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COEFFICIENT ESTIMATE PROBLEMS FOR A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS LINKED WITH THE GENERALIZED BIVARIATE FIBONACCI-LIKE POLYNOMIAL

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Abstract

In this article, using the definition of generalized bivariate Fibonacci-like polynomials that include Horadam and Chebyshev polynomials a novel subclass of bi-univalent functions are introduced. Subsequently, certain bounds are settled for the initial coefficients of the functions belonging to this new subclass. Further, the well-known Fekete-Szegö problem is addressed for the defined subclass. Finally, certain remarks are indicated for the certain private values of variables.

Keywords: Bi-univalent function, coefficient estimates, Fekete-Szegö functional, generalized bivariate Fibonaci-like polynomial

1. Introduction

The recent relations between some special polynomials and geometric function theory have made considerable effect on the researchers. Specially, Chebyshev, Faber, Horadam, Lucas, Fibonacci polynomials and their generalizations were used in definitions of some function subclasses in geometric function theory. By considering the well-known subordination concept and scientific knowledge in geometric function theory, some problems such as coefficient estimations, Fekete-Szegö and Hankel determinant problems were discussed for these new subclasses of analytic functions. Inspring by the relations between special polynomials and analytic function classes, we introduce a new subfamily of analytic and bi-univalent functions in this paper. Further, we deduce certain bounds in terms of generalized bivariate Fibonacci-like polynomials for the first two coefficients of functions belonging to this new subclass. In addition, we solve the Fekete-Szegö problems for the defined function class. Finally, we indicate several corollaries and remarks at the end of the main results.

It is worth to mention here that special polynomials generalize some special number sequences for certain values of the variables. Therefore, we obtain opportunity to investigate a couple function subclasses connected with the special polynomials and number sequences together. Namely, our main results generalize certain earlier results published before.

This paper is organized as follow: Section 1 is divided into three subsections. Some basic definitions of geometric function theory are remembered in the first subsection, while we present knowledge about the generalized bivariate Fibonacci-like polynomials in the second subsection. In third subsection, a novel function subfamily of analytic bi-univalent functions is introduced by making use of generalized bivariate Fibonacci-like polynomials. In Section 2, we determine certain bounds for the second and third coefficients of the functions belonging to the subclass introduced. Also, the Fekete-Szegö problem is discussed for this new subclass in this section. At the end of this section, we indicate certain remarks and corollaries for the initial coefficient estimations and Fekete-Szegö inequalities. Section 3 is devoted to the conclusion part.

1.1 Some basics in univalent function theory

Let $\mathcal{A}$ denote the class of all holomorphic functions of the form

$$f(t) = t + a_2 t^2 + \cdots = t + \sum_{j=2}^{\infty} a_j t^j, \tag{1}$$

in the open unit disk $\mathbb{U} = \{t \in \mathbb{C} : |t| < 1\}$ normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ show the subfamily of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{A}$. There exist some well-known subfamilies of $\mathcal{S}$ which have nice geometrical properties. Some of them are known as convex, starlike and bi-univalent functions. Convex and starlike functions are characterized as

$$\mathcal{C} = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(1 + \frac{tf''(t)}{f'(t)}\right) > 0 \text{ for } t \in \mathbb{U} \right\} \tag{2}$$

and

$$\mathcal{S}^{\star} = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{tf'(t)}{f(t)}\right) > 0 \text{ for } t \in \mathbb{U} \right\}, \tag{3}$$

respectively. Due to the well-known Koebe one quarter theorem (see [1]), it can be easily said that if $f \in \mathcal{S}$, then there exist the inverse function f^{-1} satisfying

$$f^{-1}(f(t)) = t, (t \in \mathbb{U}) \text{ and } f(f^{-1}(s)) = s, \left(|s| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(s) = s - a_2 s^2 + (2a_2^2 - a_3)s^3 - (5a_2^3 - 5a_2 a_3 + a_4)s^4 + \cdots =: g(s). \tag{4}$$

