = -\frac{\gamma}{2\bar{F}(t)}$

In the sequel, we express a lower bound for WRJI in terms of the MRL.

Theorem 5.18. Let X and Y be two non negative random variables satisfying the PHR model and decreasing PDFs such that $f(0) \leq 1$. For $\gamma = 2$, we given

$$J^w(X,Y;t) \geq -k_1 m(t), \tag{5.14}$$

where $k_1 = \bar{F}^{-2}(x)$.

In the following, we express an upper bound for WRJI in terms of the measure of inaccuracy in Equation (3.2).

Proposition 5.19. Let X and Y be nonnegative continuous random variables satisfying the PHR model. Then, we have

$$J^w(X,Y;t) \leq k_2 J^w(X,Y), \tag{5.15}$$

where $k_2 = \frac{1}{\bar{F}^{(\gamma+1)}(t)}$.

6. Non-parametric estimators

We define WRJI in Equation (3.4). In this section, we consider the estimation of this parameter. The problem of estimating $f(x)$ is more complicated than that of $\bar{F}(x)$. For this case, a method known as kernel density estimation is used. Let $(X_1, X_2, ..., X_n)$ be a random sample with pdf $f(x)$ and SF $\bar{F}(x)$. Then, an estimator for $f(x)$ can be given as

$$f_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K(\frac{x-X_i}{h_n}),$$

where h_n is a bandwidth satisfied the condition that h_n tends to 0 as n goes to infinity, and $K(x)$ is a kernel function satisfied the following conditions

$$\int_{-\infty}^{\infty} |K(x)|dx < \infty,$$
$$\sup_{-\infty<x<\infty} |K(x)| < \infty,$$
$$\lim_{x\to\infty} |xK(x)| = 0.$$

For further details on this concept, the reader can refer to [25, 34]. We consider two estimators for CDF $F(x)$. An empirical distribution function (ECDF) can be used to estimate $\bar{F}(x)$. Then, ECDF for estimating $\bar{F}(x)$ can be computed as

$$\bar{F}_n(t) = \frac{\sum_{i=1}^{n} I(X_i > t)}{n},$$

where $I(x)$ is an indicator function taking 1 for non-negative x and 0 for otherwise. As $\bar{F}_n(t)$ is a step function, some researchers considered a smoothed version of ECDF based on the kernel estimation method. Let $K(.)$ be a kernel density function. Then an estimation for $F(.)$ can be obtained as

$$\hat{F}_h(x) = \frac{1}{n}\sum_{i=1}^{n} W(\frac{x - X_i}{h_n}), \tag{6.1}$$

where h_n is a bandwidth parameter and $W(x)$ is defined as

$$W(x) = \int_{-\infty}^{x} K(t)dt.$$

Therefore, we consider two estimators for the proposed measure WRJI as

$$J_n^w(X, Y; t) = -\frac{1}{2}\int_t^{\infty} x \frac{f_n(x)}{\bar{F}_n(t)} \frac{g_n(x)}{\bar{G}_n(t)} dx, \tag{6.2}$$

and

$$J_h^w(X, Y; t) = -\frac{1}{2}\int_t^{\infty} x \frac{f_n(x)}{\hat{\bar{F}}_h(t)} \frac{g_n(x)}{\hat{\bar{G}}_h(t)} dx. \tag{6.3}$$

The choice of kernel function is not crucial for estimating $f(.)$ and $F(.)$, but it is the case for bandwidth h_n. In this paper, we apply the cross-validation method to find the best h_n for both $f(.)$ and $F(.)$. In the case of $f(.)$, the best h_n can be obtained by minimizing the mean integrated squared error as

$$h_f = \arg\min_{h_n} E\left[\int f_n^2(x)dx - \frac{2}{n}\sum_{i=1}^{n} f_{n,-i}(X_i)\right], \tag{6.4}$$

where $f_{n,-i}(X_i)$ is the kernel estimation obtained by omitting X_i. Indeed, the h_f in Equation (6.4) is the optimal bandwidth for nonnormal data and as we see the cross validation method for finding the best bandwidth is a data-driven approach. A similar argument can be applied to find the best bandwidth when we need to estimate the CDF $F(.)$. In this case, h_F can be obtained as

$$h_F = \arg\min_{h_n} \frac{1}{n}\sum_{i=1}^{n}\int \left(I(x - X_i \geq 0) - \hat{F}_{h,-i}(x)\right)^2 dx, \tag{6.5}$$

where $F_{h,-i}(x)$ is the kernel estimation of F obtained by omitting X_i. We can refer to [5], for further details and discussions on this subject.

