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Research Article

Reliability inferences in a 1-out-of-n:G multicomponent stress-strength system with unit gamma Gompertz-G_0 components

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Abstract

This paper considers reliability inferences in a system of stress-strength 1 outside of n: G when the strength systems belong to the gamma Gompertz unit distribution family (UGG). Stochastic comparisons are obtained between the survival distribution functions of this model. Additionally, some stochastic comparisons are carried out with majorized shape parameters of the unit gamma Gompertz distribution. The asymptotic and several bootstrap confidence intervals of the reliability of the stress strength are studied. In addition, the efficiency of the asymptotic and bootstrap confidence intervals is analyzed by simulation. A numerical example based on real-life data is displayed as an illustration.

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1. Introduction

Reliability of the system is the probability that the system will perform the intended task correctly when operating under certain environmental conditions. In stress-strength models, the strength of the system Y and the stress X caused by its working environment are treated as random variables. In the stress-strength system, if the random stress exceeds the random strength, then the system will fail. Therefore, the reliability of stress-strength is described by $R := P(Y > X)$. This basic idea was introduced by Birnbaum [1] and developed by Birnbaum and McCarty [2]. The stress strength model has been applied in various fields such as engineering, seismology, oceanography, hydrology, economics, and medicine; see, for example, [3] and [4], where the monograph by Kotz et al. [4] provided a comprehensive review on this topic up to 2003. Some of the applications of this model have been described by [5]-[10]. Each system usually is under different stresses during the period of its operation, therefore, so many of the systems that we deal with daily are a type of stress-strength model, and the study of their features is important. When independent random variables X and Y follow a specified distribution, the estimation of

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$R = P(Y > X)$ has been extensively discussed by many authors in the literature. Some recent contributions on the subject can be found in [11] for the inverse Pareto, [12] for the unit inverse Weibull, [13] for the inverted exponential Rayleigh, [14] and [19] for the Kumaraswamy [15] for the weighted Lindley, [16] for the two-parameter bathtub-shaped lifetime,[17] for the Burr Type X, [18] for the generalized logistic, [20] for the proportional hazard, [48] for the generalized inverted exponential, [21] for the log-logistic, [49] for the inverse Rayleigh, and [22] for the Weibull distributions. Since the components of the stress-strength system have a UGG distribution in this paper, we will briefly discuss this distribution below.

Many authors have discussed the unit distribution in recent years. These models are often used to describe various events such as proportions, percentages, and probabilities stated in $(0, 1)$. One reason for the expansion of the application of unit distributions is the increase of combined data in various fields of research, such as medicine, biology, meteorology, hydrology, economic modeling, etc. For example, several results of survival theory, such as unit survival time and system lifetime, are often greater than zero, but do not exceed sufficiently large values. Thus, they are limited to a specific region, and boundary models help to describe such data sets. In each of these situations, the random variables can be converted to the interval $(0, 1)$ using normalization or other transformation methods. Unit distributions offer the advantage of being able to be calculated using $Y = e^{-X}$ or $Y = \frac{X}{1+X}$ transformations on any statistical distribution explained in the sets $\mathbb{R}$ or $\mathbb{R}^+$. Concerning this, various unit distributions, including the Burr XII unit [23], UGG [24], and the inverse Gaussian unit [25].

The Gompertz model is a statistical distribution with a monotonic hazard ratio function. In many real-world scenarios, the hazard rate function does not follow a monotonic pattern but resembles a bathtub curve. As a result, various generalizations of the Gompertz distribution have been developed. One such generalization is the gamma Gompertz distribution, originally introduced by Sharma et al. [26]. A UGG distribution is versatile as it can have increasing, decreasing, increasing-decreasing, or decreasing-increasing density functions. Furthermore, the inverse hazard rate function of the UGG distribution can be monotonically increasing or decreasing and sometimes remain constant.

For any CDF $G_0(x)$, the unit gamma Gompertz-G_0 which we signified during this paper by $UGG - G_0(\alpha, \beta, \mu)$, has respectively CDF and the probability density function (PDF) as below,

$$G(x) = \left(\frac{\mu}{\mu - 1 + {G_0}^{-\alpha}(x)}\right)^{\beta},\ x > 0, \tag{1.1}$$

$$g(x) = \frac{\alpha\beta {G_0}^{-\alpha-1}(x)}{\mu - 1 + {G_0}^{-\alpha}(x)} g_0(x)G(x),\ x > 0, \tag{1.2}$$

where, $G_0(x)$ is the baseline CDF, $g_0(x)$ is baseline PDF, $\alpha > 0$ and $\beta > 0$ are the shape parameters, and $\mu > 0$.

Several authors have also studied multicomponent stress-strength systems. One of the multicomponent systems is the k-out-of n:G stress-strength system regarded as alive only if at least k out of n ($k < n$) strengths exceed the stress. Furthermore, the 1-out-of n:G stress-strength system demonstrates a parallel stress-strength system, which is widely used in various devices, such as the computer hard disk, brake systems, and support cables on bridges. For example, consider an aircraft with four engines such that at least one engine is required to operate for the aircraft to remain airborne. If the random variables $Y_1, ..., Y_4$ represent the strength of the engines and the random variable X represents the random stress of environmental factors such as temperature, fumes and corrosive agents. Then, due to the parallel engine system and under the stress of the aforementioned environmental factors, $R = P(\max(Y_1, ..., Y_4) > X)$ is the probability of successfully flying the aircraft. Comparisons of the minimum or maximum of two independent and heterogeneous samples, each following some specific distribution function, have received the attention of researchers in recent decades. Several stochastic orderings are proposed in the respective literature, all of which are studied based on specific distributions such as [27]-[34]. In this paper a comparison and estimation of the stress- strength reliability of the parallel system with heterogeneous UGG distribution has been carried out. In addition, we provide different sets of sufficient conditions for one system to dominate another. The results of this paper are useful for comparing the reliability of the stress strength of two parallel strength systems that are described by the UGG distributions. Unlike previous works that primarily use classical or less flexible distributions (e.g., Weibull, Rayleigh, or exponential), this paper leverages the UGG distribution. Its ability to represent a wide range of hazard rate behaviors enhances the modeling of stress-strength reliability. Using majorized shape parameters, this paper offers a refined perspective on reliability comparisons, a focus that has been largely unexplored in previous research. The performance of the methods proposed in this paper is rigorously validated through simulations and illustrated with real-life data, ensuring the practical relevance of the findings.

The remainder of this paper is organized as follows. In Section 2, some useful lemma and definitions are given, which will be used later in this paper. The formulation of the general model is presented in Section 3. In addition, under special conditions on the parameters, the reliability of the stress strength of two 1-out of n:G stress strength systems is compared. In Section 4, the special model considered is described and some stochastic orderings between the two 1-out-of n:G stress-strength systems in the UGG model are discussed. We derive the expression for $R(n, \alpha, B, \gamma) = P(Y_{n:n} > X)$ in the UGG model and develop a procedure to estimate $R(n, \alpha, B, \gamma)$ in Section 5. Furthermore, we obtain the maximum likelihood estimates (MLE) and maximum spacing (MSP) estimates of the parameters in Section 5. Section 6 provides asymptotic and bootstrap confidence intervals for $R(n, \alpha, B, \gamma)$. In Section 7, simulation studies are carried out to evaluate the performance of the asymptotic and bootstrap confidence intervals for $R(n, \alpha, B, \gamma)$. In addition, a numerical example based on real-life data is provided in Section 7. Finally, the conclusions are given in Section 8.

2. Some fundamental basic definitions and primary results

Several statistical indices, such as mean, median, skewness, and kurtosis, have previously been used in research work to compare two CDFs or PDFs. However, comparisons based on single values did not contain sufficient information. To overcome this shortcoming, several authors, such as [35], applied stochastic orders, which provide further information regarding the distribution structures. More detailed information on stochastic orders can be found in [38] and [36]. This section is devoted to the review of some notes on stochastic orders. Consider two univariate random variables of X and Y such that their following characteristics are, respectively, termed as: CDFs F and G, survival functions $\bar{F}(= 1 - F)$ and $\bar{G}(= 1 - G)$, PDFs f and g, hazard rate functions $h_f(= f/\bar{F})$ and $h_g(= g/\bar{G})$ and reversed hazard rate functions $\tilde{r}_F(= f/F)$ and $\tilde{r}_G(= g/G)$. Denote by G^{-1} the corresponding quantile function, defined by $G^{-1}(u) = \inf\{x : G(x) \geq u\},\ 0 \leq u \leq 1$. Note that stochastic orders are introduced for the sake of comparing the magnitudes of two random variables. More details of stochastic orders can be found in [38].

Definition 2.1. The vector X is said to be smaller than the vector Y in the
(i) usual stochastic order denoted by $X \leq_{st} Y$ if $\bar{F}(t) \leq \bar{G}(t)$ for all t.
(ii) hazard rate order denoted by $X \leq_{hr} Y$ if $\bar{G}(t)/\bar{F}(t)$ increases in t. If X and Y are absolutely continuous, then $X \leq_{hr} Y$ is equivalent to $h_F(t) \geq h_G(t)$ for all t.
(iii) reversed hazard rate order denoted by $X \leq_{rhr} Y$ if $G(t)/F(t)$ increases in t. If X and Y are absolutely continuous, then $X \leq_{rhr} Y$ is equivalent to $\tilde{r}_F(t) \leq \tilde{r}_G(t)$ for all t.

(iv) likelihood ratio order denoted by $X \leq_{lr} Y$ if $g(t)/f(t)$ increases in t for which the ratio is well defined.
(v) mean residual life order denoted by $X \leq_{MRL} Y$ if $\int_x^{\infty} \bar{G}(u)du / \int_x^{\infty} \bar{F}(u)du$ is increasing in x.
(vi) convex transform order (denoted by $X \leqslant_c Y$) if $G^{-1}(F(x))$ is convex in x on the support of F.
(vii) star order (denoted by $X \leqslant_* Y$) if $\dfrac{G^{-1}(F(x))}{x}$ increases in $x > 0$.
(viii) super-additive order (denoted by $X \leqslant_{su} Y$) if $G^{-1}(F(x+y)) \geq G^{-1}(F(x)) + G^{-1}(F(y))$, $\forall\ x \geq 0$,
$y \geq 0$.
(ix) dispersive order (denoted by $X \leqslant_{disp} Y$) if and only if $G^{-1}(F(x)) - x$ increases in x.

Suppose that I^n is a n-dimensional Euclidean space, where $I \subseteq \mathbb{R}$ and two real vectors $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in I^n$ and $\boldsymbol{y} = (y_1, y_2, ..., y_n) \in I^n$ with ascending elements $\{x_{(1)}, x_{(2)}, ..., x_{(n)}\}$ and $\{y_{(1)}, y_{(2)}, ..., y_{(n)}\}$. Consider the following definition derived by [37].

Definition 2.2. The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$ (denoted $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, 2, ..., n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Definition 2.3. A real-valued function $\varphi : I^n \to \Re$ is said to be Schur-Convex (Schur-Concave)on I^n if $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$, implies $\varphi(\boldsymbol{x}) \geq (resp. \leq)\varphi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in I^n$.