It is well-known that if both functions f and f^{-1} are univalent in $\mathbb{U}$, then the function $f \in \mathcal{A}$ is called bi-univalent function in $\mathbb{U}$. In general, bi-univalent functions' class is shown by Σ and characterized by

$$\Sigma = \{f \in \mathcal{S}: f^{-1} \in \mathcal{S} \text{ for } t \in \mathbb{U}\}. \tag{5}$$

There have been a few significant coefficient estimations on the analytic bi-univalent functions the years between 1967 and 1985. Lewin [5] proved the first estimation on $|a_2|$ in 1967, while Brannan and Clunie [6] presented a bound for the same coefficient of the bi-univalent functions in 1980, respectively. Further, Netenyahu [7] proved that $\max|a_2| = \frac{4}{3}$ in 1969 and Tan [8] deduced $|a_2| \leq 1.485$ for $f \in \Sigma$ in 1984. Also, Brannan and Taha [9] studied on some subclasses of bi-univalent functions and gave certain coefficient bounds in 1985. In geometric function theory, finding a coefficient estimate on $|a_n|$ for $n \in \mathbb{N}, n \geq 3,$ is still an open problem. After 2010, researches on bi-univalent functions has gained a new aspect and speed since the most attractive and pioneering paper (see [4]) on bi-univalent functions was published by Srivastava *et* al. in 2010. This pioneering paper [4] include certain perfect examples of bi-univalent functions and a short history about the class Σ. After publication of this pioneering work many researchers focused on this topic and published numerious results in recent years, see for examples [10-22, 34-47]. The authors make use generally a technique known as subordination in the mentioned articles. This technique may be reminded as follow (see [23]):

If the functions $\mathfrak{f}$ and $\mathfrak{g} \in \mathcal{A}$, then $\mathfrak{f}$ is said to be subordinate to $\mathfrak{g}$ if there exist a Schwarz function $\mathfrak{w}$ such that

$$\mathfrak{w}(0) = 0, |\mathfrak{w}(t)| < 1 \text{ and } \mathfrak{f}(t) = \mathfrak{g}\big(\mathfrak{w}(t)\big) \quad (t \in \mathbb{U}).$$

This subordination is shown by

$$\mathfrak{f} \prec \mathfrak{g} \quad \text{or} \quad \mathfrak{f}(t) \prec \mathfrak{g}(t) \quad (t \in \mathbb{U}).$$

If $\mathfrak{g}$ is univalent in $\mathbb{U}$, then this subordination is equivalent to

$$\mathfrak{f}(0) = \mathfrak{g}(0), \;\; \mathfrak{f}(\mathbb{U}) \subset \mathfrak{g}(\mathbb{U}).$$

A fascinated problem in geometric function theory is known as Fekete-Szegö problem. This problem is related to coefficients of the functions $f \in \mathcal{S}$ and in [2] Fekete and Szegö proved the following sharp result for $f \in \mathcal{S}$:

$$|a_3 - \tau a_2^2| \leq \begin{cases} 4\tau - 3, & \tau \geq 1 \\ 1 + 2e^{\left(\frac{-2\tau}{1-\tau}\right)}, & 0 \leq \tau < 1 \\ 3 - 4\tau, & \tau < 0. \end{cases}$$

The fundamental inequality $|a_3 - \tau a_2^2| \leq 1$ is acquired when $\tau \to 1$. The coefficients' combination of the form

$$F_\tau(f) = a_3 - \tau a_2^2$$

on the functions $f \in \mathcal{A}$ has a considerable impact on geometric function theory. Finding bounds for $|F_\tau(f)|$ is a maximization problem. A paper published by Zaprawa [3] has been a cornerstone to solve Fekete-Szegö problem for bi-univalent functions in the recent years.

