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Research Article

Componentwise linearity of dominating ideals of path graphs

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Abstract

We show the componentwise linearity of dominating ideals of path graphs by describing a linear quotient order of their minimal generating sets. We also give formulas for their Betti numbers, regularity and projective dimension.

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1. Introduction

In the 1960s, Berge and Ore developed a mathematical formulation for the concept of graph domination, which has since gathered significant attention from researchers. Its applications span across diverse fields such as computer science, operations research, linear algebra, and optimization. For additional related concepts regarding graph domination, readers are directed to [2]. In relation to the concept of domination in graphs, the definitions of closed neighborhood ideals and dominating ideals were introduced by Sharifan and Moradi recently in [7]. The closed neighborhood ideal of a simple graph G, denoted by $NI(G)$, is the squarefree monomial ideal generated by monomials corresponding to the closed neighborhoods of vertices of G, and the dominating ideal of G, denoted by $DI(G)$, is the squarefree monomial ideal generated by monomials corresponding to the dominating sets of G. It is observed in [7, Lemma 2.2] that for any graphs G, the ideals $DI(G)$ and $NI(G)$ are Alexander dual of each other. Different graphs can correspond to the same closed neighborhood ideals and dominating ideals as noticed in [5, Example 2.2].

In comparison to the edge ideals and cover ideals associated with graphs which are well known and extensively studied, relatively little is known in the case of closed neighborhood ideals and dominating ideals of graphs. In [1], Farber introduced strongly chordal graphs and proved that a graph G is strongly chordal if and only if the neighborhood hypergraph of G is totally balanced. Rephrasing this in the algebraic language, it is established in [6] that the dominating ideals of strongly chordal graphs are componentwise linear. Since path graphs are strongly chordal, one can conclude that the dominating ideals of path graphs are componentwise linear. In our work, we show that the dominating ideals of path graphs

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have linear quotients by precisely giving a linear quotient order of their minimal generating set. Invoking [4, Theorem 8.3.15], we obtain another proof for the componentwise linearity of dominating ideals of path graphs. Utilizing a well-known result of Sharifan and Varbaro [8, Corollary 2.7], we also compute the total and graded Betti numbers of the dominating ideals of path graphs.

The breakdown of the content of this article is as follows: Section 2 contains all the required ingredients from commutative algebra and graph theory. In Section 3, we describe a recursive order on the generating set of dominating ideals of path graphs which gives linear quotients as shown in Theorem 3.5. This order is used in Section 4, to describe the total and graded Betti numbers of dominating ideals of path graphs (See Theorem 4.3 and Theorem 4.4). In Theorem 4.2, we also compute projective dimension and regularity of dominating ideals of path graphs and recover the formulas given in [7, Theorem 2.6].

2. Required terminologies

We first recall some definitions and notions from commutative algebra. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and I be a homogeneous ideal of S. Let

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{p,j}(I)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}(I)} \xrightarrow{\phi_0} I \to 0$$

be the minimal graded free resolution of I, see [4, page 265]. For each i and j, $\beta_{i,j}(I)$ is the (i,j)*-th graded Betti number of* I, and the i*-th Betti number of* I is $\beta_i(I) = \sum_{j\in\mathbb{Z}} \beta_{i,j}$. The CastelnuovoMumford regularity (or simply regularity) of I, denoted by $\operatorname{reg}(I)$, is

$$\operatorname{reg}(I) = \max\{j \mid \beta_{i,i+j}(I) \neq 0\},$$

and the projective dimension of I is the length of its minimal graded free resolution, given by

$$\operatorname{proj\,dim}(I) = \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$

The ideal I is said to have d*-linear* resolution if $\beta_{i,j}(I) = 0$ for all i and all $j - i \neq d$. Let I_d be the ideal generated by all homogeneous polynomials of degree d in I. The ideal I is called a *componentwise linear ideal* if I_d has a linear resolution, for each d.

For a given monomial ideal $I \subset S$, we denote the unique minimal generating set of I by $G(I)$. Let $G(I) = \{u_1, \dots, u_r\}$. The ideal I is said to have *linear quotients* with respect to the ordering $u_1, \dots, u_r$ if the colon ideal $(u_1, \dots, u_{i-1}) : (u_i)$ is generated by a subset of variables, for each $i = 2, \dots, r$. We refer such an order of $G(I)$ as a *linear quotient order*. If an ideal admits linear quotients, then it is componentwise linear, see [4, Proposition 8.2.15]. However, the converse of this statement does not hold, for example, see [3, Remark 2.15]. For more information on ideals with linear quotients and componentwise linear ideals, we refer reader to [4, Theorem 8.2.15] and [3].