Lemma 2.4. *The function* $h(x) = \frac{xp^{-x}}{\mu - 1 + p^{-x}}$ *is increasing and convex in* x*, for all* $x > 0$*,* $\mu > 1$*, and* $0 < p < 1$*.*

In what follows, let us denote

$$\mathcal{M}_{Y_{n:n}, Y_i, G_0, \alpha_i, \beta_i, X, F, R(n, \boldsymbol{\alpha}, \boldsymbol{\beta})} = \{Y_{n-r+1:n}, Y_i \sim UGG\text{–}G_0(\alpha_i, \beta_i, \mu), Y_0 \sim G_0(.), X \sim F, R(n, \boldsymbol{\alpha}, \boldsymbol{\beta})\},$$

as the model function of the 1-out-of-n:G stress-strength system associated with the INID components lifetimes Y_i, where $Y_i \sim UGG - G_0(\alpha_i, \beta_i, \mu)$, for $i = 1, 2, ..., n$ and the strength system is subjected to a random stress X with the CDF F. Also, assume that the strength-system and the stress strength reliability of the aforementioned system are $Y_{n:n}$ and $R(n, \boldsymbol{\alpha}, \boldsymbol{\beta})$, respectively.

3. Description general model and majorization ordering results

Consider a 1-out-of-n:G strength system of order n consisting of independent and heterogeneous UGG components with lifetimes $Y_i \sim UGG - G_0(\alpha_i, \beta_i, \mu)$, for $i = 1, 2, ..., n$. The system is subjected to random stress X with CDF F. Also, suppose that Y_i and X are independent random variables for $i = 1, 2, ..., n$. It is obvious that $Y_{n:n} = \max(Y_1, ..., Y_n)$ is the strength of the 1-out-of-n:G system. The CDF and PDF of the random strength of the aforementioned 1-out-of-n:G system is given by

$$G_{n:n}(y) = \prod_{i=1}^{n} \left(\frac{\mu}{\mu - 1 + {G_0}^{-\alpha_i}(y)} \right)^{\beta_i}, \tag{3.1}$$

and

$$g_{n:n}(y) = G_{n:n}(y) \sum_{i=1}^{n} \alpha_i \beta_i \frac{g_0(y)}{G_0(y)} \frac{{G_0}^{-\alpha_i}(y)}{b - 1 + {G_0}^{-\alpha_i}(y)}, \tag{3.2}$$

respectively. Then, the stress-strength reliability of the 1-out-of-n:G stress-strength system is given by

$$\begin{aligned} R(n,\boldsymbol{\alpha},\boldsymbol{\beta}) &= P(Y_{n:n} > X) \\ &= 1 - \int_0^1 R(u,n,\boldsymbol{\alpha},\boldsymbol{\beta})du, \end{aligned} \tag{3.3}$$

where $R(u,n,\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{i=1}^{n} \left(\frac{\mu}{\mu-1+G_0{}^{-\alpha_i}(F^{-1}(u))}\right)^{\beta_i}$ and $F^{-1}(u) = \inf\{x : F(x) \geq u\}$ for $0 \leq u \leq 1$ is the corresponding quantile function. In the following, under special parameters conditions, the stress strength reliability of 1-out-of-n:G stress strength systems with independent and heterogeneous UGG components are compared under identical baseline CDFs.

Result 3.1. *Let $\mathcal{M}_{Y_{n:n},Y_i,G_0,\alpha_i,\beta_i,X,F,R(n,\boldsymbol{\alpha},\boldsymbol{\beta})}$ and $\mathcal{M}_{Z_{n:n},Z_i,G_0,\gamma_i,\beta_i,X,F,R(n,\boldsymbol{\gamma},\boldsymbol{\beta})}$ be two model functions of two 1-out-of-n:G stress-strength systems. If $\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta} \in D_+(\varepsilon_+)$, then $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$ implies $R(n,\boldsymbol{\alpha},\boldsymbol{\beta}) \geq R(n,\boldsymbol{\gamma},\boldsymbol{\beta})$.*

Proof. Let $r_{1:n}(x)$ be the reversed hazard rate function of $Y_{1:n}$, then we have

$$r_{n:n}(y) = \sum_{i=1}^{n} \alpha_i\beta_i \frac{g_0(y)}{G_0(y)} \frac{G_0{}^{-\alpha_i}(y)}{b-1+G_0{}^{-\alpha_i}(y)}.$$

According to Theorem A.3 in [37], it suffices to indicate that the random variable hazard rate function $Y_{1:n}$, that is,

$$r_{n:n}(y) = \sum_{i=1}^{n} \alpha_i\beta_i \frac{g_0(y)}{G_0(y)} \frac{G_0{}^{-\alpha_i}(y)}{b-1+G_0{}^{-\alpha_i}(y)} = \frac{g_0(y)}{G_0(y)} \sum_{i=1}^{n} \beta_i\psi(\alpha_i),$$

is a Schur-convex function in $\boldsymbol{\alpha}$, where $\psi(\alpha_i) = \frac{\alpha_i G_0{}^{-\alpha_i}(x)}{\mu-1+G_0{}^{-\alpha_i}(x)}$. Based on Lemma 2.4, it can be concluded that $\psi(\alpha_i)$ is increasing and convex with respect to α_i for $i = 1,\ldots,n$. Therefore, according to Theorem 3.1 (a)(i)(3.2 (b)(i)) of [32], $r_{n:n}(y)$ is a Schur-convex function in $\boldsymbol{\alpha}$ on $D_+(\varepsilon_+)$. Thus, according to Theorem A.8 in [37], we have $Y_{n:n} \geq_{rhr} Z_{n:n}$ and so $Y_{n:n} \geq_{st} Z_{n:n}$. Finally, by Equation (3.3) the proof is complete. □

Result 3.1 states that in a 1-out-of-n:G stress-strength system with independent components, the higher heterogeneity of the shape parameter (in majorization order) results in the higher stress-strength reliability of the 1-out-of-n:G stress-strength system in the reversed hazard rate order.

Result 3.2. *Let $\mathcal{M}_{Y_{n:n},Y_i,G_0,\alpha_i,\beta_i,X,F,R(n,\boldsymbol{\alpha},\boldsymbol{\beta})}$ and $\mathcal{M}_{Z_{n:n},Z_i,G_0,\alpha_i,\delta_i,X,F,R(n,\boldsymbol{\alpha},\boldsymbol{\delta})}$ be two model functions of two 1-out-of-n:G stress strength systems. If $\boldsymbol{\delta},\boldsymbol{\beta},\boldsymbol{\alpha} \in \varepsilon_+$, and $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$, then $R(n,\boldsymbol{\alpha},\boldsymbol{\beta}) \geq R(n,\boldsymbol{\alpha},\boldsymbol{\delta})$.*

Proof. The reversed hazard rate function of the random variable $Y_{n:n}$ can be expressed as

$$r_{n:n}(x) = f(x) \sum_{i=1}^{n} u_i\psi(\beta_i),$$

where, for $i = 1, ..., n$, $\psi(\beta_i) = \beta_i$ and $u_i = \frac{\alpha_i F^{-\alpha_i-1}(x)}{b-1+F^{-\alpha_i}(x)}$. Due to the lemma 2.4, it can be observed that u_i is an increasing function in α_i, for $i = 1, ..., n$. Therefore, $\boldsymbol{\alpha} \in \varepsilon_+$ implies $\boldsymbol{u} = (u_1, ..., u_n) \in \varepsilon_+$. Furthermore, if $\boldsymbol{u} \in \varepsilon_+$ and $\boldsymbol{\beta} \in \varepsilon_+$, according to Theorem 3.2 (b) (i) in [32], $\tilde{r}_{n:n}(x)$ is a Schur-convex function in ε_+ and Thus, according to Theorem A.8 in [37], we have $Y_{n:n} \geq_{rhr} Z_{n:n}$ and so $Y_{n:n} \geq_{st} Z_{n:n}$. Finally, by Equation 3.3 the proof is complete. □

In the following, the comparison of the reliability of the strength of the stress of two parallel systems with independent and heterogeneous $UGG-F$ components is carried out under non-identical baseline distribution functions.

Result 3.3. *Let* $\mathcal{M}_{Y_{n:n},Y_i,G_0,\alpha_i,\beta_i,X,F,R(n,\boldsymbol{\alpha},\boldsymbol{\beta})}$ *and* $\mathcal{M}_{Z_{n:n},Z_i,H_0,\gamma_i,\beta_i,X,F,R(n,\boldsymbol{\gamma},\boldsymbol{\beta})}$ *be two model functions of* 1*-out-of-n:G stress strength systems. If*$\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta} \in D_+(\varepsilon_+)$ *and* $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\gamma}$*, then* $Y_0 \geq_{st} Z_0$ *implies* $R(n,\boldsymbol{\alpha},\boldsymbol{\beta}) \geq R(n,\boldsymbol{\gamma},\boldsymbol{\beta})$*.*

Proof. Let $W_i \sim UGG - G_0(\gamma_i,\beta_i,\mu)$ for $i = 1,2,\ldots,n$. For $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in A_n$ and $(\boldsymbol{\gamma},\boldsymbol{\beta}) \in A_n$, in the proof of the result 3.1, $Y_{n:n} \geq_{st} W_{n:n}$ was shown. Therefore, we have

$$\prod_{i=1}^{n}\Big(\frac{\mu}{\mu-1+{G_0}^{-\alpha_i}(y)}\Big)^{\beta_i} \leq \prod_{i=1}^{n}\Big(\frac{\mu}{\mu-1+{G_0}^{-\gamma_i}(y)}\Big)^{\beta_i}. \tag{3.4}$$

According to the definition of the usual stochastic order and based on the relationship $Y_0 \geq_{st} Z_0$, for all $\beta_i,\ \gamma_i > 0$, we have

$$\left(b-1+{G_0}^{-\gamma_i}(y)\right)^{\beta_i} \geq \left(\mu-1+{H_0}^{-\gamma_i}(y)\right)^{\beta_i}.$$

Therefore, Equation 3.4 can be rewritten as

$$\prod_{i=1}^{n}\Big(\frac{\mu}{\mu-1+{G_0}^{-\alpha_i}(y)}\Big)^{\beta_i} \leq \prod_{i=1}^{n}\Big(\frac{\mu}{\mu-1+{G_0}^{-\gamma_i}(y)}\Big)^{\beta_i} \leq \prod_{i=1}^{n}\Big(\frac{mu}{\mu-1+{H_0}^{-\gamma_i}(y)}\Big)^{\beta_i},$$

which results in $G_{n:n}(y) \leq H_{n:n}(y)$, where $H_{n:n}(y)$ is the CDF of the random variable $Z_{n:n}$. This implies $Y_{n:n} \geq_{st} Z_{n:n}$. Therefore, by Equation 3.3 the proof is complete. □

Result 3.4. *Let* $\mathcal{M}_{Y_{n:n},Y_i,G_0,\alpha_i,\beta_i,X,F,R(n,\boldsymbol{\alpha},\boldsymbol{\beta})}$ *and* $\mathcal{M}_{Z_{n:n},Z_i,H_0,\gamma_i,\beta_i,X,F,R(n,\boldsymbol{\gamma},\boldsymbol{\beta})}$ *be two model functions of* 1*-out-of-n:G stress strength systems. If*$\boldsymbol{\delta},\boldsymbol{\beta},\boldsymbol{\alpha} \in \varepsilon_+$ *and* $\boldsymbol{\beta} \overset{m}{\succeq} \boldsymbol{\delta}$*, then* $Y_0 \geq_{st} Z_0$ *implies* $R(n,\boldsymbol{\alpha},\boldsymbol{\beta}) \geq R(n,\boldsymbol{\alpha},\boldsymbol{\delta})$*.*