1.2 Generalized bivariate Fibonacci-like polynomial

Special polynomials such as Chebyshev, Horadam, Faber, Fibonacci, Lucas and their generalizations have significant cruciality in applied sciences. These polynomials generalize many special sequences of numbers. Especially, Fibonacci numbers is one of the crucial special numbers. There exist a few generalizations of Fibonacci numbers since it has a common usage in the applied sciences (see [24]). A significant generalization of Fibonacci numbers is generalized bivariate Fibonacci polynomial. For historical development and detailed knowledge of the generalized bivariate Fibonacci polynomial one can refer to [25] and its references. Further, the authors introduced a novel generification of Fibonacci polynomial that is called generalized bivariate Fibonacci-like polynomial in [25].

Let $a, b, p, q \in \mathbb{Z}^+$ and $x, y \in \mathbb{R}$. The generalized bivariate Fibonacci-like polynomials are defined by the recurrence relation:

$$V_n(x,y) = pxV_{n-1}(x,y) + qyV_{n-2}(x,y), \qquad n \geq 2, \tag{6}$$

where $V_0(x,y) = a$, $V_1(x,y) = b$ and $px \neq 0$, $qy \neq 0$, $p^2x^2 + 4qy \neq 0$. The generating functions of generalized bivariate Fibonacci-like polynomials is (see [25])

$$\begin{aligned} V^{(x,y)}(t) &= \sum_{n=0}^{\infty} V_n(x,y)t^n = \frac{a + (b - apx)t}{1 - pxt - qyt^2} \\ &= a + bt + (bpx + aqy)t^2 + \cdots = V_0(x,y) + V_1(x,y)t + V_2(x,y)t^2 + \cdots. \end{aligned} \tag{7}$$

Remark 1.1 Choosing differently the parameters p, q, a, b and y, we get the next polynomial sequences:

1. $V_n(x,y)$ polynomial reduces to Bivariate Fibonacci polynomial $F_n(x,y)$ for $a = 0$ and $p = q = b = 1$.

2. $V_n(x,y)$ polynomial reduces to Fibonacci polynomial $F_n(x)$ for $a = 0$ and $p = q = b = y = 1$.

3. $V_n(x,y)$ polynomial reduces to Pell polynomial $P_n(x)$ for $a = 0$, $p = 2$ and $q = b = y = 1$.

4. $V_n(x,y)$ polynomial reduces to Bivariate Lucas polynomial $L_n(x,y)$ for $a = 2$, $b \to x$ and $p = q = 1$.

5. $V_n(x,y)$ polynomial reduces to Chebyshev polynomial of the second kind $U_n(x)$ for $a = q = 1$, $b \to 2x$, $p = 2$ and $y = -1$.

6. $V_n(x,y)$ polynomial reduces to Horadam polynomial $H_{n+1}(l)$ for $b \to bl$ and $y = 1$.

Hereafter, we will use V_n instead of $V_n(x, y)$ for brevity.

1.3 The Function Class $\mathfrak{F}_\Sigma^\alpha(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y})$

In this part, a new subclass of bi-univalent functions is introduced as follow:

Definition 1.2 Let $\alpha \in [0,1]$. A function $f \in \Sigma$ of the form (1) is said to be in the class $\mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$ if the next conditions hold true:

$$\alpha\left(1 + \frac{tf''(t)}{f'(t)}\right) + (1 - \alpha)\left(\frac{tf'(t)}{f(t)}\right) \prec V^{(x,y)}(t) + 1 - a \tag{8}$$

and

$$\alpha\left(1 + \frac{sg''(s)}{g'(s)}\right) + (1 - \alpha)\left(\frac{sg'(s)}{g(s)}\right) \prec V^{(x,y)}(s) + 1 - a, \tag{9}$$

where $p^2x^2 + 4qy > 0$, $t, s \in \mathbb{U}$ and the function g is of the form (4).