The *support* of a monomial $u \in S$, denoted by $\operatorname{supp}(u)$, is the set of variables that divide u. Given a squarefree monomial ideal $I \subset S$, the Alexander dual of I, denoted by $I^\vee$, is

$$I^\vee = \bigcap_{u\in G(I)} (x_i \ : \ x_i \in \operatorname{supp}(u)).$$

Next, we recall some definitions and notions from graph theory. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For each vertex $v \in V(G)$, the *closed neighborhood* of v in G is the set

$$N_G[v] = \{u \in V(G) : \{u, v\} \in E(G)\} \cup \{v\}.$$

When there is no confusion about the underlying graph, we will denote $N_G[v]$ simply by $N[v]$. A vertex $v \in V(G)$ is said to *dominate* a vertex $v' \in V(G)$ if $v \in N[v']$. In particular, every vertex dominates itself. A subset $S \subseteq V(G)$ is called a *dominating set* of G if $S \cap N[v] \neq \emptyset$, for all $v \in V(G)$. A dominating set is called minimal if it does not properly contain any other dominating set of G. The domination in graphs is a well-studied topic in graph theory. We refer reader to [2] for more information on this topic.

Let G be a simple graph with $V(G) = \{x_1, \dots, x_n\}$. We identify the vertices of G as variables of S to simplify the notation. In [7], Moradi and Sharifan introduced the notion of closed neighborhood ideals and dominating ideals of graphs as follows: for any graph G, the *closed neighborhood ideal* of G is

$$NI(G) = (\prod_{x_j \in N[x_i]} x_j : x_i \in V(G)),$$

and the *dominating ideal* of G is

$$DI(G) = (\prod_{x_i \in T} x_i : T \text{ is a minimal dominating set of } G).$$

It is shown in [7, Lemma 2.2] that $DI(G)$ is the Alexander dual of $NI(G)$.

3. Linear quotient order of dominating ideals of path graphs

For a given $n > 1$, let P_n be the the *path graph* with vertices $x_1, \dots, x_n$ and edges $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}$. For $n = 1$, we set P_1 as a graph with an isolated vertex x_1. In this section we will construct a recursive order of $G(DI(P_n))$ which gives linear quotients. To do this, we first give a recursive presentation of dominating sets of P_n. Throughout the following text, for any non-empty set A and for any element x, we set $xA = \{xy |\ y \in A\}$. If A is empty then we set xA to be an empty set as well.

Remark 3.1. *Let $I_n = DI(P_n)$, and set*

$$\begin{aligned} A_1 &= \emptyset, & B_1 &= \{x_1\}, \\ A_2 &= \{x_1\}, & B_2 &= \{x_2\}, \\ A_3 &= \{x_2\}, & B_3 &= \{x_1x_3\}, \\ A_4 &= \{x_1x_3, x_2x_3\}, & B_4 &= \{x_1x_4, x_2x_4\}. \end{aligned}$$

It can be readily verified with simple computations that

$$\begin{aligned} G(I_1) &= A_1 \cup B_1 = \{x_1\}; \\ G(I_2) &= A_2 \cup B_2 = \{x_1, x_2\}; \\ G(I_3) &= A_3 \cup B_3 = \{x_2, x_1x_3\}; \\ G(I_4) &= A_4 \cup B_4 = \{x_1x_3, x_2x_3, x_1x_4, x_2x_4\}. \end{aligned}$$

For $n \geq 5$, we set $A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4})$ and $B_n = x_nG(I_{n-2})$. It can be easily verified that

$$G(I_5) = A_5 \cup B_5 = x_4G(I_2) \cup x_4(x_3A_1) \cup x_5G(I_3) = \{x_1x_4, x_2x_4, x_2x_5, x_1x_3x_5\}$$

and similarly,

$$\begin{aligned} G(I_6) = A_6 \cup B_6 &= x_5G(I_3) \cup x_5(x_4A_2) \cup x_6G(I_4) \\ &= \{x_2x_5, x_1x_3x_5, x_1x_4x_5, x_1x_3x_6, x_2x_3x_6, x_1x_4x_6, x_2x_4x_6\}. \end{aligned}$$

In the following theorem, we give a recursive way to construct dominating ideals for path graphs.