4. Special Models

Consider a 1-out-of-n:G strength system of order n consisting of independent and heterogeneous UGG components with lifetimes $Y_i \sim UGG - G_0(\alpha,\beta_i,\mu)$, for $i = 1,2,...,n$. The system is subjected to random stress X with CDF G_0. Also, suppose that Y_i and X are independent random variables for $i = 1,2,...,n$. Consider $\mathcal{M}^s{}_{Y_{n:n},Y_i,G_0,\alpha,\beta_i,X,G_0,R(n,\alpha,B,\mu)}$ as the model function of the aforementioned 1-out-of-n:G stress-strength system, where $B = \sum_{i=1}^{n}\beta_i$. Then stress-strength reliability of the aforementioned system is given by

$$\begin{aligned} R(n,\alpha,B,\mu) &= P(X < Y_{n:n}) \\ &= 1 - \int_0^\infty \prod_{i=1}^{n}\Big(\frac{\mu}{\mu-1+G_0^{-\alpha}(x)}\Big)^{\beta_i} dG_0(x) \\ &= 1 - \int_0^1 \Big(\frac{\mu}{\mu-1+u^{-\alpha}}\Big)^{B} du. \end{aligned} \tag{4.1}$$

We now study the stochastic ordering between $Y_{n:n}$ and X. Using relation (3.2), we have

$$\frac{g_{n:n}(y)}{g_0(y)} = \alpha B\mu^{-1}{G_0}^{-\alpha_i-1}(y){G_{n:n}}^{\frac{1}{\beta}+1}(y). \tag{4.2}$$

This gives

$$\frac{d}{dy}\Big[\frac{g_{n:n}(y)}{g_0(y)}\Big] = \alpha B\mu^{-1}{G_0}^{-\alpha-1}(y){G_{n:n}}^{\frac{1}{B}+1}(y)\Big[-(\alpha+1)r_0(y) + (\frac{1}{B}+1)r_{n:n}(y)\Big], \tag{4.3}$$

where, $r_0(y) = \frac{g_0(y)}{G_0(y)}$ and $r_{n:n}(y) = \frac{g_{n:n}(y)}{G_{n:n}(y)}$. Therefore, $\alpha = \frac{1}{B}$ and $\mu \geq 1$ involve $Y_{n:n} \leq_{lr} X$.

Based on what has already been stated, the following results are obtained.

Result 4.1. *Let* $\mathcal{M}^s{}_{Y_{n:n},Y_i,G_0,\alpha,\beta_i,X,G_0,R(n,\alpha,B,\mu)}$ *be the model function of the* 1*-out-of-n:G stress-strength system. Then, if* $\alpha = \frac{1}{B}$ *and* $\mu \geq 1(\mu < 1)$,

i) $Y_{n:n} \leq_{hr} X(Y_{n:n} \geq_{hr} X)$;
ii) $Y_{n:n} \leq_{rh} X(Y_{n:n} \geq_{hr} X)$;
iii) $Y_{n:n} \leq_{st} X(Y_{n:n} \geq_{hr} X)$;
iv) $Y_{n:n} \leq_{MRL} X(Y_{n:n} \geq_{hr} X)$;
v) $R(n, \alpha, \boldsymbol{\beta}, \mu) \leq (\geq)\frac{1}{2}$.

Next, we investigate the stochastic orderings with respect to the parameters β and α. Let $\mathcal{M}^s{}_{Y_{n:n},Y_i,G_0,\alpha,\beta_i,X,G_0,R_1(n,\alpha,B,\mu_1)}$ and $\mathcal{M}^s{}_{Z_{n:n},Z_i,G_0,\gamma,\delta_i,X,G_0,R_2(n,\gamma,D,\mu_2)}$ be two model functions of two 1-out-of-n:G stress-strength systems, where $D = \sum_{i=1}^{n} \delta_i$. Then,

$$\frac{g_{n:n}(y)}{h_{n:n}(y)} = \frac{\alpha B\mu_2}{\gamma D\mu_1} \frac{{G_0}^{\gamma}(y)}{{G_0}^{\alpha}(y)} \frac{{G_{n:n}}^{\frac{1}{B}+1}(y)}{{H_{n:n}}^{\frac{1}{D}+1}(y)}. \tag{4.4}$$

Therefore, we have

$$\begin{aligned}\frac{d}{dy}\Big[\frac{g_{n:n}(y)}{h_{n:n}(y)}\Big] = \frac{\alpha B\mu_2}{\gamma D\mu_1}\Big[&(\gamma+1)r_0(y) - (\frac{1}{D}+1)s_{n:n}(y)\\ &- (\alpha+1)r_0(y) + (\frac{1}{B}+1)r_{n:n}(y)\Big],\end{aligned} \tag{4.5}$$

where, $s_{n:n}(y) = \frac{h_{n:n}(y)}{H_{n:n}(y)}$. If $\alpha = \frac{1}{B}$ and $\gamma = \frac{1}{D}$, then we have

$$\begin{aligned}\frac{d}{dy}\Big[\frac{g_{n:n}(y)}{h_{n:n}(y)}\Big] = \frac{\mu_2}{\mu_1}\Bigg[&(\gamma+1)r_0(y)(1 - \frac{{G_0}^{-\gamma}}{\mu_2 - 1 + {G_0}^{-\gamma}(y)})\\ &+ (\alpha+1)r_0(y)(\frac{{G_0}^{-\alpha}}{\mu_2 - 1 + {G_0}^{-\alpha}(y)} - 1)\Bigg].\end{aligned} \tag{4.6}$$

Thus $\alpha = \frac{1}{B}$, $\gamma = \frac{1}{D}$, $\mu_1 < 1$, and $\mu_2 \geq 1$ implies $Y_{n:n} \geq_{lr} Z_{n:n}$.
By the above-mentioned results, the following result is achieved.

Result 4.2. *Let* $\mathcal{M}^s{}_{Y_{n:n},Y_i,G_0,\alpha,\beta_i,X,G_0,R_1(n,\alpha,B,\mu_1)}$ *and* $\mathcal{M}^s{}_{Z_{n:n},Z_i,G_0,\gamma,\delta_i,X,G_0,R_2(n,\gamma,D,\mu_2)}$ *be two model functions of two* 1*-out-of-n:G stress-strength systems. Then, if* $\alpha = \frac{1}{B}$, $\gamma = \frac{1}{D}$, $\mu_1 < 1(\mu_1 \geq 1)$, *and* $\mu_2 \geq 1(\mu_2 < 1)$,

i) $Y_{n:n} \geq_{hr} Z_{n:n}(Y_{n:n} \leq_{hr} Z_{n:n})$;
ii) $Y_{n:n} \geq_{rh} Z_{n:n}(Y_{n:n} \leq_{rh} Z_{n:n})$;
iii) $Y_{n:n} \geq_{st} Z_{n:n}(Y_{n:n} \leq_{st} Z_{n:n})$;
iv) $Y_{n:n} \geq_{MRL} Z_{n:n}(Y_{n:n} \leq_{MRL} Z_{n:n})$;
v) $R_1(n, \alpha, B, \mu_1) \geq R_2(n, \gamma, D, \mu_2)(R_1(n, \alpha, B, \mu_1) \leq R_2(n, \gamma, D, \mu_2))$.

The next result shows that some stochastic orders are preserved under the aforementioned models.

Result 4.3. *Let* $\mathcal{M}^s{}_{Y_{n:n},Y_i,G_0,\alpha,\beta_i,X,G_0,R_1(n,\alpha,B,\mu)}$ *and* $\mathcal{M}^s{}_{Z_{n:n},Z_i,H_0,\alpha,\beta_i,X,H_0,R_2(n,\alpha,B,\mu)}$ *be two model functions of two* 1*-out-of-n:G stress-strength systems. Then we have*

i) $Y_0 \leqslant_c Z_0$ *if, and only if,* $Y_{n:n} \leqslant_c Z_{n:n}$.
ii) $Y_0 \leqslant_* Z_0$ *if, and only if,* $Y_{n:n} \leqslant_* Z_{n:n}$.
iii) $Y_0 \leqslant_{su} Z_0$ *if, and only if,* $Y_{n:n} \leqslant_{su} Z_{n:n}$.
iv) $Y_0 \leqslant_{disp} Z_0$ *if, and only if,* $Y_{n:n} \leqslant_{disp} Z_{n:n}$.

Proof. By assumptions, we have

$$G_{n:n}{}^{-1}(H_{n:n}(y)) = G_0{}^{-1}\left[\left(1-\mu(1-\frac{\mu-1+H_0^{-\alpha}(y)}{\mu})\right)^{-\frac{1}{\alpha}}\right]. \tag{4.7}$$

The results follow from (4.7) and parts **(vi)** to **(ix)** of Definition 2.1. □

5. Point estimation of the the 1-out-of-n:G stress-strength reliability in *UGG* models

In this section, we propose two point estimators for the 1-out-of-n:G stress-strength reliability in *UGG* models using ML and MPS methods.

5.1. ML estimation

Here, we will find the ML estimation of the 1-out-of-n:G stress-strength reliability when the strength components belong to the family of UGG distribution with an exponential baseline distribution. In addition, we consider that the strength system is subjected to independent stress with an exponential distribution. Let $\mathcal{M}_{W,Y_i,G_0,\alpha,\beta_i,X,F,R(n,\alpha,B,\gamma)}$ be a model functions of a 1-out-of-n:G stress-strength system, where $G_0(y) = 1 - e^{-\lambda y}$ with known mean $\frac{1}{\lambda}$, $F(x) = 1 - e^{-\gamma x}$ with unknown mean $\frac{1}{\gamma}$, $W = Y_{n:n}$, and

$$\begin{aligned} R(n,\alpha,B,\gamma) &= P(W > X) \\ &= 1 - \int_0^1 \left(\frac{\mu}{\mu - 1 + \left(1-(1-u)^{\frac{\lambda}{\gamma}}\right)^{-\alpha}}\right)^B du. \end{aligned} \tag{5.1}$$

Therefore, the two probability density functions associated with the aforementioned stress-strength model are given, respectively, by

$$g_W(w) = \frac{B\alpha\lambda}{\mu} e^{-\lambda w}(1-e^{-\lambda w})^{-\alpha-1}\left(\frac{\mu}{\mu-1+(1-e^{-\lambda w})^{-\alpha}}\right)^{B+1}, \; y>0,\; B>0,\; \lambda>0,\; \mu, \tag{5.2}$$

and

$$f(x) = \gamma e^{-\gamma x}, \;\; x > 0, \;\; \gamma > 0. \tag{5.3}$$

Let $W_1, \ldots, W_{m_1}$ be a random sample of size m_1 distributed as W with PDF in (5.2) and $X_1, \ldots, X_{m_2}$ be a random sample of size m_2 from X with PDF in (5.3). Then, the likelihood function of the observed sample is easily provided by

$$\begin{aligned} L(B,\alpha,\gamma) &= \left(\frac{B\alpha\lambda}{\mu}\right)^{m_1} e^{-\lambda\sum_{j=1}^{m_1} w_j} \prod_{j=1}^{m_1}(1-e^{-\lambda w_j})^{-\alpha-1} \\ &\times \prod_{j=1}^{m_1}\left(\frac{\mu}{\mu-1+(1-e^{-\lambda w_j})^{-\alpha}}\right)^{B+1} \gamma^{m_2} e^{-\gamma\sum_{k=1}^{m_2} x_k}. \end{aligned} \tag{5.4}$$