Remark 1.3 It is important to note that the above subclass of analytic bi-univalent functions reduces to certain subclasses of functions for certain values of the parameters a, b, p, q, x, y and α. Also, we would like to emphasize here that some of the obtained subclasses were studied previously by several authors.

1. For $\alpha = 1$, the class $\mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$ reduces to the class $\mathfrak{F}_\Sigma^1(a, b, p, q, x, y)$ of bi-convex functions subordinate to the generalized bivariate Fibonacci-like polynomial which satisfy the followings:

$$1 + \frac{tf''(t)}{f'(t)} \prec V^{(x,y)}(t) + 1 - a$$

and

$$1 + \frac{sg''(s)}{g'(s)} \prec V^{(x,y)}(s) + 1 - a.$$

This class of functions is introduced and studied in [10, Definition 1.5].

2. For $\alpha = 0$, the class $\mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$ reduces to the class $\mathfrak{F}_\Sigma^0(a, b, p, q, x, y)$ of bi-starlike functions subordinate to the generalized bivariate Fibonacci-like polynomial which satisfy the followings:

$$\frac{tf'(t)}{f(t)} \prec V^{(x,y)}(t) + 1 - a$$

and

$$\frac{sg'(s)}{g(s)} \prec V^{(x,y)}(s) + 1 - \alpha.$$

This class of functions is defined and invesigated in [10, Definition 1.4].

3. For $x \in (1/2,1]$, $y = -1$, $a = q = 1$, $p = 2$ and $b \to 2x$ the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ reduces to the class $\mathfrak{F}_\Sigma^\alpha(1,2x,2,1,x,-1) \equiv H_\Sigma(\alpha,x)$ subordinate to the Chebyshev polynomial of the second kind $U_n(x)$ which introduced by Altınkaya and Yalçın in [26, Definition 1].

4. For $\alpha = 0$, $x \in (1/2,1]$, $y = -1$, $a = q = 1$, $p = 2$ and $b \to 2x$ the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ reduces to the class $\mathfrak{F}_\Sigma^0(1,2x,2,1,x,-1) \equiv S_\Sigma^\star(x)$ of bi-starlike functions subordinate to the Chebyshev polynomial of the second kind $U_n(x)$ which introduced in [13, Definition 1.4], [15, Definition 1.3], [27, Remark 1, (iv.)], [28, Remark 2.5, (i.)], [29, Remark 1.5] and [30, Special case:xiv]. Also, for $\alpha = 1$, $x \in (1/2,1]$, $y = -1$, $a = q = 1$, $p = 2$ and $b \to 2x$ the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ reduces to the class $\mathfrak{F}_\Sigma^1(1,2x,2,1,x,-1) \equiv K_\Sigma(x)$ of bi-convex functions subordinate to the Chebyshev polynomial of the second kind $U_n(x)$ which introduced in [28, Remark 2.5, (ii.)] and [30, Special case:xiii].

5. For $b \to bx, y = 1$, the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ reduces to the class $\mathfrak{F}_\Sigma^\alpha(a,bx,p,q,x,1) \equiv \mathcal{M}_\Sigma(\alpha,x)$ subordinate to Horadam polynomials which satisfies the followings:

$$\alpha\left(1 + \frac{tf''(t)}{f'(t)}\right) + (1-\alpha)\left(\frac{tf'(t)}{f(t)}\right) \prec \Pi(x,t) + 1 - a$$

and

$$\alpha\left(1 + \frac{sg''(s)}{g'(s)}\right) + (1-\alpha)\left(\frac{sg'(s)}{g(s)}\right) \prec \Pi(x,s) + 1 - a,$$

where $\Pi(x,t)$ denotes generating function of the Horadam polynomial. This class of functions introduced and studied by Abirami, Magesh and Yamini in [31].

6. For $\alpha = 1$, $b \to bx, y = 1$, the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ reduces to the class $\mathfrak{F}_\Sigma^1(a,bx,p,q,x,1) \equiv \mathcal{K}_\Sigma(x)$ of subordinate to Horadam polynomials which introduced by Abirami, Magesh and Yamini in [31].