7. Simulation

We run a simulation with 10000 iterations to compute $J_n^w(X, Y; t)$ and $J_h^w(X, Y; t)$. We consider some distributions such as exponential ($\exp(\lambda)$) and beta ($\text{beta}(\alpha, \beta)$) with sample size n varies in $\{30, 50\}$, and then the bias and the mean squared error (MSE) of estimates are provided. The Gaussian kernel function is used and the bandwidths h_n are considered via the method of cross validation as proposed in Section 6 for estimating of PDF $f(.)$ and CDF $F(.)$, respectively, based on two estimators $J_n^w(X, Y; t)$ and $J_h^w(X, Y; t)$. The results of simulation are provided in Table 1. For different values of time t given in Table 1, in the case of exponential, we compare the proposed estimators J_n^w and J_h^w, when we consider an $\exp(\lambda = 1)$ as the actual distribution and $\exp(\lambda)$ for $\lambda = 2, 5, 7$ as the one that assigned by an experimenter. Similar comparison is provided for the case of beta distribution in Table 1. In this case, a beta(1, 1) is considered as the actual distribution.

Table 1. Bias and MSE for $J_n^w(X,Y;t)$ and $J_h^w(X,Y;t)$ for exponential and beta distributions.

exponential			$\lambda=2$		$\lambda=5$		$\lambda=7$	
t	n		J_n^w	J_h^w	J_n^w	J_h^w	J_n^w	J_h^w
0.01	30	bias	0.0513	0.0404	0.0551	0.0518	0.0456	0.0432
		MSE	0.0027	0.0018	0.0035	0.0027	0.0021	0.0019
	50	bias	0.0490	0.0386	0.0543	0.0512	0.0452	0.0431
		MSE	0.0025	0.0016	0.0030	0.0026	0.0021	0.0019
0.05	30	bias	0.0561	0.0456	0.0671	0.0637	0.0588	0.0562
		MSE	0.0033	0.0023	0.0045	0.0041	0.0035	0.0032
	50	bias	0.0537	0.0439	0.0665	0.0634	0.0581	0.0557
		MSE	0.0030	0.0021	0.0044	0.0040	0.0034	0.0031
0.10	30	bias	0.0618	0.0512	0.0820	0.0787	0.0744	0.0716
		MSE	0.0047	0.0029	0.0068	0.0062	0.0056	0.0052
	50	bias	0.0586	0.0499	0.0808	0.0777	0.0739	0.0714
		MSE	0.0036	0.0027	0.0065	0.0061	0.0055	0.0051
beta			$(\alpha,\beta)=(1,4)$		$(\alpha,\beta)=(5,3)$		$(\alpha,\beta)=(6,6)$	
t	n		J_n^w	J_h^w	J_n^w	J_h^w	J_n^w	J_h^w
0.01	30	bias	0.0342	0.0170	0.1028	0.0510	0.0824	0.0410
		MSE	0.0012	0.0004	0.0111	0.0035	0.0824	0.0023
	50	bias	0.0326	0.0170	0.0963	0.0491	0.0773	0.0397
		MSE	0.0011	0.0004	0.0097	0.0031	0.0063	0.0020
0.10	30	bias	0.0455	0.0244	0.1024	0.0537	0.0824	0.0447
		MSE	0.0022	0.0008	0.0112	0.0040	0.0073	0.0027
	50	bias	0.0426	0.0230	0.0946	0.0512	0.0771	0.0426
		MSE	0.0019	0.0007	0.0095	0.0034	0.0063	0.0023
0.30	30	bias	0.0802	0.0571	0.1156	0.0831	0.0955	0.0679
		MSE	0.0072	0.0039	0.0149	0.0082	0.0101	0.0055
	50	bias	0.0726	0.0517	0.1054	0.0756	0.0880	0.0633
		MSE	0.0058	0.0031	0.0123	0.0067	0.0084	0.0046

From Table 1, it is seen that the MSE of $J_n^w(X,Y;t)$ and $J_n^w(X,Y;t)$ decreases as sample size n increases. In both cases exponential and beta, the MSE of the estimator $J_h^w(X,Y;t)$ is less than that of $J_n^w(X,Y;t)$. The estimator based on the kernel method outperforms the one based on the empirical estimation of CDF. So, in practice, we recommend using approach based on the kernel estimations.