The corresponding log-likelihood function is

$$\ell(B,\alpha,\gamma) = m_1(\ln B + \ln\alpha + \ln\lambda - \ln\mu) - \lambda\sum_{j=1}^{m_1} w_j - (\alpha+1)\sum_{j=1}^{m_1}\ln(1-e^{-\lambda w_j}) + \\ + (B+1)m_1\ln\mu - (B+1)\sum_{j=1}^{m_1}\ln\left(\mu - 1 + (1-e^{-\lambda w_j})^{-\alpha}\right) + m_2\ln\gamma - \gamma\sum_{k=1}^{m_2} x_k. \tag{5.5}$$

Denote $\boldsymbol{\theta} = (B,\alpha,\gamma)$. Taking the first order partial derivatives with respect to B, α, and γ and setting them to zero, we get the following system of score equations

$$\begin{cases} \frac{\partial l(\boldsymbol{\theta})}{\partial B} = \frac{m_1}{B} + m_1\ln\mu - \sum\limits_{j=1}^{m_1}\ln\left(\mu - 1 + (1-e^{-\lambda w_j})^{-\alpha}\right) = 0, \\ \frac{\partial l(\boldsymbol{\theta})}{\partial \gamma} = \frac{m_2}{\gamma} - \sum\limits_{k=1}^{m_2} x_k = 0, \\ \frac{\partial l(\boldsymbol{\theta})}{\partial \alpha} = \frac{m_1}{\alpha} - \sum\limits_{j=1}^{m_1}\ln(1-e^{-\lambda w_j}) + (B+1)\sum\limits_{j=1}^{m_1}\frac{(1-e^{-\lambda w_j})^{-\alpha}\ln(1-e^{-\lambda w_j})}{\mu-1+(1-e^{-\lambda w_j})^{-\alpha}} = 0. \end{cases} \tag{5.6}$$

The MLEs of B, α, and γ denoted by $\hat{B}$, $\hat{\alpha}$, and $\hat{\gamma}$ are the solutions to the above system of score equations which maximize the likelihood function (5.4). From (5.6), we attain

$$\hat{B}(\alpha) = -\frac{m_1}{\sum\limits_{j=1}^{m_1}\ln\left(\frac{\mu}{\mu-1+(1-e^{-\lambda w_j})^{-\alpha}}\right)}, \tag{5.7}$$

and

$$\hat{\gamma} = \frac{m_2}{\sum\limits_{k=1}^{m_2} x_k}. \tag{5.8}$$

By substituting (5.7) and (5.8) in (5.6) we have

$$\frac{\partial l(\boldsymbol{\theta})}{\partial \alpha} = \frac{m_1}{\alpha} - \sum_{j=1}^{m_1}\ln(1-e^{-\lambda w_j}) + \frac{m_1 + \sum\limits_{j=1}^{m_1}\ln\left(\mu-1+(1-e^{-\lambda w_j})^{-\alpha}\right) - m_1\ln\mu}{\sum\limits_{j=1}^{m_1}\ln\left(\mu-1+(1-e^{-\lambda w_j})^{-\alpha}\right) - m_1\ln\mu} \\ \times\sum_{j=1}^{m_1}\frac{(1-e^{-\lambda w_j})^{-\alpha}\ln(1-e^{-\lambda w_j})}{\mu-1+(1-e^{-\lambda w_j})^{-\alpha}} = 0. \tag{5.9}$$

By solving the system of non-linear equations in (5.9), $\hat{B}$, $\hat{\alpha}$, and $\hat{\gamma}$ will be obtained. Therefore, the MLE of $R(n,\alpha,B,\gamma)$ can be obtained by

$$\hat{R}(n,\alpha,B,\gamma) = 1 - \int_0^1\left(\frac{\mu}{\mu - 1 + \left(1-(1-u)^{\frac{\lambda}{\hat{\gamma}}}\right)^{-\hat{\alpha}}}\right)^{\hat{B}} du. \tag{5.10}$$

In the next subsection, various methods are used to find the confidence interval for $R(n,\alpha,B,\gamma)$.

5.2. MSP estimation

MSP estimation was proposed by [39], and subsequently investigated by [40] based on an approximation of the Kullback-Leibler divergence. The estimation method can effectively replace the ML method in a problem in which the likelihood is unbounded. Based on the literature [41], the following method is briefly given to calculate the MSP estimator of the parameter. Suppose $X_1, \dots, X_n$ is a random sample from population X with CDF $F_\theta(x)$, where θ is an unknown parameter vector. Let $F(x)$ be the true CDF of the population X, $(X_{1:n}, ..., X_{n:n})$ be the order statistic of this sample, and its observations is $(x_{1:n}, ..., x_{n:n})$. According to the literature [39], and subsequently investigated by [40], the Kullback-Leibler divergence between $F_\theta(x)$ and $F(x)$ is approximated as

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\ln\left|\frac{F(x_{i:n})-F(x_{i-1:n})}{F_\theta(x_{i:n})-F_\theta(x_{i-1:n})}\right|, \tag{5.11}$$

where $F_\theta(x_{0:n}) \equiv 0$ and $F_\theta(x_{n+1:n}) \equiv 1$. The MSP estimators for the unknown parameter vector $\boldsymbol{\theta}$ can be obtained by minimizing the above equation. Furthermore, the above minimization problem can be transformed into the following maximization expression:

$$Q(\boldsymbol{\theta}) = \sum_{i=1}^{n+1}\ln\left[F(x_{i:n})-F(x_{i-1:n})\right]. \tag{5.12}$$

In this section, Let $W_1, \dots, W_{m_1}$ be a random sample of size m_1 distributed as W with the PDF in (5.2), $X_1, \dots, X_{m_2}$ be a random sample of size m_2 from X with the PDF in (5.3), and $\boldsymbol{\theta} = (B, \alpha, \gamma)$. Based on the above method for calculating MSP estimator of parameter, the following expression is given by

$$\begin{aligned} Q(\boldsymbol{\theta}) = &\sum_{i=1}^{m_1+1}\ln\left[\left(\frac{\mu}{\mu-1+(1-e^{-\lambda w_{i:m_1}})^{-\alpha}}\right)^B - \left(\frac{\mu}{\mu-1+(1-e^{-\lambda w_{i-1:m_1}})^{-\alpha}}\right)^B\right] \\ &+ \sum_{i=1}^{m_2+1}\ln\left[e^{-\gamma x_{i-1:m_2}} - e^{-\gamma x_{i:m_2}}\right], \end{aligned} \tag{5.13}$$

where $w_{0:m_1} = 0$, $x_{0:m_2} = 0$, $w_{m_1+1:m_1} = +\infty$, and $x_{m_2+1:m_2} = +\infty$. The partial derivatives of Equation (5.13) with respect to parameters B, α, and γ are given by

$$\begin{cases} \frac{\partial Q(\boldsymbol{\theta})}{\partial B} = \sum_{i=1}^{m_1+1}\frac{C_{1i}{}^B\ln C_{1i} - C_{2i}{}^B\ln C_{2i}}{C_{1i}{}^B - C_{2i}{}^B} = 0, \\ \frac{\partial Q(\boldsymbol{\theta})}{\partial \gamma} = \sum_{i=1}^{m_2+1}\frac{x_{i:m_2}e^{-\gamma x_{i:m_2}} - x_{i-1:m_2}e^{-\gamma x_{i-1:m_2}}}{e^{-\gamma x_{i-1:m_2}} - e^{-\gamma x_{i:m_2}}} = 0, \\ \frac{\partial Q(\boldsymbol{\theta})}{\partial \alpha} = \frac{B}{\mu}\sum_{i=1}^{m_1+1}\frac{C_{1i}{}^{B+1}\ln(1-e^{-\lambda w_{i:m_1}}) - C_{2i}{}^{B+1}\ln(1-e^{-\lambda w_{i-1:m_1}})}{C_{1i}{}^B - C_{2i}{}^B} = 0, \end{cases} \tag{5.14}$$

where $C_{1i} = \left(\frac{\mu}{\mu-1+(1-e^{-\lambda w_{i:m_1}})^{-\alpha}}\right)$ and $C_{2i} = \left(\frac{\mu}{\mu-1+(1-e^{-\lambda w_{i-1:m_1}})^{-\alpha}}\right)$. Let $\tilde{B}$, $\tilde{\alpha}$, and $\tilde{\gamma}$ be the MSP estimators of parameters B, α, and γ, respectively, then $\tilde{B}$, $\tilde{\alpha}$, and $\tilde{\gamma}$ can be obtained by solving the above equation (5.14). The MSP estimation $\tilde{R}(n, \alpha, B, \gamma)$ can be obtained by substituting $\tilde{B}$, $\tilde{\alpha}$, and $\tilde{\gamma}$ into $R(n, \alpha, B, \gamma)$.

6. Confidence intervals for $R(n, \alpha, B, \gamma)$

In this section, various methods are used to find confidence interval for the multicomponent stress-strength reliability parameter $R(n, \alpha, B, \gamma)$. It is difficult to obtain the exact distribution of $\hat{R}(n, \alpha, B, \gamma)$. Therefore, it is difficult to obtain the exact confidence interval of $R(n, \alpha, B, \gamma)$. For this reason, we consider an asymptotic confidence interval and two parametric bootstrap methods for the construction of the confidence interval of $R(n, \alpha, B, \gamma)$, which was first proposed by Efron [42].

6.1. Asymptotic confidence interval

In this subsection, the asymptotic interval of $R(n,\alpha,B,\gamma)$ is obtained using the asymptotic distribution of MLE $\hat{\boldsymbol{\theta}} = (\hat{\alpha},\hat{B},\hat{\gamma})$. Let $\boldsymbol{\theta} = (\alpha,B,\gamma)^{\intercal}$. The Hessian is the matrix of second derivatives of the likelihood with respect to the parameters and defined by

$$\mathbf{H}(\boldsymbol{\theta}) = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\intercal}} = \begin{pmatrix} \frac{\partial^2 l(\boldsymbol{\theta})}{\partial B^2} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial B\partial\alpha} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial B\partial\gamma} \\ \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha\partial B} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha^2} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha\partial\gamma} \\ \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\gamma\partial B} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\gamma\partial\alpha} & \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\gamma^2} \end{pmatrix}, \tag{6.1}$$

where

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha^2} = -\frac{m_1}{\alpha^2} - (B+1)(\mu-1)\sum_{j=1}^{m_1}\frac{(1-e^{-\lambda w_j})^{-\alpha}\Big(\ln(1-e^{-\lambda w_j})\Big)^2}{\Big(\mu-1+(1-e^{-\lambda w_j})^{-\alpha}\Big)^2},$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial B\partial\alpha} = \sum_{j=1}^{m_1}\frac{(1-e^{-\lambda w_j})^{-\alpha}\ln(1-e^{-\lambda w_j})}{\mu-1+(1-e^{-\lambda w_j})^{-\alpha}},$$

$$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha\partial B} = \sum_{j=1}^{m_1}\frac{(1-e^{-\lambda w_j})^{-\alpha}\ln(1-e^{-\lambda w_j})}{\mu-1+(1-e^{-\lambda w_j})^{-\alpha}},$$