7. For $\alpha = 0$, $b \to bx$ and $y = 1$, the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ reduces to the class $\mathfrak{F}_\Sigma^0(a,bx,p,q,x,1) \equiv \mathcal{W}_\Sigma(x)$ related to Horadam polynomials which mentioned by Srivastava, Altınkaya and Yalçın in [19, Remark 2].

Remark 1.4 It is worthy noting that the class $\mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ defined in Definition 1.2 is non-empty set. Fixing certain values to the variables we get the following example:

Example 1.5 The function $u(t) = -2\log\left(1 - \frac{t}{2}\right) = t + \frac{t^2}{4} + \frac{t^3}{12} + \cdots$ is in the class $\mathfrak{F}_\Sigma^1(0,1,1,1,1/3,0)$. From Figure 1. (a) and (b), it is easy to see that the function $u(t)$ and its inverse $v(s) = \frac{2\left(e^{\frac{s}{2}}-1\right)}{e^{\frac{s}{2}}} = s - \frac{s^2}{4} + \frac{s^3}{24} + \cdots$ are univalent in $\mathbb{U}$. Therefore, the function $u(t) \in \Sigma$.

Now, replacing $u(t) = -2\log\left(1 - \frac{t}{2}\right)$, $v(s) = \frac{2\left(e^{\frac{s}{2}} - 1\right)}{e^{\frac{s}{2}}}$, $\alpha = b = p = q = 1$, $a = y = 0$ and $x = 1/3$ in (8) and (9) we have

$$u_1(t) = \frac{2}{2 - \mathrm{t}} \prec u_2(t) = \frac{2t + 3}{3 - t} \tag{10}$$

and

$$v_1(s) = 1 - \frac{s}{3} \prec v_2(s) = \frac{2s + 3}{3 - s}. \tag{11}$$

It is easy to check that the relations (10) and (11) hold true for $t, s \in \mathbb{U}$. These relations can be proven from [48, Lemma 2.1, p.36]. In fact, the function $u_2(t) = \frac{2t+3}{3-t}$ (or $v_2(s) = \frac{2s+3}{3-s}$) is analytic and univalent in $\mathbb{U}$, $u_1(0) = u_2(0) = v_1(0) = v_2(0) = 1$, $u_1(\mathbb{U}) \subset u_2(\mathbb{U})$ (see Figure 1. (c)) and $v_1(\mathbb{U}) \subset v_2(\mathbb{U})$ (see Figure 1. (d)).

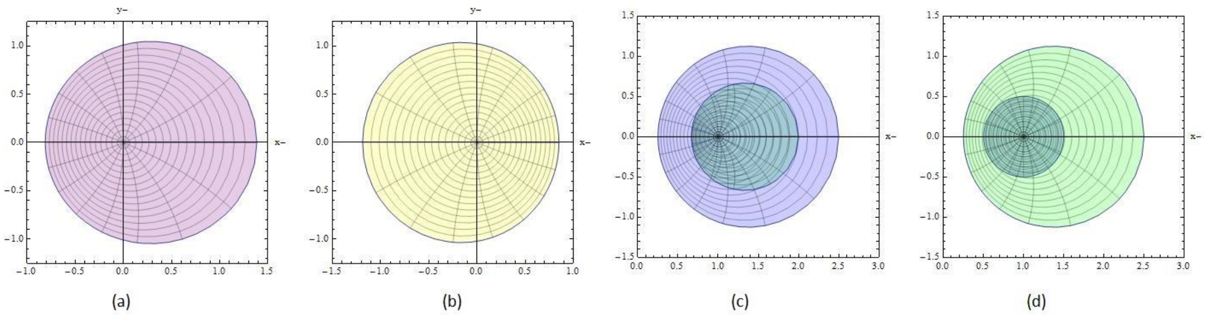

Figure 1. (a) $u(\mathbb{U})$ (b) $v(\mathbb{U})$ (c) $u_1(\mathbb{U}) \subset u_2(\mathbb{U})$ (d) $v_1(\mathbb{U}) \subset v_2(\mathbb{U})$

2. Coefficient estimates and Fekete-Szegö inequality for the function class $\mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$

In this section, the upper bound estimates for the coefficients a_2 and a_3 of functions in the class $\mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$ are presented. Also, we determine bound for the Fekete-Szegö functional of functions in the class $\mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$.