8. Real data

Here, we consider two real datasets to show the behavior of the estimator given in the previous part.

First real dataset:

The following dataset is given by [21] represented the remission times (in months) for 128 patients with bladder cancer. This dataset is as

2.09, 3.48, 6.94, 0.08, 4.87, 23.63, 8.66, 13.11, 3.52, 0.20, 2.23, 25.74, 4.98, 9.02, 13.29, 6.97, 2.26, 3.57, 0.40, 7.09, 5.06, 9.22, 13.80, 3.64, 0.50, 0.81, 2.46, 2.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 5.32, 2.62, 3.82, 12.07, 7.32, 14.77, 32.15, 10.06, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 17.14, 36.66, 4.26, 15.96, 4.23, 1.05, 2.69, 8.65, 5.41, 10.75, 16.62, 7.62, 1.19, 2.75, 43.01, 11.25, 7.63, 5.41, 17.12, 1.26, 46.12, 2.83, 5.49, 4.33, 7.66, 3.36, 21.73, 22.69, 6.93, 4.50, 12.63, 2.07, 8.37, 79.05, 2.87, 5.62, 1.35, 11.64, 17.36, 7.87, 3.02, 4.34, 1.40, 7.93, 6.25, 5.71, 6.76, 12.02, 11.79, 18.10, 1.46, 2.02, 3.31, 4.51, 4.40, 5.85, 8.26, 6.54, 8.53, 12.03, 11.98, 19.13, 1.76, 20.28, 2.02, 3.36, 3.25.

For this dataset, we consider three candidate distributions. One of them is the log-logistic distribution with parameters α and λ which has the following PDF and CDF in Equations (8.1) and (8.2) respectively, as

$$g_{LL}(x;\alpha,\lambda) = \alpha\lambda^{-\alpha}x^{\alpha-1}\left(1+(\frac{x}{\lambda})^{\alpha}\right)^{-2}, \tag{8.1}$$

and

$$G_{LL}(x;\alpha,\lambda) = 1-\left(1+(\frac{x}{\lambda})^{\alpha}\right)^{-1}. \tag{8.2}$$

We denote the log-logistic distribution by LL(α,λ). Recently, two generalization of the LL distribution were introduced. The alpha power transformed log-logistic (APLL(α,λ,a)) was introduced by [1] which has the following CDF in Equation (8.3) as

$$F_{APLL}(x;\alpha,\lambda,a) = \frac{a^{G_{LL}(x;\alpha,\lambda)}-1}{a-1}. \tag{8.3}$$

Also, Alfaer et al. [2] introduced an extended version of log-logistic distribution (ExLL(α,λ,a)) which has the following CDF in Equation (8.4) as

$$F_{ExLL}(x;\alpha,\lambda,a) = 1-\left(\frac{1-G_{LL}(x;\alpha,\lambda)}{1-(1-a)G_{LL}(x;\alpha,\lambda)}\right)^{a}. \tag{8.4}$$

We fit the three above distributions to these data. The MLE of the parameters of the above distributions (LL, APLL and ExLL) are given in Table 2. Also the Kolmogorov-Smirnov (K-S) statistics as well as its p-value of the above distributions are given in Table 3. From Table 3, all of the distributions can be fitted to this dataset at the type I error

Table 2. MLE of parameters of proposed distributions.

parameter	LL	APLL	ExLL
$\hat{\alpha}$	1.7251	1.7118	1.4276
$\hat{\lambda}$	6.0898	4.9174	20.0321
$\hat{a}$	–	2.0976	2.0701

Table 3. K-S as well as p-value of proposed distributions.

statistic	LL	APLL	ExLL
K-S	0.0399	0.0400	0.0351
p-value	0.9870	0.9866	0.9975

rate 0.05. In the following, we examine two situations. In the first situation, we consider the LL as the actual distribution of the data and APLL as a distribution assigned by the experimenter. In the second situation, we consider the LL as the actual distribution of the data and ExLL as a distribution assigned by the experimenter. For two situations, we compute $J_n^w(LL, \hat{F}; t)$, $J_h^w(LL, \hat{F}; t)$ and $J^w(LL, F_0; t)$, where $F_0(.)$ can be either APLL or ExLL distribution. The values of two estimators J_n^w and J_h^w and true value $J^w(LL, F_0; t)$ are depicted in Figure 5 as a function of t.