$\frac{\partial^2 l(\boldsymbol{\theta})}{\partial B^2} = -\frac{m_1}{B^2}$, $\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\gamma^2} = -\frac{m_2}{\gamma^2}$, and $\frac{\partial^2 l(\boldsymbol{\theta})}{\partial B\partial\gamma} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\alpha\partial\gamma} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\gamma\partial B} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial\gamma\partial\alpha} = 0$. It can be demonstrated that the likelihood function satisfies the regularity conditions prepared in [51], pp. 384-385. The observed Fisher information matrix can be presented as

$$\mathbb{I}_n(\hat{\boldsymbol{\theta}}) = -H(\hat{\boldsymbol{\theta}}) = -\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\intercal}}\Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}. \tag{6.2}$$

Let $m = m_1 + m_2$, $k_1 = \lim_{m_2\to+\infty}\lim_{m_1\to+\infty}\frac{m_1}{m_1+m_2}$, and $k_1 = \lim_{m_2\to+\infty}\lim_{m_1\to+\infty}\frac{m_2}{m_1+m_2}$. We define the Fisher information matrix of θ based on the model function $\mathcal{M}_{W,Y_i,G_0,\alpha,\beta_i,X,F,R(n,\alpha,B,\gamma)}$ as follows

$$\mathbb{I}(\boldsymbol{\theta}) = \lim_{\substack{m_1\to\infty\\ m_2\to\infty}} E\Big(-\frac{1}{m}\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^{\intercal}}\Big) = \begin{pmatrix} \mathbb{I}_{11} & \mathbb{I}_{12} & \mathbb{I}_{13} \\ \mathbb{I}_{21} & \mathbb{I}_{22} & \mathbb{I}_{23} \\ \mathbb{I}_{31} & \mathbb{I}_{32} & \mathbb{I}_{33} \end{pmatrix}, \tag{6.3}$$

where

$$\mathbb{I}_{11} = \frac{k_1}{B^2}, \quad \mathbb{I}_{12} = \Psi(B,\alpha,\mu,2), \quad \mathbb{I}_{13} = \mathbb{I}_{31} = 0, \quad \mathbb{I}_{33} = \frac{k_2}{\gamma^2}, \quad \mathbb{I}_{21} = \Psi(B,\alpha,\mu,2),$$

$$\mathbb{I}_{13} = \mathbb{I}_{31} = 0, \quad \mathbb{I}_{22} = \frac{k_1}{\alpha^2} - (B+1)(\mu-1)\Psi(B,\alpha,\mu,3),$$

and

$$\Psi(B,\alpha,\mu,c) = -k_1 B\alpha\mu^B\int_0^1\frac{u^{-2\alpha-1}(\ln u)^{c-1}}{(\mu-1+u^{-\alpha})^{B+c}}du.$$

Applying the above-mentioned notation, we obtain the following asymptotic normality of the maximum likelihood estimates $\hat{\theta} = (\hat{\alpha},\hat{B},\hat{\gamma})^{\intercal}$, of $\theta = (\alpha,B,\gamma)$.

Theorem 6.1. *If model* (3.1) *holds, the MLE* $(\hat{\alpha},\hat{B},\hat{\gamma})^{\intercal}$ *of* $(\alpha,B,\gamma)^{\intercal}$ *weakly converges to the following multivariate normal distribution:*

$$\sqrt{m}\begin{pmatrix}\hat{\alpha}-\alpha\\ \hat{B}-B\\ \hat{\gamma}-\gamma\end{pmatrix} \xrightarrow{D} \boldsymbol{N}(\mathbf{0},\mathbb{I}^{-1}(\boldsymbol{\theta})), \tag{6.4}$$

where $\mathbb{I}^{-1}(\boldsymbol{\theta})$ *is the inverse of the Fisher information matrix* $\mathbb{I}(\boldsymbol{\theta})$*.*

Since $\mathbb{I}(\boldsymbol{\theta})$ includes integrals, one has to apply a numerical procedure to evaluate these integrals to use this asymptotic normality. Practically, it is convenient to substitute the Fisher information matrix $\mathbb{I}(\boldsymbol{\theta})$ by

$$-\frac{1}{m}H(\hat{\boldsymbol{\theta}}) = -\frac{1}{m}\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\intercal}\Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}. \tag{6.5}$$

In fact, $H(\hat{\boldsymbol{\theta}})$ is a consistent estimator of $\mathbb{I}(\boldsymbol{\theta})$ since

$$\lim_{\substack{m_1\to\infty \\ m_2\to\infty}} -\frac{1}{m_1+m_2}\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\intercal} = \lim_{\substack{m_1\to\infty \\ m_2\to\infty}} -\frac{1}{m_1+m_2}\mathbb{I}_m(\boldsymbol{\theta}) = \mathbb{I}(\boldsymbol{\theta}). \tag{6.6}$$

To construct the asymptotic normality of $R(n,\alpha,B,\gamma)$ represented in (5.1), we define

$$V(\alpha,B,\gamma) = \Big(\frac{\partial R(n,\alpha,B,\gamma)}{\partial B}, \frac{\partial R(n,\alpha,B,\gamma)}{\partial \alpha}, \frac{\partial R(n,\alpha,B,\gamma)}{\partial \gamma}\Big), \tag{6.7}$$

where

$$\frac{\partial R(n,\alpha,B,\gamma)}{\partial B} = -\mu^B \int_0^1 \Psi(u,\alpha,\lambda,\gamma,\mu,0)\ln\left(\frac{\mu}{\mu-1+\left(1-(1-u)^{\frac{\lambda}{\gamma}}\right)^{-\alpha}}\right)du,$$

$$\frac{\partial R(n,\alpha,B,\gamma)}{\partial \alpha} = -B\mu^B \int_0^1 \left(1-(1-u)^{\frac{\lambda}{\gamma}}\right)^{-\alpha}\ln\left(1-(1-u)^{\frac{\lambda}{\gamma}}\right)\Psi(u,\alpha,\lambda,\gamma,\mu,1)du,$$

$$\frac{\partial R(n,\alpha,B,\gamma)}{\partial \gamma} = \frac{\alpha B\lambda\mu^B}{\gamma^2}\int_0^1 \ln(1-u)(1-u)^{\frac{\lambda}{\gamma}}\left(1-(1-u)^{\frac{\lambda}{\gamma}}\right)^{-\alpha-1}\Psi(u,\alpha,\lambda,\gamma,\mu,1)du,$$

and

$$\Psi(u,\alpha,\lambda,\gamma,\mu,c) = \left(\mu-1+\left(1-(1-u)^{\frac{\lambda}{\gamma}}\right)^{-\alpha}\right)^{-B-c}.$$

Applying the Delta method on the MLE of $R(n,\alpha,B,\gamma)$, we obtain

$$\sqrt{m}\hat{R}(n,\alpha,B,\gamma) = \sqrt{m}R(n,\alpha,B,\gamma) + V(\alpha,B,\gamma)\sqrt{m}\begin{pmatrix}\hat{\alpha}-\alpha \\ \hat{B}-B \\ \hat{\gamma}-\gamma\end{pmatrix} + o_p(1). \tag{6.8}$$

Using the results of Theorem 6.1, we obtain the following theorem.

Theorem 6.2. *Suppose that model* (1.1) *holds, we have*

$$\sqrt{m}\Big(\hat{R}(n,\alpha,B,\gamma) - R(n,\alpha,B,\gamma)\Big) \xrightarrow{D} \boldsymbol{N}\bigg(\mathbf{0}, V(\alpha,B,\gamma)\mathbb{I}^{-1}V^\intercal(\alpha,B,\gamma)\bigg), \tag{6.9}$$

where $V^\intercal(\alpha,B,\gamma)$ *is the transpose of* $V(\alpha,B,\gamma)$ *and* $\mathbb{I}^{-1}$ *is the inverse of information matrix presented in* (6.3)*.*

To build confidence intervals for $R(n,\alpha,B,\gamma)$, we apply the following consistent estimated variance

$$\widehat{Var}(\hat{R}(n,\alpha,B,\gamma)) = \frac{1}{m}V(\hat{\alpha},\hat{B},\hat{\gamma})\mathbb{I}^{-1}V^\intercal(\hat{\alpha},\hat{B},\hat{\gamma}). \tag{6.10}$$

Applying the consistent estimated variance provided in (6.10) in addition to the asymptotic normal distributions in Theorem 6.2, we are able to construct inferences on $R(n,\alpha,B,\gamma)$ by building confidence intervals. Now, the asymptotic confidence interval $100(1-\frac{\alpha}{2})$ of $R(n,\alpha,B,\gamma)$ is obtained as

$$\Big(\hat{R}(n,\alpha,B,\gamma) - z_{\frac{\alpha}{2}}\widehat{Var}(\hat{R}(n,\alpha,B,\gamma)), \hat{R}(n,\alpha,B,\gamma) + z_{\frac{\alpha}{2}}\widehat{Var}(\hat{R}(n,\alpha,B,\gamma))\Big),$$

where $z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ quantile of the standard normal distribution.

6.2. Bootstrap confidence intervals

In the previous subsection, the behavior of inferences about $R(n, \alpha, B, \gamma)$ based on extended asymptotic theories depends greatly on the approximation of the MLE sampling distribution of the parameters of interest to the normal distribution. Sometimes, such a regular approximation requires a large sample size which is probably impractical in actual problems. In this section, we look at two bootstrap intervals, that is, bootstrap-t and bootstrap percentile intervals, which best need a viable sample size to receive an appropriate estimate of the CDF of the unique populations. The bootstrap technique offered through [42], is a resampling system that has had great success in solving many complicated statistical problems. In this paper, parametric and non-parametric bootstrap techniques have been applied to generate random samples primarily based on which to build confidence intervals for the parameters of interest. Let $W_1, \ldots, W_{m_1} \sim G_W(w)$ and $X_1, \ldots, X_{m_2} \sim F(x)$ be the two identically independently distributed (i.i.d.) random samples. Applying the method in Section 5, we were able to gain the MLE of B, α, and γ indicated by $\hat{B}$, $\hat{\alpha}$, and $\hat{\gamma}$, respectively. The i.i.d. samples $W_1^*, \ldots, W_{m_1}^* \sim G_W(w; \hat{B}, \hat{\alpha})$ and $X_1^*, \ldots, X_{m_2}^* \sim F(x; \hat{\gamma})$ are named parametric bootstrap samples. Let $\hat{G}_{W,m_1}$ and $\hat{F}_{2,m_2}$ be the empirical CDFs determined by $W_1, \ldots, W_{m_1}$ and $X_1, \ldots, X_{m_2}$, respectively. The simple random samples with replacement $W_1^*, \ldots, W_{m_1}^* \sim \hat{G}_{W,m_1}$ and $X_1^*, \ldots, X_{m_2}^* \sim \hat{F}_{m_2}$ are named non-parametric bootstrap samples.

Steps for constructing bootstrap estimates of parameters

The next algorithm is applied to compute parametric and non-parametric bootstrap estimates $\hat{B}_{1,b}^*$, $\hat{\alpha}_{1,b}^*$, and $\hat{\gamma}_{2,b}^*$ for $b = 1, \ldots, B$. Several bootstrap confidence intervals will be obtained based on these bootstrap estimates of parameters.