Theorem 2.1 Let $f(t) \in \mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$. Then,

$$|a_2| \le \frac{b\sqrt{b}}{\sqrt{(1+\alpha)|b^2 - (bpx + aqy)(1+\alpha)|}}, \tag{12}$$

and

$$|a_3| \le \frac{b^2}{(1+\alpha)^2} + \frac{b}{2(1+2\alpha)}. \tag{13}$$

Proof. If $f(t) \in \mathfrak{F}_\Sigma^\alpha(a, b, p, q, x, y)$, then, there exist analytic functions $\rho, \kappa \colon \mathbb{U} \to \mathbb{U}$ given by

$$\rho(t) = \sum_{k=1}^{\infty} \rho_k t^k, \tag{14}$$

$$\kappa(s) = \sum_{k=1}^{\infty} \kappa_k s^k \tag{15}$$

such that

$$|\rho(t)| = |\rho_1 t + \rho_2 t^2 + \rho_3 t^3 + \cdots| < 1 \tag{16}$$

and

$$|\kappa(s)| = |\kappa_1 s + \kappa_2 s^2 + \kappa_3 s^3 + \cdots| < 1. \tag{17}$$

Inequalities (16) and (17) imply that $|\rho_k| \leq 1$ and $|\kappa_k| \leq 1$ for $k \in \mathbb{N}$ and $t,\ s \in \mathbb{U}$. By Definition 1.2, it may be written that

$$\alpha\left(1 + \frac{tf''(t)}{f'(t)}\right) + (1-\alpha)\frac{tf'(t)}{f(t)} = V^{(x,y)}\big(\rho(t)\big) + 1 - a \tag{18}$$

and

$$\alpha\left(1 + \frac{sg''(s))}{g'(s)}\right) + (1-\alpha)\frac{sg'(s)}{g(s)} = V^{(x,y)}\big(\kappa(s)\big) + 1 - a. \tag{19}$$

Now, using the infinite series representations of the functions $f(t)$, $g(s)$ and $V^{(x,y)}(t)$ given by (1), (4) and (7), respectively, the equations (18) and (19) reduce to

$$1 + (1+\alpha)a_2 t + [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]t^2 + \cdots = 1 + V_1\rho(t) + V_2\rho^2(t) + \cdots, \tag{20}$$

and

$$1 - (1+\alpha)a_2 s + [(3+5\alpha)a_2^2 - 2(1+2\alpha)a_3]s^2 + \cdots = 1 + V_1\kappa(s) + V_2\kappa^2(s) + \cdots. \tag{21}$$

Replacing $\rho(t)$ and $\kappa(s)$ in (20) and (21), a basic computation yields that

$$1 + (1+\alpha)a_2 t + [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]t^2 + \cdots$$

$$= 1 + V_1\rho_1 t + (V_1\rho_2 + V_2\rho_1^2)t^2 + \cdots, \tag{22}$$

and

$$1 - (1+\alpha)a_2 s + [(3+5\alpha)a_2^2 - 2(1+2\alpha)a_3]s^2 + \cdots$$

$$= 1 + V_1\kappa_1 s + (V_1\kappa_2 + V_2\kappa_1^2)s^2 + \cdots. \tag{23}$$

Comparing the coefficients of the equations (22) and (23), it can be written the next equalities:

$$(1+\alpha)a_2 = V_1\rho_1 \tag{24}$$

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = V_1\rho_2 + V_2\rho_1^2 \tag{25}$$