From Figure 5, it is seen that both J_n^w and J_h^w well estimate the value of J^w in both plots. Indeed, in both cases, J_h^w fits true value J^w much better than J_n^w. So, we can use the estimator J_h^w for our practical situations. Also, as expected, the values of J_n^w and J_h^w as well as J^w are decreasing function of t in our two situations.

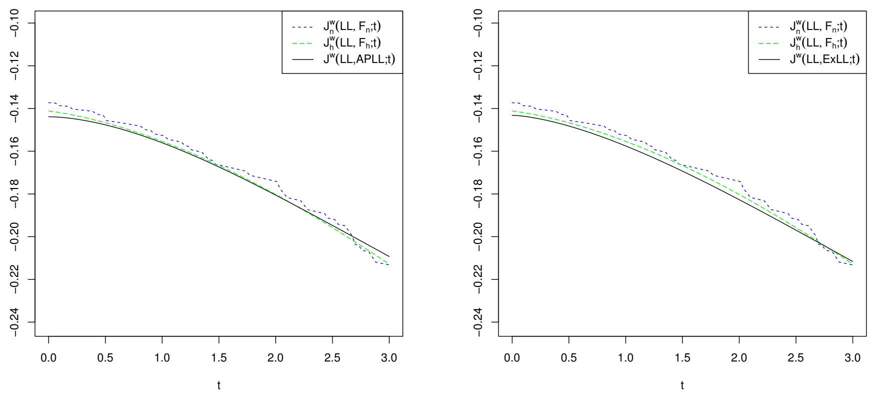

Figure 5. The plot of J_n^w, J_h^w and J^w: left plot is LL as an actual and APLL as a distribution assigned by the experimenter and right plot is LL as an actual and ExLL as a distribution assigned by the experimenter.

Also, in the following Figure 6, we plot the values of $J^w(LL, F_0; t)$ for both model APLL and ExLL as well as $J_h^w(LL, F_h; t)$ when LL is considered as the actual distribution of the data. Figure 6 shows that when APLL distribution is assigned by the experimenter the obtained inaccuracy measure is lower than the case when the experimenter uses the ExLL model for these data. From Figure 6, we see that the estimated J_h^w is closer to the $J^w(LL, APLL; t)$ than the $J^w(LL, ExLL; t)$ when we consider LL as the actual distribution for these data. Therefore, APLL model provides a better approximation to these data when LL is the actual distribution in the sense that the SF $\bar{F}_{APLL}(x)$ is closer to the actual model $\bar{G}_{LL}(x)$ than SF $\bar{F}_{ExLL}(x)$.

Second real dataset:

The following dataset is given by [4] represented the the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. This dataset is as

0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

For this dataset, three candidate distributions are considered. One of them is the Weibull distribution (WEI) with parameters λ and γ which has the following CDF in Equation (8.5) as

$$G_{WEI}(x; \lambda, \gamma) = 1 - \exp^{-\lambda x^{\gamma}}. \tag{8.5}$$

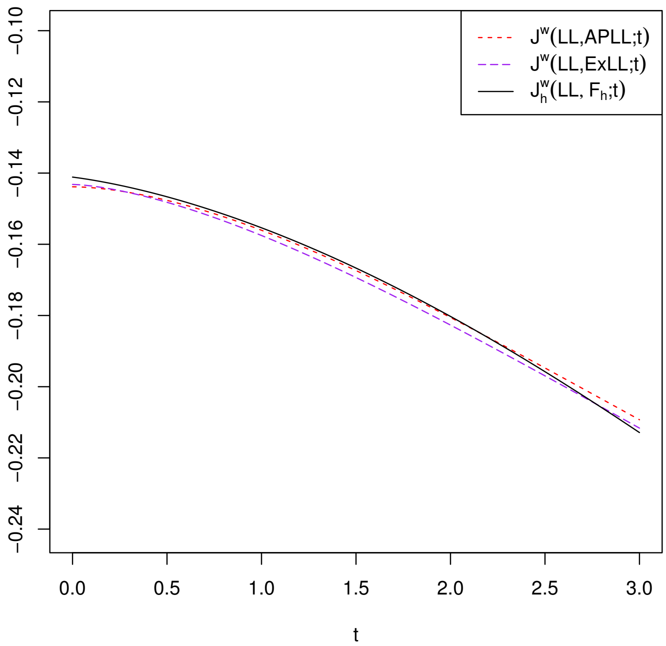

Figure 6. The plot of J_h^w, $J^w(LL, APLL; t)$ and $J^w(LL, ExLL; t)$: LL model is used as an actual and APLL and ExLL models are assigned by the experimenter.