Theorem 3.2. Let $I_n = DI(P_n)$ and $n \geq 5$. Then $G(I_n) = A_n \cup B_n$, where

$$A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}x_{n-2}A_{n-4} \quad \text{and} \quad B_n = x_nG(I_{n-2}).$$

Moreover, the sets $x_{n-1}G(I_{n-3})$, $x_{n-1}x_{n-2}A_{n-4}$, and $x_nG(I_{n-2})$ are pairwise disjoint.

Proof. It immediately follows from the definition of minimal dominating set and the construction of A_n and B_n that $x_{n-1}G(I_{n-3})$, $x_{n-1}x_{n-2}A_{n-4}$, and $x_nG(I_{n-2})$ are pairwise disjoint. To prove $G(I_n) = A_n \cup B_n$, we apply induction on n. The case $n = 5$ can be verified from Remark 3.1. Assume that $n > 5$. First we show that $A_n \cup B_n \subseteq G(I_n)$. Let $w \in B_n$, then $w = x_nw'$ for some $w' \in G(I_{n-2})$. Since $\operatorname{supp}(w')$ is a minimal dominating set of P_{n-2} and x_n dominates x_{n-1} and itself in P_n, we obtain that $\operatorname{supp}(w)$ is a dominating set of P_n. The minimality of $\operatorname{supp}(w)$ follows from the minimality of $\operatorname{supp}(w')$. This gives $w \in G(I_n)$.

Now, let $w \in A_n$. Then $w = x_{n-1}w''$ for some $w'' \in G(I_{n-3})$ or $w'' \in x_{n-2}A_{n-4}$. If $w'' \in G(I_{n-3})$, then $\operatorname{supp}(w'')$ is a minimal dominating set of P_{n-3}. Furthermore, x_{n-1} dominates x_{n-2}, x_n and itself in P_n. It yields that $\operatorname{supp}(x_{n-1}w'')$ is a dominating set of P_n. The minimality of $\operatorname{supp}(w)$ follows from the minimality of $\operatorname{supp}(w'')$. This gives $w \in G(I_n)$. On the other hand, if $w'' \in x_{n-2}A_{n-4}$, then there exists $w''' \in A_{n-4}$ such that $w = x_{n-1}x_{n-2}w'''$. By the induction hypothesis, we have $A_{n-4} \subset G(I_{n-4})$, hence $\operatorname{supp}(w''')$ is a minimal dominating set of P_{n-4}. The remaining vertices $x_{n-3}, x_{n-2}, x_{n-1}, x_n$ of P_n are minimally dominated by x_{n-2} and x_{n-1}. Thus, $\operatorname{supp}(x_{n-1}x_{n-2}w''')$ is a minimal dominating set of P_n and $w \in G(I_n)$.

Next we show that $G(I_n) \subseteq A_n \cup B_n$. Since $N[x_n] = \{x_n, x_{n-1}\}$, for any $w \in G(I_n)$, either x_n divides w or x_{n-1} divides w. However x_n and x_{n-1} do not divide w at the same time, by the virtue of minimality of $\operatorname{supp}(w)$ as a dominating set of P_n.

First, assume that x_n divides w. Then $w = x_nw'$ for some monomial w'. Since x_n dominates itself and x_{n-1}, the set $\operatorname{supp}(w') = \operatorname{supp}(w) \setminus \{x_n\}$ is a dominating set of P_{n-2}. The minimality of $\operatorname{supp}(w')$ follows from the minimality of $\operatorname{supp}(w)$. It gives $w' \in G(I_{n-2})$ and $w \in B_n$.

Now, assume that x_{n-1} divides w. Then $w = x_{n-1}w''$ for some monomial w''. Below we show that $w \in A_n$. To do this, we consider the following cases:

Case 1: Assume that $x_{n-2} \notin \operatorname{supp}(w'')$. Since x_{n-1} dominates only x_n, x_{n-2} and itself, for each $i = 1, \ldots, n-3$, the vertex x_i must be dominated by $\operatorname{supp}(w'')$. This shows that $\operatorname{supp}(w'')$ is a dominating set of P_{n-3}. The minimality of $\operatorname{supp}(w'')$ follows from the minimality of $\operatorname{supp}(w)$, and we obtain $w'' \in G(I_{n-3})$. This shows that $w \in x_{n-1}G(I_{n-3})$.