and

$$-(1+\alpha)a_2 = V_1\kappa_1 \tag{26}$$

$$(3+5\alpha)a_2^2 - 2(1+2\alpha)a_3 = V_1\kappa_2 + V_2\kappa_1^2. \tag{27}$$

From the above equalities, one can easily obtain by summing equalities (24) and (26) that

$$\rho_1 = -\kappa_1, \tag{28}$$

since $V_1 = b \neq 0$. Also, summing squares of (24) and (26) we get,

$$a_2^2 = \frac{V_1^2(\rho_1^2 + \kappa_1^2)}{2(1+\alpha)^2}. \tag{29}$$

In addition, by adding (25) and (27) we have

$$2(1+\alpha)a_2^2 = V_1(\rho_2 + \kappa_2) + V_2(\rho_1^2 + \kappa_1^2). \tag{30}$$

Further, from (29) and (30) it can be concluded that

$$a_2^2 = \frac{V_1^3(\rho_2 + \kappa_2)}{2(1+\alpha)[V_1^2 - V_2(1+\alpha)]}. \tag{31}$$

Since $V_1 = b,\ V_2 = bpx + aqy$ and $|\rho_k| \leq 1, |\kappa_k| \leq 1$ for $k \in \mathbb{N}$, a straightforward calculation yields

$$|a_2| \leq \frac{b\sqrt{b}}{\sqrt{(1+\alpha)|b^2 - (bpx + aqy)(1+\alpha)|}}.$$

On the other hand, if we subtract (25) from (27), we get

$$a_3 = \frac{V_1(\rho_2 - \kappa_2)}{4(1+2\alpha)} + a_2^2, \tag{32}$$

since equation (28). Replacing (29) in (32) and putting $V_1 = b$, we have

$$a_3 = \frac{b(\rho_2 - \kappa_2)}{4(1+2\alpha)} + \frac{b^2(\rho_1^2 + \kappa_1^2)}{2(1+\alpha)^2}. \tag{33}$$

Now, using triangle ineaquality and basic computations we arrive at

$$|a_3| \le \frac{b^2}{(1+\alpha)^2} + \frac{b}{2(1+2\alpha)}.$$

Example 2.2 The function $u(t) = -2\log\left(1-\frac{t}{2}\right)$ fits the requirements of Theorem 2.1. Setting $\alpha = b = p = q = 1,\ a = y = 0$ and $x = \frac{1}{3}$ we obtain that $|a_2| = \frac{1}{4} \le \frac{\sqrt{6}}{2}$ and $|a_3| = \frac{1}{12} \le \frac{5}{12}$.

Theorem 2.3 Let $f(t) \in \mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$ and $\tau \in \mathbb{R}$. Then,

$|a_3 - \tau a_2^2| \le$

$$\begin{cases} \dfrac{b}{2(1+2\alpha)}, & |1-\tau| \le \dfrac{(1+\alpha)|b^2-(bpx+aqy)(1+\alpha)|}{2b^2(1+2\alpha)} \\ \dfrac{b^3|1-\tau|}{(1+\alpha)|b^2-(bpx+aqy)(1+\alpha)|}, & |1-\tau| \ge \dfrac{(1+\alpha)|b^2-(bpx+aqy)(1+\alpha)|}{2b^2(1+2\alpha)}. \end{cases} \quad (34)$$

Proof. Suppose that $\tau \in \mathbb{R}$. If we consider (32), then we can write that

$$a_3 - \tau a_2^2 = \frac{V_1(\rho_2 - \kappa_2)}{4(1+2\alpha)} + (1-\tau)a_2^2. \quad (35)$$

Now, subsituting (31) in (35), we have

$$a_3 - \tau a_2^2 = \frac{V_1(\rho_2 - \kappa_2)}{4(1+2\alpha)} + (1-\tau)\frac{V_1^3(\rho_2+\kappa_2)}{2(1+\alpha)[V_1^2 - V_2(1+\alpha)]}. \quad (36)$$