Algorithm 1. Algorithm outline to compute bootstrap estimates of parameters

Algorithm:

(1) Choose bootstrap samples with sizes n_1 and n_2 from the equivalent bootstrap populations, i.e., $W_1^*, \ldots, W_{m_1}^* \sim G_W(w; \hat{B}, \hat{\alpha})$ or $\hat{G}_{W,m_1}$ and $X_1^*, \ldots, X_{m_2}^* \sim F(x; \hat{\gamma})$ or $\hat{F}_{m_2}$, respectively.

(2) Apply the method explained in Section 5 to estimate bootstrap MLEs $\hat{B}^*$, $\hat{\alpha}^*$, and $\hat{\gamma}^*$ depend on $W_1^*, \ldots, W_{m_1}^*$ and $X_1^*, \ldots, X_{m_2}^*$ and compute the MLEs according to the following pattern

$$\hat{R}^*(n, \hat{\alpha}^*, \hat{B}^*, \hat{\gamma}^*) = 1 - \int_0^1 \left(\frac{\mu}{\mu - 1 + \left(1 - (1-u)^{\frac{\lambda}{\hat{\gamma}^*}}\right)^{-\hat{\alpha}^*}} \right)^{\hat{B}^*} du. \tag{6.11}$$

(3) Repeat steps 1 and 2, B times and save the MLEs of parameters into their equivalent sets of bootstrap estimates: $\hat{B}_b^*$, $\hat{\alpha}_b^*$, $\hat{\gamma}_b^*$ and $\hat{R}_b^*(n, \hat{\alpha}_b^*, \hat{B}_b^*, \hat{\gamma}_b^*)$ for $b = 1, \ldots, B$.

In the following, two different kinds of bootstrap confidence intervals for the parameters of interest are proposed. For the sake of simplicity of display, we reduce our writing only to $R(n, \alpha, B, \gamma)$. The steps of building confidence intervals for the other three parameters of interest B, α, and γ are similar to $R(n, \alpha, B, \gamma)$. Suppose that $\hat{R}_b^*(n, \hat{\alpha}_b^*, \hat{B}_b^*, \hat{\gamma}_b^*)$ for $b = 1, \ldots, B$ are the bootstrap estimates of R. Moreover, assume that $\hat{R}(n, \hat{\alpha}, \hat{B}, \hat{\gamma})$ is the MLE obtained from the original dataset, and the confidence level is considered to be $100(1 - \alpha)\%$.

Bootstrap-t confidence interval

The bootstrap-t confidence interval imitates the method of building standard-t confidence intervals. Two parts of the confidence interval, i.e. t-like critical value, and the standard error of $\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma})$, are computed from the bootstrap estimates $\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b)$ for $b=1,\ldots,B$. The bootstrap standard error is determined by

$$SE^*(\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma})) = \sqrt{\frac{1}{B}\sum_{b=1}^{B}(\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b) - \overline{\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b)})^2},$$

where

$$\overline{\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b)} = \frac{1}{B}\sum_{b=1}^{B}\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b).$$

To obtain the t-like critical value, stated by $\hat{t}^*_\alpha$, we first standardize $\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b)$ for $b=1,\ldots,B$ by applying

$$z^*_b(R) = \frac{\hat{R}^*_b(n,\hat{\alpha}^*_b,\hat{B}^*_b,\hat{\gamma}^*_b) - \hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma})}{SE^*(\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma}))}.$$

The t-like critical value $\hat{t}^*_\alpha$ based on the bootstrap estimate is determined as

$$\frac{\#\{z^*_b(R) \le \hat{t}^*_\alpha\}}{B} = \alpha.$$

Therefore, the bootstrap-t confidence interval can be described as

$$\Big(\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma}) - \hat{t}^*_{1-\frac{\alpha}{2}}SE^*(\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma})), \hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma}) + \hat{t}^*_{\frac{\alpha}{2}}SE^*(\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma}))\Big),$$

where $\hat{t}^*_{1-\frac{\alpha}{2}}$ and $\hat{t}^*_{\frac{\alpha}{2}}$ are the the $(\frac{\alpha}{2})$-th and $(1-\frac{\alpha}{2})$-th percentile values of $z^*_b(R)$, respectively.

Bootstrap percentile confidence interval

We need to construct a confidence interval based on the bootstrap distribution. Suppose that $\hat{H}^*_B(t) = Pr(\hat{R}^*_B(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_B) \le t)$ where $\hat{H}^*_B$ is the bootstrap CDF of $\hat{R}^*_B(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_b)$. If the bootstrap distribution achieved by the Mont Carlo simulation is, then we have $\hat{H}^*_B(t) = \frac{\#(\hat{R}^*_B(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_B)\le t)}{B}$. [52] established a $100(1-\alpha)\%$ approximate bootstrap percentile confidence interval $100(1-\alpha)\%$ for R as

$$(\hat{R}^{*(\frac{\alpha}{2})}(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_B), \hat{R}^{*(1-\frac{\alpha}{2})}(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_B))$$

where $\hat{R}^{*(\frac{\alpha}{2})}(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_B)$ be the $\frac{\alpha}{2}$-th percentile of the distribution of $\hat{R}^*_B(n,\hat{\alpha}^*_B,\hat{B}^*_B,\hat{\gamma}^*_B)$.

7. A simulation study

In this section, we complete simulation studies on the performance of some considerable estimators of $R(n,\alpha,B,\gamma)$, established in the preceding sections, based on small samples. The calculations in this paper are performed using the open source statistical computer package R (v.4.3.3) on the Windows platform. We apply the following reverse transformation algorithm to generate random samples according to the models (5.2) and (5.3). It is known that for CDF $G_W(w)$ the random variable stated by $W = G_W^{-1}(U)$ has distribution $G_W(w)$, where U is a uniform random variable defined on $(0,1)$. Note that, under (5.2), the random number W represented by

$$W = G_W^{-1}(U) = -\frac{1}{\lambda}\ln\left(1 - \left(\mu(U^{-\frac{1}{\beta}} - 1) + 1\right)^{-\frac{1}{\alpha}}\right). \tag{7.1}$$

Let $W \sim g_W(w)$ and $X \sim f(x)$, as determined in (5.2) and (5.3). We first simulate 1000 random samples from $g_W(w)$ and $f(x)$, respectively. For a pair of two samples from (5.2) and (5.3), we can perform the approaches prepared in Sections 5 and 6 to obtain the MLE of $R(n,\alpha,B,\gamma)$ along with the asymptotic confidence intervals of $R(n,\alpha,B,\gamma)$. In addition, some plots are provided that show the sampling distributions of the MLE suggested by $R(n,\alpha,B,\gamma)$ along with serial plots of MSE versus the number of simulations to study the stability of the simulation results. Since the parameters λ and μ are known, we select a constant $\lambda=\mu=2$ throughout this simulation study. The sample size is one of the main factors that affect the performance of the estimators. Like always, we also want to analyze the influence of sample size on different suggested estimators of $R(n,\alpha,B,\gamma)$. In Figure 1, we have graphed the values of $\mathrm{MSE}(\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma}))$ versus $R(n,\alpha,B,\gamma)$, for some different values of m_1 and m_2. Figure 1 shows that the estimator ML has more error when $R(n,\alpha,B,\gamma)$ tends to 0.5. Furthermore, for $m_1=m_2$ it is symmetric around the point $R(n,\alpha,B,\gamma)=0.5$ and departs from symmetry when $m_1<m_2$ or $m_1>m_2$. The MSE of the estimator is first increasing, then decreasing, and reaches its maximum at the point $R(n,\alpha,B,\gamma)\simeq\frac{m_1}{m_1+m_2}$.

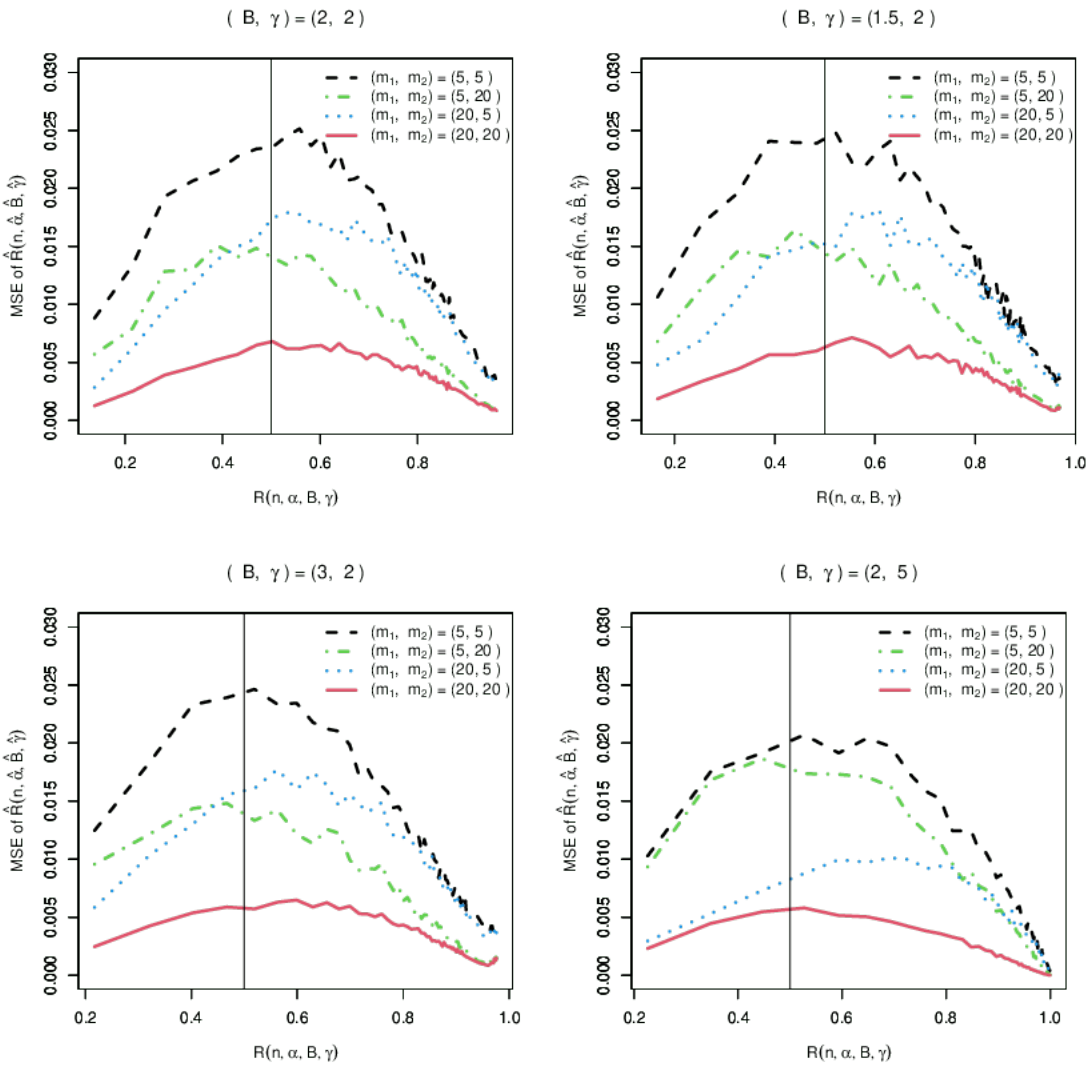

Figure 1. Plots of MSE of $\hat{R}(n,\hat{\alpha},\hat{B},\hat{\gamma})$ versus $R(n,\alpha,B,\gamma)$.