Case 2: Assume that $x_{n-2} \in \operatorname{supp}(w'')$. Then $w = x_{n-1}w'' = x_{n-1}x_{n-2}u$ for some monomial u. Then $x_{n-3} \notin \operatorname{supp}(u)$, otherwise, $\operatorname{supp}(w) \setminus \{x_{n-2}\}$ is a dominating set of P_n, a contradiction to the minimality of $\operatorname{supp}(w)$. This shows that $\operatorname{supp}(u)$ is a dominating set of P_{n-4}. The minimality of $\operatorname{supp}(u)$ follows from the minimality of $\operatorname{supp}(w)$ and hence $u \in G(I_{n-4})$. By the induction hypothesis we have $G(I_{n-4}) \subseteq A_{n-4} \cup B_{n-4}$. Note that $x_{n-4} \notin \operatorname{supp}(u)$, otherwise, $\operatorname{supp}(w) \setminus \{x_{n-2}\}$ is a dominating set of P_n, a contradiction to the minimality of $\operatorname{supp}(w)$. This shows that $u \notin B_{n-4}$ because every element in B_{n-4} is a multiple of x_{n-4}. Therefore $u \in A_{n-4}$ and $w \in x_{n-1}x_{n-2}A_{n-4} \subset A_n$. This completes the proof. $\square$

Remark 3.3. *Set $I_n = DI(P_n)$. For each $n \geq 1$, we order the elements of $G(I_n)$ by first listing the elements of A_n and then listing the elements of B_n. In particular, for $1 \leq n \leq 4$, we order the elements in A_n and B_n as given in Remark 3.1. For $n \geq 5$, from Theorem 3.2 we have*

$$\begin{aligned} G(I_n) &= A_n \cup B_n \\ &= x_{n-1}G(I_{n-3}) \cup x_{n-1}x_{n-2}A_{n-4} \cup x_nG(I_{n-2}). \end{aligned}$$

Let $G(I_{n-3}) = \{u_1, \ldots, u_t\}$, $A_{n-4} = \{a_1, \ldots, a_k\}$ and $G(I_{n-2}) = \{v_1, \ldots, v_s\}$. An ordering of $G(I_n)$ for $n \geq 5$ is given below:

$$x_{n-1}u_1, \ldots, x_{n-1}u_t, x_{n-1}x_{n-2}a_1, \ldots, x_{n-1}x_{n-2}a_k, x_nv_1, \ldots, x_nv_s.$$

For example, in Remark 3.1 the elements of $G(I_5)$ and $G(I_6)$ are listed in the order described above. Throughout the following text, we let $\mathbf{A_n}$ and $\mathbf{B_n}$ be the ideals generated by the elements of A_n and B_n, respectively.

Next we will show that $DI(P_n)$ has linear quotients with respect to the ordering of the generators given in the remark above. To do this, we first state the following simple observation.

Lemma 3.4. *Let $I \subset S = K[x_1, \dots, x_n, x, y]$ be a monomial ideal with $G(I) = \{u_1, \dots, u_m\}$ such that $x, y \notin \operatorname{supp}(u_i)$ for all $i = 1, \dots, m$. Then the following statements hold.*

(i) *Let w be a monomial in S with $x \notin \operatorname{supp}(w)$. Then any generator of $xI : (w)$ is divisible by x.*
(ii) *For any $u_i \in G(I)$, we have $xI : (yu_i) = (x)$.*

Proof. (i) It is easy to see that every generator of $xI : (w)$ is of the form $xu_i/\gcd(xu_i, w)$ for some i, for example see [4, Proposition 1.2.2]. Using the assumption $x \notin \operatorname{supp}(w)$, we conclude that x does not divide $\gcd(xu_i, w)$, as required.

(ii) It follows from (i) that every generator of $xI : (yu_i)$ is divisible by x. Moreover, we have $(x) = (xu_i) : (yu_i) \subseteq xI : (yu_i)$. This gives $xI : (yu_i) = (x)$. □

Now we give the main theorem of this section.

Theorem 3.5. *For any $n \geq 1$, $DI(P_n)$ has linear quotients.*

Proof. Set $I_n = DI(P_n)$. We show that the order of $G(I_n)$ described in Remark 3.3 is a linear quotient order. We proceed by applying induction on n. It is easy to verify the assertion for $1 \leq n \leq 5$ by following straightforward computations. Let $n > 5$ and for all $1 \leq k < n$ assume that I_k has linear quotients with the order as in Remark 3.3.