Putting into place $V_1 = b, V_2 = bpx + aqy$ in (36) implies that

$$a_3 - \tau a_2^2 = b\left\{\left[\Delta(\tau) + \frac{1}{4(1+2\alpha)}\right]\rho_2 + \left[\Delta(\tau) - \frac{1}{4(1+2\alpha)}\right]\kappa_2\right\}, \quad (37)$$

where $\Delta(\tau) = \frac{(1-\tau)b^2}{2(1+\alpha)[b^2(bpx+aqy)(1+\alpha)]}$. Taking modulus both sides of (37) and using triangular inequality with the facts that $|\rho_2| \le 1$, $|\kappa_2| \le 1$ we arrive at (34).

Putting $\tau = 1$ in Theorem 2.3 we get the following:

Corollary 2.4 Let $f(t) \in \mathfrak{F}_\Sigma^\alpha(a,b,p,q,x,y)$. Then,

$$|a_3 - a_2^2| \le \frac{b}{2(1+2\alpha)}. \quad (38)$$

Remark 2.5 Taking $\alpha = 0$ and $\alpha = 1$ in Theorem 2.1 we obtain [10, Remark 2.2 (i.) and (ii.)]. Also, for $\alpha = 0$ and $\alpha = 1$ in Theorem 2.3 we obtain [10, Remark 4.2 (ii.) and (iii.)].

Remark 2.6 Taking $b \to bx$ and $y = 1$ in Theorem 2.1 and Theorem 2.3 we obtain the results in [31, Theorem 2]. If we put $b \to bx$ and $y = 1$ and $\alpha = 1$ in Theorem 2.1 and Theorem 2.3, then the results coincide with the results in [31, Corollary 3].

Remark 2.7 Taking $b \to bx$ and $y = 1$ and $\alpha = 0$ in Theorem 2.1 and Theorem 2.3 we obtain in [19, Corollary 1 and Corollary 3], respectively.

Remark 2.8 Taking $x \in (1/2,1],\ y = -1,\ b \to 2x,\ a = q = 1$ and $p = 2$ in Theorem 2.1 we obtain [26, Theorem 2] and [32, Corollary 8]. Also, for the same values Theorem 2.3 reduces to [26, Theorem 3].

Remark 2.9 Taking $\alpha = 0,\ x \in (1/2,1],\ y = -1,\ b \to 2x,\ a = q = 1$ and $p = 2$ in Theorem 2.1 we obtain the bound on $|a_2|$ in [28, Corollary 5.2]. Also, Taking $\alpha = 1, x \in (1/2,1],\ y = -1,\ b \to 2x,\ a = q = 1$ and $p = 2$ in Theorem 2.1 we obtain the bound on $|a_2|$ in [28, Corollary 5.4].

Remark 2.10 Taking $\alpha = 0,\ x \in (1/2,1],\ y = -1,\ b \to 2x,\ a = q = 1$ and $p = 2$ in Theorem 2.1 we obtain [14, Corollary 2.2], [13, Corollary 2.3] and [15, Corollary 2.1].

Remark 2.11 Taking $\alpha = 0,\ x \in (1/2,1],\ y = -1,\ b \to 2x,\ a = q = 1$ and $p = 2$ in Theorem 2.1 and Theorem 2.3 we obtain the results of [27, Corollary 4] and [33, Corollary 2].

3. Conclusion

In this paper, with the aid of generalized bivariate Fibonacci-like polynomial a novel subclass of functions both analytic and bi-univalent is introduced. At first, initial coefficient estimates are discussed and then the well-known Fekete-Szegö problem is solved for the subclass defined. We note that private choosing of the variables p, q, x, y, a, b and α our findings cover certain results in the field.

In the future works, this study can be extended to various conic regions and also Hankel determinant problems may be investigated for the defined subclass.

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