Assume that m_1 and m_2 be the sample sizes that generated from (5.2) and (5.3), respectively. We want to assess the performance of the estimators suggested of α, B, and γ with values $(\alpha,B,\gamma,m_1,m_2)=(2,2,2,5,5)$, $(\alpha,B,\gamma,m_1,m_2)=(2,2,2,5,20)$,

$(\alpha, B, \gamma, m_1, m_2) = (2, 2, 2, 20, 5)$, $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 2, 20, 20)$, $(\alpha, B, \gamma, m_1, m_2) = (3, 1.5, 2, 5, 5)$, $(\alpha, B, \gamma, m_1, m_2) = (3, 1.5, 2, 5, 20)$, $(\alpha, B, \gamma, m_1, m_2) = (3, 1.5, 2, 20, 5)$, $(\alpha, B, \gamma, m_1, m_2) = (3, 1.5, 2, 20, 20)$, $(\alpha, B, \gamma, m_1, m_2) = (1.5, 3, 2, 5, 5)$, $(\alpha, B, \gamma, m_1, m_2) = (1.5, 3, 2, 5, 20)$, $(\alpha, B, \gamma, m_1, m_2) = (1.5, 3, 2, 20, 5)$, $(\alpha, B, \gamma, m_1, m_2) = (1.5, 3, 2, 20, 20)$, $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 5, 5, 5)$, $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 5, 5, 20)$, $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 5, 20, 5)$, and $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 5, 20, 20)$. From Figures 2, we can see that the simulated MSEs of $\hat{R}(n, \hat{\alpha}, \hat{B}, \hat{\gamma})$ under different choices of $(\alpha, B, \gamma, m_1, m_2)$ become stable when the number of simulations reaches approximately 395. As expected, the MSEs showed themselves to be smaller for a larger sample size.

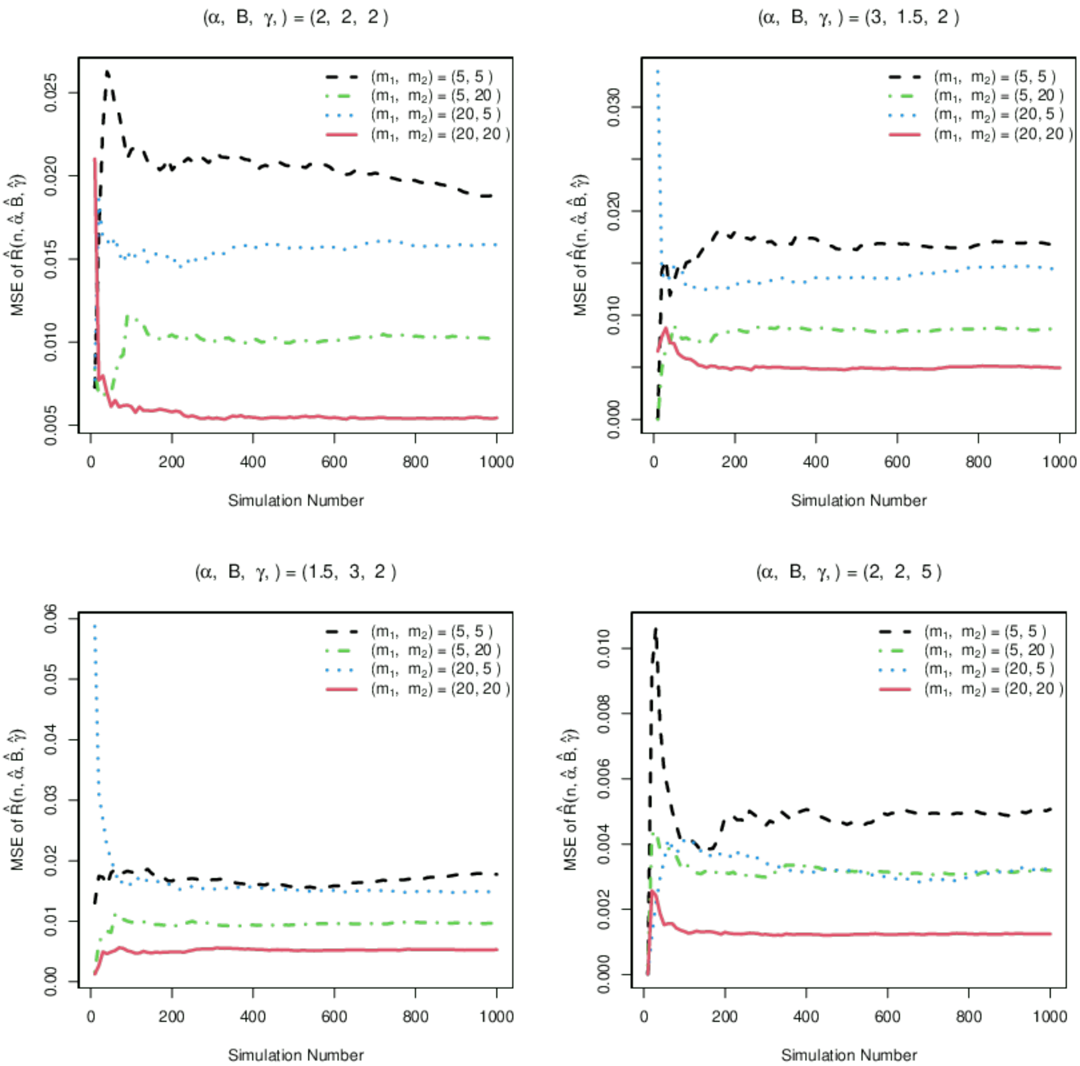

Figure 2. Plots of MSE of $\hat{R}$ versus the number of simulations.

In Figure 3, we show the sampling distributions of MLE for $\hat{R}(n, \hat{\alpha}, \hat{B}, \hat{\gamma})$ based on simulation using different values of population parameters and sample sizes. It can be observed that for $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 2, 5, 20)$, $(2, 2, 2, 20, 5)$, $(3, 1.5, 2, 5, 20)$, $(3, 1.5, 2, 20, 5)$, $(1.5, 3, 2, 5, 20)$, $(1.5, 3, 2, 20, 5)$, $(2, 2, 5, 5, 20)$, and $(2, 2, 5, 20, 5)$ the sampling distribution of $\hat{R}(n, \hat{\alpha}, \hat{B}, \hat{\gamma})$ are skewed. Also, for $(\alpha, B, \gamma, m_1, m_2) = (2, 2, 2, 20, 20)$, $(3, 1.5, 2, 20, 20)$, $(1.5, 3, 2, 20, 20)$, and $(2, 2, 5, 20, 20)$ the sampling distributions of $\hat{R}(n, \hat{\alpha}, \hat{B}, \hat{\gamma})$ are approximately symmetric at $R(n, \alpha, B, \gamma)$.

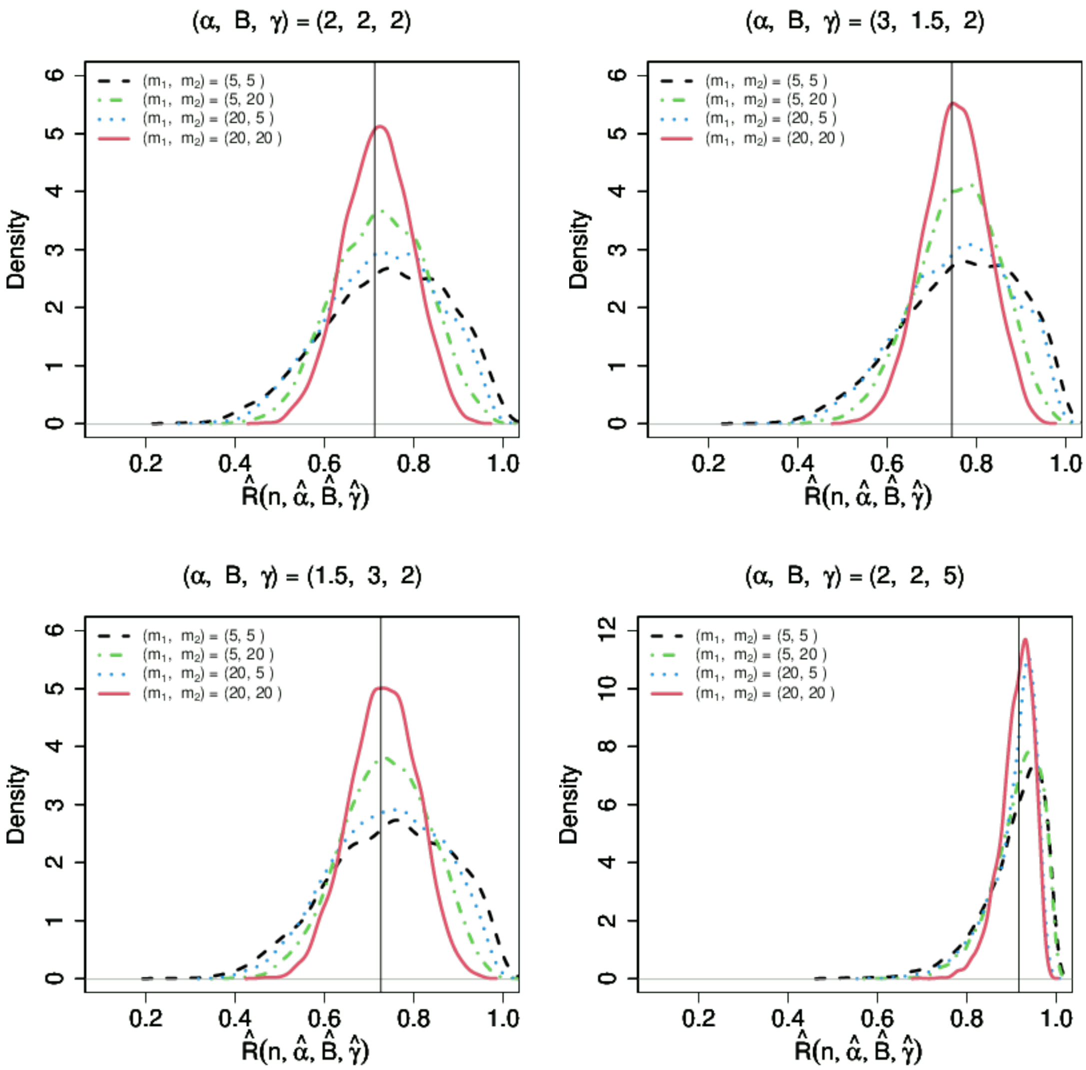

Figure 3. Sampling distribution of $\hat{R}$.