First, we show that $\mathbf{A_n} = x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}$ has linear quotients. By the induction hypothesis $\mathbf{A_{n-4}}$ has linear quotients because $G(I_{n-4}) = A_{n-4} \cup B_{n-4}$. Let $A_{n-4} = \{a_1, \dots, a_k\}$ where the generators are listed in the linear quotient order. We know that $x_{n-1}I_{n-3}$ and $x_{n-1}x_{n-2}\mathbf{A_{n-4}}$ have linear quotients because I_{n-3} and $\mathbf{A_{n-4}}$ have linear quotients. Moreover, for $i = 2, \dots, k$, we have

$$[x_{n-1}I_{n-3} + (x_{n-1}x_{n-2}a_1, \dots, x_{n-1}x_{n-2}a_{i-1})] : (x_{n-1}x_{n-2}a_i) =$$
$$x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i) + (x_{n-1}x_{n-2}a_1, \dots, x_{n-1}x_{n-2}a_{i-1}) : (x_{n-1}x_{n-2}a_i).$$

Therefore, we only need to show that $x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i)$ has linear quotients, for all $i = 1, \dots, k$. We claim that for all $i = 1, \dots, k$,

$$x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i) = (x_{n-3}, x_{n-4}). \tag{3.1}$$

Proof of claim: For $5 < n \leq 10$, the above claim can be verified with straightforward computation. The reason we let $n > 10$ in the following argument is to avoid non-positive indices.

Note that $x_{n-1}I_{n-3} : (x_{n-1}x_{n-2}a_i) = I_{n-3} : (x_{n-2}a_i)$. Using Theorem 3.2 we obtain

$$I_{n-3} = x_{n-4}I_{n-6} + x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-3}I_{n-5}$$

which gives

$$\begin{aligned} I_{n-3} : (x_{n-2}a_i) = x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) \\ + x_{n-3}I_{n-5} : (x_{n-2}a_i). \end{aligned} \tag{3.2}$$

Since $A_{n-4} = x_{n-5}G(I_{n-7}) \cup \ x_{n-5}x_{n-6}A_{n-8}$, we separate the discussion in the following two cases: $a_i \in x_{n-5}G(I_{n-7})$ or $a_i \in x_{n-5}x_{n-6}A_{n-8}$.

Case 1: Let $a_i \in x_{n-5}G(I_{n-7})$. From Lemma 3.4 and Theorem 3.2, we obtain

$$\begin{aligned} x_{n-3}I_{n-5} : (x_{n-2}a_i) &= (x_{n-3}\mathbf{A_{n-5}} + x_{n-3}x_{n-5}I_{n-7}) : (x_{n-2}a_i) \\ &= (x_{n-3}). \end{aligned}$$

Note that $x_{n-5}G(I_{n-7}) = x_{n-5}(A_{n-7} \cup B_{n-7})$. If $a_i \in x_{n-5}A_{n-7}$, then again from Lemma 3.4, we obtain

$$x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) = (x_{n-4}).$$

On the other hand, if $a_i \in x_{n-5}B_{n-7} = x_{n-5}x_{n-7}G(I_{n-9})$, then by using the expansion $I_{n-6} = x_{n-7}I_{n-9} + x_{n-7}x_{n-8}\mathbf{A_{n-10}} + \mathbf{B_{n-6}}$ obtained from Theorem 3.2, and as an application of Lemma 3.4, we have

$$\begin{aligned} x_{n-4}I_{n-6} : (x_{n-2}a_i) &= (x_{n-4}x_{n-7}I_{n-9} + x_{n-4}x_{n-7}x_{n-8}\mathbf{A_{n-10}} + x_{n-4}\mathbf{B_{n-6}}) : (x_{n-2}a_i) \\ &= (x_{n-4}). \end{aligned}$$

Then, again from Lemma 3.4 we obtain

$$x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) = (x_{n-4}).$$

Therefore, from (3.2) we conclude that $I_{n-3} : (x_{n-2}a_i) = (x_{n-3}, x_{n-4})$ and the claim holds.