We denote this distribution by WEI(λ, γ). The other two candidates are the gamma exponentiated-exponential (GEE) [33] and exponential-exponential geometric (EEG) [32]models. The GEE$(\lambda, \alpha, \theta)$ has the following PDF in Equation (8.6) as

$$f_{GEE}(x; \lambda, \alpha, \theta) = \frac{\alpha\theta}{\Gamma(\lambda)} \exp^{-\theta x}(1 - \exp^{-\theta x})^{\alpha-1}(-\alpha log(1 - \exp^{-\theta x}))^{\lambda-1}. \tag{8.6}$$

Also, the EEG(α, θ, p) has the following PDF in Equation (8.7) as

$$f_{EEG}(x; \alpha, \theta, p) = \frac{\alpha\theta(1-p)\exp^{-\theta x}(1 - \exp^{-\theta x})^{\alpha-1}}{(1 - p + p(1 - \exp^{-\theta x})^{\alpha})^2}. \tag{8.7}$$

We fit the three above distributions to these data. The MLE of the parameters of the above distributions (WEI, GEE and EEG) are given in Table 4. Also the Kolmogorov-

Table 4. MLE of the parameters of the proposed distributions.

parameter	WEI	GEE	EEG
$\hat{\gamma}$	1.7962	–	–
$\hat{\lambda}$	0.2934	1.2899	–
$\hat{\alpha}$	–	3.4676	3.5144
$\hat{\theta}$	–	0.9118	1.1081
$\hat{p}$	–	–	0.0343

Smirnov (K-S) statistics as well as its p-value of the above distributions are given in Table 5. From Table 5, all of the distributions can be fitted to this dataset at the type I error rate 0.05. As in the first real dataset, we examine two situations. In the first situation, we consider the GEE as the actual distribution of the data and WEI as a distribution assigned by the experimenter. In the second situation, we consider the GEE as the actual distribution of the data and EEG as a distribution assigned by the experimenter. For two

Table 5. K-S as well as p-value of the proposed distributions.

statistic	WEI	GEE	EEG
K-S	0.0982	0.0870	0.0883
p-value	0.4902	0.6458	0.6284

situations, we compute $J_n^w(GEE, \hat{F}; t)$, $J_h^w(GEE, \hat{F}; t)$ and $J^w(GEE, F_0; t)$, where $F_0(.)$ can be either WEI or EEG distribution. The values of the two estimators J_n^w and J_h^w and true value $J^w(GEE, F_0; t)$ are depicted in Figure 7 as a function of t.

From Figure 7, it is seen that J_h^w fits true value J^w much better than J_n^w. So, we can use the estimator J_h^w for our practical situations.

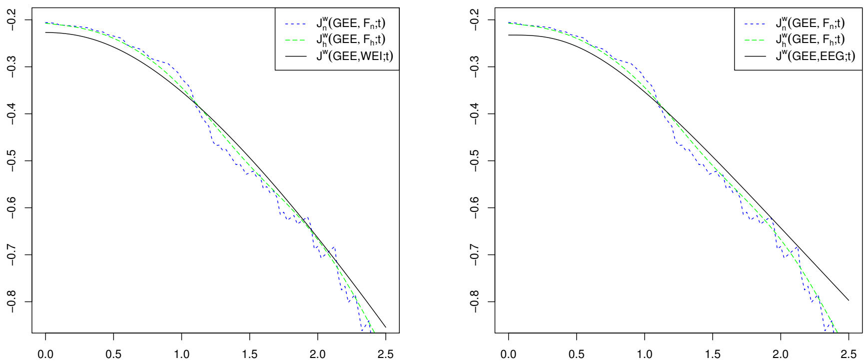

Figure 7. The plot of J_n^w, J_h^w and J^w: left plot is GEE as an actual and WEI as a distribution assigned by the experimenter and right plot is GEE as an actual and EEG as a distribution assigned by the experimenter.

Similar to the first real dataset, in the following Figure 8, we plot the values of $J^w(GEE, F_0; t)$ for both model WEI and EEG as well as $J_h^w(GEE, F_h; t)$ when GEE is considered as the actual distribution of the data. Figure 8 shows that when WEI model is assigned by the experimenter the obtained inaccuracy measure is lower than the case when the experimenter uses EEG model for these data. From Figure 8, we see that the estimated J_h^w is closer to the $J^w(GEE, WEI; t)$ than the $J^w(GEE, EGG; t)$ when we consider GEE as the actual distribution for these data.