Table 1. Comparison of two point estimations for some value of the parameters.

m_1	m_2	γ	α	B	$R(n,\alpha,B,\gamma)$	$\tilde{R}(n,\alpha,B,\gamma)$	MSE(MSP)	$\hat{R}(n,\alpha,B,\gamma)$	MSE(ML)
5	5	1.5	3	2	0.7063	0.6977	0.0193	0.6952	0.0198
5	5	2	2	2	0.7123	0.7281	0.0188	0.7196	0.0196
5	5	5	3	3	0.9785	0.9699	0.0112	0.9621	0.0132
5	20	1.5	3	2	0.7063	0.7191	0.0087	0.7083	0.0092
5	20	2	2	2	0.7123	0.6960	0.0102	0.7032	0.0154
5	20	5	3	3	0.9785	0.9622	0.0005	0.9715	0.0012
20	5	1.5	3	2	0.7063	0.6872	0.0176	0.7021	0.0195
20	5	2	2	2	0.7123	0.7370	0.0151	0.7185	0.0176
20	5	5	3	3	0.9785	0.9812	0.0011	0.9705	0.0034
20	20	1.5	3	2	0.7063	0.7295	0.0057	0.7287	0.0068
20	20	2	2	2	0.7123	0.7334	0.0053	0.7391	0.0062
20	20	5	3	3	0.9785	0.9784	0.0002	0.9801	0.0012

The MSEs of the ML and MSP estimators in Table 1 are compared, it can be seen that the MSE of the MSP estimator is slightly smaller than the MSE of the ML estimator, which concludes that the MSP estimator is overall slightly better than the ML estimator in this stress- strength model.

To obtain and make a comparison between the different bootstrap confidence intervals, we select 1000 parametric and nonparametric bootstrap samples using the method prepared in Section 6.2 from each of the simulations and find both parametric and nonparametric bootstrap-t, bootstrap-q confidence intervals. In Table 2, for all combinations of $m_1 = 5, 20$, $m_2 = 5, 20$ and $R(n,\alpha,B,\gamma) = 0.184, 0.336, 0.506, 0.725, 0.861, 0.994,$ we report the average length (AL) and the coverage proportions (CP) of asymptotic, parametric and nonparametric bootstrap confidence intervals which include the true value of the corresponding parameter. The CP and AL of parametric bootstrap percentile confidence intervals, respectively, P.CP.B.t and P.AL.B.t imply the CP and AL of parametric bootstrap-t confidence intervals, respectively, N-P.CP.P. and N-P.AL.P. represent the CP and AL of nonparametric bootstrap percentile confidence intervals, respectively, N-P.CP.B.t and N-P.AL.B.t stand for CP and AL of nonparametric bootstrap-t confidence intervals, respectively

Table 2. The values of CP and AL of the aforementioned asymptotic, parametric and non-parametric confidence intervals for R.

$R(n,\alpha,B,\gamma)$	m_1	m_2	CP.A.	AL.A.	P.CP.P.	P.AL.P.	N-P.CP.P.	N-P.AL.P.	P.CP.B.t	P.AL.B.t	N-P.CP.B.t	N-P.AL.B.t
0.184	5	5	0.940	0.397	0.969	0.288	0.961	0.319	0.946	0.270	0.945	0.284
0.184	5	20	0.912	0.331	0.958	0.293	0.955	0.298	0.926	0.225	0.925	0.231
0.184	20	5	0.933	0.249	0.981	0.219	0.952	0.224	0.950	0.170	0.941	0.185
0.184	20	20	0.948	0.185	0.973	0.147	0.970	0.156	0.954	0.118	0.959	0.115
0.336	5	5	0.934	0.549	0.964	0.457	0.960	0.509	0.936	0.340	0.939	0.390
0.336	5	20	0.921	0.434	0.967	0.377	0.956	0.367	0.939	0.289	0.943	0.301
0.336	20	5	0.920	0.420	0.970	0.351	0.965	0.364	0.946	0.272	0.945	0.271
0.336	20	20	0.938	0.267	0.972	0.225	0.974	0.232	0.950	0.173	0.949	0.173
0.506	5	5	0.940	0.600	0.966	0.496	0.962	0.533	0.941	0.375	0.940	0.406
0.506	5	20	0.928	0.428	0.971	0.388	0.973	0.392	0.944	0.292	0.941	0.297
0.506	20	5	0.935	0.492	0.972	0.424	0.965	0.432	0.947	0.336	0.946	0.329
0.506	20	20	0.943	0.296	0.974	0.252	0.973	0.273	0.949	0.196	0.953	0.197
0.725	5	5	0.930	0.533	0.972	0.431	0.966	0.444	0.930	0.342	0.936	0.365
0.725	5	20	0.904	0.382	0.961	0.300	0.963	0.317	0.924	0.257	0.920	0.248
0.725	20	5	0.909	0.473	0.964	0.391	0.967	0.402	0.938	0.299	0.931	0.301
0.725	20	20	0.940	0.279	0.985	0.234	0.988	0.243	0.955	0.184	0.957	0.190
0.861	5	5	0.934	0.395	0.972	0.318	0.978	0.361	0.935	0.254	0.921	0.261
0.861	5	20	0.923	0.273	0.966	0.212	0.972	0.241	0.966	0.177	0.965	0.180
0.861	20	5	0.920	0.359	0.963	0.296	0.965	0.307	0.945	0.251	0.944	0.248
0.861	20	20	0.944	0.213	0.986	0.181	0.990	0.182	0.959	0.134	0.961	0.139
0.994	5	5	0.945	0.216	0.981	0.180	0.985	0.175	0.919	0.137	0.910	0.139
0.994	5	20	0.939	0.112	0.973	0.094	0.976	0.100	0.931	0.073	0.935	0.076
0.994	20	5	0.948	0.220	0.971	0.199	0.978	0.184	0.940	0.136	0.943	0.139
0.994	20	20	0.954	0.110	0.989	0.095	0.995	0.097	0.952	0.073	0.957	0.078

From Table 2, by empirical evidence, it is observed that:
(i) AL approximately reduces by raising the sample size and the maximum of the AL takes place at the middle point $R(n,\alpha,B,\gamma) = 0.5$.
(ii) the CP of bootstrap percentile confidence intervals is greater than all others.
(iii) the AL the bootstrap-t confidence intervals is better than others.
(iv) there is no meaningful difference between parametric and nonparametric confidence intervals from the point of view of the length and coverage proportion of the intervals.

8. An illustrative example

In this section, we suggest a numerical example based on a real-life data set to illustrate the performance of the procedure considered. The data sets used in this article represent the monthly water capacity of the Shasta reservoir in California, USA, especially the month of April and the mean annual capacity from 1992 to 2015. (See [46] for more details). To take precautions against excessive drought, the following scenario can be constructed. In the five-year period, if the maximum of the water capacity of the reservoir on each April is more than the average water capacity of the previous year (which is the preceding year of the five years period), it is claimed that there will be no excessive drought in the months of October and November afterward. Let X_1 be the mean annual capacity of 1992, Y_{1k}, $k = 1, \dots, 5$ be the capacity of April from 1993 to 1997, X_2 is the mean annual capacity of 1998, Y_{2k}, $k = 1, \dots, 5$ are the capacity of April from 1999 to 2003. When we continue this data process up to 2015, we obtain $n = 4$. To remove (or reduce) the dependency between Y_{ik} and X_i , the years of X_i are not used to obtain the data Y_{ik}. For computational ease, all values divided by the total capacity of Shasta reservoir 4.552.000 acre-foot and these transformed data are obtained as

$$\boldsymbol{Y} = \begin{pmatrix} 0.9366 & 0.7763 & 0.9150 & 0.9463 & 0.8649 \\ 0.9350 & 0.9124 & 0.8831 & 0.9439 & 0.9966 \\ 0.9243 & 0.8913 & 0.8570 & 0.6490 & 0.6587 \\ 0.9372 & 0.9754 & 0.8322 & 0.5292 & 0.5894 \end{pmatrix}, \qquad \boldsymbol{X} = \begin{pmatrix} 0.4529 \\ 0.8222 \\ 0.6730 \\ 0.7985 \end{pmatrix}.$$

Thus, we obtain the 1-out-of-n:G multicomponent stress-strength system with observed stress data $\boldsymbol{X}$ and observed strength data $\boldsymbol{W} = (0.9463, 0.9966, 0.9243, 0.9463, 0.9754)$. This data set has been used in hydrology research to analyze droughts. First, we test to see whether the exponential distribution function is appropriate to fit the stress data set or not. For modeling the stress data via the exponential distribution, we use the fitdist(...) command in the fitdistrplus R package. We find that the exponential distribution with scale parameter $\gamma = 1.456346$ is quite well adapted to the stress data set. We use the Kolmogorov-Smirnov (K-S) test to verify the claim, the corresponding p-value is 0.2182. For modeling the strength data via the exponential distribution, we use the mpsexpg (...) command in the MPS R package [50]. One of the outputs of this command is the p-value of Chi-square goodness-of-fit tests based on the maximum product spacing approach with Morans log spacing statistic. It should be mentioned that this test is not a classical chi-square test. For more details on this test, see [44]. Also, the first output of this command is the estimated parameter vector, which is obtained with the maximum product spacing approach. It would appear that the $UGG - G_0(70.718681, 1, 1)$ model is totally good for fitting the strength data set, where G_0 is the exponential distribution function with mean 4.729113. For computing the p-value, we applied the command mpsexpg(...) in MPS R Package. In this package, the significance level for the aforementioned goodness-of-fit test is reported. The corresponding p-values of the aforementioned goodness-of-fit tests for strength data is 0.8955. It should be mentioned that, for $\mu = 1$ and $\beta = 1$, the special case of the UGG model is the proportional reversed hazard rate model with cdf $G(x) = G_0^{\alpha}(x),\ \alpha > 0$.

Table 3. The values AL of asymptotic and various parametric and non-parametric bootstrap confidence intervals of R based on real dataset.

α	AL.A.	P .AL.P.	P .AL.B.t	N- P .AL.P.	N-PAL.B.t
0.05	0.2984 (0.5023,0.8007)	0.2873 (0.6440,0.9313)	0.0769 (0.6893,0.7662)	0.2622 (0.5270,0.7892)	0.1949 (0.6676,0.8625)
0.1	0.2891 (0.5511,0.8402)	0.2721 (0.6950,0.9671)	0.0633 (0.6994,0.7627)	0.2614 (0.6831, 0.9445)	0.1886 (0.7237,0.9123)

From Table 3, it appears that, there is a significant difference between bootstrap percentile, bootstrap-t, and asymptotic confidence intervals in terms of AL. The AL of the bootstrap-t confidence interval is less than others. Therefore, the use of the bootstrap-t confidence interval for R is recommended. Also, all confidence intervals for R don't contain the value 0.5 implying that there is a significant difference in the aforementioned two data sets. It should be mentioned that $\hat{R} = 0.7592$ and $\tilde{R} = 0.7628$. Therefore, based on the results, there will be no excessive drought in the months of October and November afterwards with a probability of 0.7592.

9. Summary and conclusion

In this paper, the stress-strength reliability of a 1-out-of n:G stress-strength system associated with the UGG models is investigated. First of all by using the properties of the populations parameters and functional form of UGG models the stochastic orders between stress and strength random variables are assessed and the exact expression for reliability of a 1-out-of n:G stress-strength system is obtained. The asymptotic distribution of the MLE of stress-strength reliability is determined based on exponential baseline distribution functions. Furthermore, the parametric and non-parametric bootstrap-t and bootstrap quantile confidence intervals are proposed. Their performances on some sample sizes with respect to AL and CP are analyzed by using a simulation study. In the view point of CP, the bootstrap percentile confidence is suitable among the other confidence intervals in the simulation study. A numerical example based on real-life data was taken to demonstrate the performance of the recommended approaches. In the real-life data, we suggest the use of bootstrap-t confidence interval for 1-out-of n:G stress-strength reliability which is also confirmed the results of the simulation study.

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