Case 2: Let $a_i \in x_{n-5}x_{n-6}A_{n-8}$. Theorem 3.2 gives

$$\begin{aligned} I_{n-6} &= \mathbf{A_{n-6}} + \mathbf{B_{n-6}} = \mathbf{A_{n-6}} + x_{n-6}I_{n-8} \\ &= \mathbf{A_{n-6}} + x_{n-6}\mathbf{A_{n-8}} + x_{n-6}\mathbf{B_{n-8}}. \end{aligned}$$

Thanks to Lemma 3.4, we obtain

$$\begin{aligned} x_{n-4}I_{n-6} : (x_{n-2}a_i) &= (x_{n-4}\mathbf{A_{n-6}} + x_{n-4}x_{n-6}\mathbf{A_{n-8}} + x_{n-4}x_{n-6}\mathbf{B_{n-8}}) : (x_{n-2}a_i). \\ &= (x_{n-4}). \end{aligned}$$

Hence

$$\begin{aligned} I_{n-3} : (x_{n-2}a_i) &= x_{n-4}I_{n-6} : (x_{n-2}a_i) + x_{n-4}x_{n-5}\mathbf{A_{n-7}} : (x_{n-2}a_i) + x_{n-3}I_{n-5} : (x_{n-2}a_i) \\ &= (x_{n-4}) + x_{n-3}I_{n-5} : (x_{n-2}a_i). \end{aligned}$$

From Theorem 3.2, we have the expansion

$$\begin{aligned} I_{n-5} &= \mathbf{A_{n-5}} + \mathbf{B_{n-5}} \\ &= x_{n-6}I_{n-8} + x_{n-6}x_{n-7}\mathbf{A_{n-9}} + x_{n-5}I_{n-7} \\ &= x_{n-6}\mathbf{A_{n-8}} + x_{n-6}\mathbf{B_{n-8}} + x_{n-6}x_{n-7}\mathbf{A_{n-9}} + x_{n-5}I_{n-7}. \end{aligned}$$

Once again, as a direct application of Lemma 3.4, we obtain

$$x_{n-3}I_{n-5} : (x_{n-2}a_i) = (x_{n-3}).$$

This completes the proof of our claim.

Let $G(I_{n-2}) = \{v_1, \ldots, v_s\}$ where the generators are listed in the linear quotient order. Next, we show that $\mathbf{A_n} + (x_nv_1, \ldots, x_nv_{i-1}) : (x_nv_i)$ has linear quotients for all $i = 2, \ldots, s$.

Since I_{n-2} has linear quotients, it follows that x_nI_{n-2} also has linear quotients. Therefore, it is enough to show that $\mathbf{A_n} : (x_nv_i)$ has linear quotients for each $i = 1, \dots, s$. We claim that

$$\mathbf{A_n} : (x_nv_i) = (x_{n-1}). \tag{3.3}$$

Proof of claim: For $5 < n \leq 7$, the above claim can be verified with straightforward computation. The reason we let $n > 7$ in the following argument is to avoid the non-positive indices in the following text.

Since $G(I_{n-2}) = A_{n-2} \cup B_{n-2}$, we first consider the case when $v_i \in A_{n-2} = x_{n-3}G(I_{n-5}) \cup x_{n-3}x_{n-4}A_{n-6}$. After a repeated use of Theorem 3.2, we obtain

$$\begin{aligned} I_{n-3} &= x_{n-4}I_{n-6} + x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-3}I_{n-5} \\ &= x_{n-4}\mathbf{A_{n-6}} + x_{n-4}\mathbf{B_{n-6}} + x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-3}I_{n-5}. \end{aligned}$$

Above equality together with Lemma 3.4 gives $x_{n-1}I_{n-3} : (x_nv_i) = (x_{n-1})$. Therefore, in this case,

$$\begin{aligned} \mathbf{A_n} : (x_nv_i) &= (x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}) : (x_nv_i) \\ &= x_{n-1}I_{n-3} : (x_nv_i) + x_{n-1}x_{n-2}\mathbf{A_{n-4}} : (x_nv_i) \\ &= (x_{n-1}) \end{aligned}$$

as required. Next, let $v_i \in B_{n-2} = x_{n-2}G(I_{n-4}) = x_{n-2}A_{n-4} \cup x_{n-2}B_{n-4}$. If $v_i \in x_{n-2}A_{n-4}$, then Lemma 3.4 gives

$$\mathbf{A_n} : (x_nv_i) = [x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}] : (x_nv_i) = (x_{n-1}).$$