As we mentioned previously, the potential impact of WRJI extends to aid in model selection. So, organizations can make better decisions to select the optimized model in dynamic situations.

9. Discussion and conclusion

The WRJI measure is a broadened concept of extropy that serves as a powerful tool for measuring errors in experimental results. It combines an uncertainty measure and a discrimination measure between two distributions to quantify inaccuracies in statements about probabilities of events in an experiment. This measure is particularly useful in statistical inference, estimation, and reliability studies for modeling lifetime data. In lifetime studies, where data is often truncated, WRJI extends information-theoretic concepts to ordered situations and record values, enabling better characterization of probability distributions and identification of the most appropriate model for lifetime data. Traditional methods for finding the best model, such as goodness-of-fit procedures and probability plots, may fall short, making the WRJI measure a valuable tool in this context.

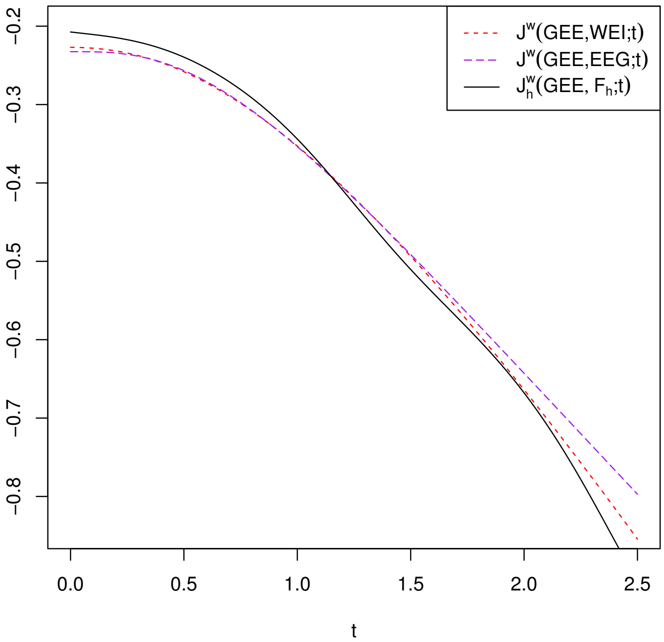

Figure 8. The plot of J_h^w, $J^w(GEE, WEI; t)$ and $J^w(GEE, EEG; t)$: GEE model is used as an actual and WEI and EEG models are assigned by the experimenter.

In summary, the proposed WRJI measure significantly improves:

Quantitative Measurement: WRJI provides a numerical value that quantifies the degree of inaccuracy between two variables X and Y. This allows for a more precise and objective assessment of the relationship between the variables, enabling researchers to make data-driven decisions.

Weighted Approach: By incorporating weighted factors, WRJI accounts for the importance or significance of each observation in the calculation of inaccuracy. This ensures that outliers or extreme values are appropriately weighted, leading to a more accurate representation of the relationship between X and Y.

Comparative Analysis: WRJI facilitates the comparison of inaccuracy between X and Y with other variables or datasets. This comparative analysis helps identify which variables have stronger or weaker relationships, informing decision-making processes and model selection.

Interpret-ability: The numerical value provided by WRJI is easily interpretable, making it accessible to researchers, policymakers, and stakeholders. This clarity in communication enhances the understanding of inaccuracy and supports informed decision-making.

Diagnostic Tool: WRJI serves as a diagnostic tool to identify potential issues or discrepancies in the relationship between X and Y. By measuring inaccuracy, researchers can pinpoint areas that require further investigation or refinement, improving the accuracy and reliability of their analysis.

Practical Applications: The introduction of WRJI in reliability modeling and decision-making has significant practical implications across various industries such as transportation, energy, finance, healthcare, and so on.
Given the importance of the defined measure above, in this paper, by considering the concept of residual extropy inaccuracy measure, its weighted version was proposed. Under the assumption that the reference distribution G and true distribution F satisfy the PHR model, it has been shown that the proposed measure determines the lifetime distribution uniquely. Moreover, upper and lower bounds and some inequalities concerning WRJI are determined. Two non-parametric estimators based on the kernel density estimation method for the proposed measures were also obtained. The performance of the estimators were also discussed using some simulation studies. A real data set was used for illustrating our estimators.

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