If $v_i \in x_{n-2}B_{n-4} = x_{n-2}x_{n-4}G(I_{n-6})$, then Lemma 3.4 gives

$$\begin{aligned} x_{n-1}I_{n-3} : (x_nv_i) &= [x_{n-1}x_{n-4}I_{n-6} + x_{n-1}x_{n-4}x_{n-5}\mathbf{A_{n-7}} + x_{n-1}\mathbf{B_{n-3}}] : (x_nv_i) \\ &= (x_{n-1}), \end{aligned}$$

and we again retrieve $\mathbf{A_n} : (x_nv_i) = [x_{n-1}I_{n-3} + x_{n-1}x_{n-2}\mathbf{A_{n-4}}] : (x_nv_i) = (x_{n-1})$. This completes the proof. □

Using Theorem 3.5, we retrieve the following result from [6, Theorem 2.8].

Corollary 3.6. *For any $n \geq 1$, $DI(P_n)$ is a componentwise linear ideal.*

Proof. By Theorem 3.5, $DI(P_n)$ has linear quotients. Thus, by [4, Theorem 8.2.15], $DI(P_n)$ is componentwise linear. □

4. Betti numbers of dominating ideals of path graphs

In this section, we give a recursive formula to compute the Betti numbers of dominating ideals of path graphs. To do this, we recall the following result of Sharifan and Varbaro from [8] which gives the Betti numbers, regularity and projective dimension of an ideal with linear quotients.

Theorem 4.1 ([8], Corollary 2.7)**.** *Let I be a monomial ideal with linear quotients with respect to $u_1, \dots, u_r$ where $G(I) = \{u_1, \dots, u_r\}$. Let n_p be the number of minimal generators of $(u_1, \dots, u_{p-1}) : u_p$ for $p = 1, \dots, r$. Then*

$$\beta_{i,i+j}(I) = \sum_{1 \leq p \leq r, \deg(u_p) = j} \binom{n_p}{i}, \qquad \beta_i(I) = \sum_{p=1}^{r} \binom{n_p}{i},$$

$$\operatorname{reg}(I) = \max\{\deg(u_p) : p = 1, \dots, r\},$$

$$\operatorname{proj\,dim}(I) = \max\{n_p : p = 1, \dots, r\}.$$

In [7, Theorem 2.6], authors computed the regularity and projective dimension of $NI(P_n)$. Using $NI(P_n)^\vee = DI(P_n)$ and invoking Terai's well-known result [9, Corollary 0.3] one can formulate the regularity and projective dimension of $DI(P_n)$. However, in the following result, we describe the regularity and projective dimension of $DI(P_n)$ in the terms of n as an application of Theorem 4.1 and Theorem 3.5.

Theorem 4.2. *For any $n \geq 2$, following hold.*

(1) $\operatorname{reg}(DI(P_n)) = \lceil \frac{n}{2} \rceil = \operatorname{proj}\dim(NI(P_n)) + 1$,
(2) $\operatorname{proj}\dim(DI(P_n)) = \lfloor \frac{n}{2} \rfloor = \operatorname{reg}(NI(P_n)) - 1$.

Proof. A well known result of Terai [9, Corollary 0.3] states that for any squarefree monomial ideal $\operatorname{reg}(I) = \operatorname{proj}\dim(I^\vee)+1$, and from [7, Lemma 2.2], we have $NI(P_n)^\vee = DI(P_n)$. Therefore, to prove the assertion, it is enough to compute regularity and projective dimension of $DI(P_n)$. Let $n \geq 2$ and $I_n = DI(P_n)$. It follows from Theorem 4.1 that the regularity of I_n is

$$\max\{|S| : S \text{ is a minimal dominating set of } P_n\}.$$

We first observe that for any minimal dominating set A of P_n, we have $|A| \leq \lceil \frac{n}{2} \rceil$. To verify this, assume $n \geq 4$, as the statement is trivially true for $1 \leq n \leq 3$. Write $n = 4q + r$ where $q \geq 1$ and $0 \leq r \leq 3$. For each $i = 0, \ldots, q-1$, define $A_i = \{x_{4i+1}, x_{4i+2}, x_{4i+3}, x_{4i+4}\}$. Additionally, if $r \geq 1$, let $B_r = \{x_{4q+k} : 1 \leq k \leq r\}$. Now, let A be a minimal dominating set of P_n. The minimality of A ensures that $|A \cap A_i| \leq 2$ for each i. If $r = 0$, then $|A| \leq n/2$, as required. For $r = 1$ or $r = 2$, we have $|A \cap B_r| \leq 1$, and for $r = 3$, $|A \cap B_r| \leq 2$, again due to the minimality of A. In all cases, it follows that $|A| \leq \lceil \frac{n}{2} \rceil$.

From above discussion, we conclude $\operatorname{reg}(I_n) \leq \lceil \frac{n}{2} \rceil$. On the other hand, it is easy to see that for any n, the set $\{x_i : i \text{ is odd and } i \leq n\}$ is a minimal dominating set of P_n. This gives us $\operatorname{reg}(I_n) = \lceil \frac{n}{2} \rceil$, as required.

To prove $\operatorname{proj}\dim(I_n) = \lfloor \frac{n}{2} \rfloor$, we apply the induction on n. For $2 \leq n \leq 6$, the equality can be verified using Theorem 4.1. For $n > 6$, following (3.1), (3.3) and the linear quotient order of $DI(P_n)$ given in Theorem 3.5, we obtain

$$\operatorname{proj}\dim(I_n) = \max\{\operatorname{proj}\dim(I_{n-3}), \operatorname{proj}\dim(\mathbf{A_{n-4}}) + 2, \operatorname{proj}\dim(I_{n-2}) + 1\}.$$

Moreover, using $\mathbf{A_{n-4}} \subset I_{n-4}$ and the induction hypothesis, we obtain $\operatorname{proj}\dim(\mathbf{A_{n-4}}) \leq \operatorname{proj}\dim(I_{n-4}) = \lfloor \frac{n-4}{2} \rfloor$, $\operatorname{proj}\dim(I_{n-3}) = \lfloor \frac{n-3}{2} \rfloor$, and $\operatorname{proj}\dim(I_{n-2}) = \lfloor \frac{n-2}{2} \rfloor$. This gives us the desired formula. □

Using Theorem 4.1, and the linear quotient order of $DI(P_n)$ from Theorem 3.5, first we list Betti numbers of $I_n = DI(P_n)$, for $n = 1, \ldots, 6$.

Table 1. Betti numbers of $I_n = DI(P_n)$

n	$\beta_0(I_n)$	$\beta_1(I_n)$	$\beta_2(I_n)$	$\beta_3(I_n)$
1	1	-	-	-
2	2	1	-	-
3	2	1	-	-
4	4	4	1	-
5	4	4	1	-
6	7	11	6	1

Now, we give recursive formulas for the total and graded Betti numbers of $DI(P_n)$, for $n > 6$. To simplify the notation in the subsequent text, we use the following definition.

Let J be a monomial ideal with linear quotients and $u_1, \ldots, u_s$ be the linear quotient order of the generators of J. We call the colon ideal $(u_1, \ldots, u_{k-1}) : u_k$ the *k-th colon* of J. It follows from Theorem 3.5 that I_n has linear quotients with respect to the order of generators given in Remark 3.3. For each n, we denote by $s_k^{(n)}$, the size of k-th colon of I_n.

Theorem 4.3. *Let $I_n = DI(P_n)$ with $n > 6$. Then*

$$\begin{aligned}\beta_i(I_n) = \beta_i(I_{n-3})& \\ &+ \beta_i(I_{n-2}) + \beta_{i-1}(I_{n-2}) \\ &+ \beta_i(I_{n-4}) + 2\beta_{i-1}(I_{n-4}) + \beta_{i-2}(I_{n-4}) \\ &- \beta_i(I_{n-6}) - 3\beta_{i-1}(I_{n-6}) - 3\beta_{i-2}(I_{n-6}) - \beta_{i-3}(I_{n-6}).\end{aligned}$$

Proof. Recall from Theorem 3.2 that $G(I_n) = A_n \cup B_n$, where

$$A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4}), \text{ and } B_n = x_nG(I_{n-2}).$$

Let $|A_n| = t$, and $G(I_{n-2}) = \{v_1, \ldots, v_s\}$. We recall the equality in (3.3) from the proof of Theorem 3.5 that states $\mathbf{A_n} : (x_nv_i) = (x_{n-1})$, for all $i = 1, \ldots, s$. Let $n \geq 3$ and $|G(I_n)| = r$. It follows from Theorem 3.5 that I_n has linear quotients with respect to the order of generators given in Remark 3.3. For each n, we denote by $s_k^{(n)}$, the size of k-th colon of I_n. Then

$$\begin{aligned}\beta_i(I_n) &= \beta_i(\mathbf{A_n}) + \sum_{k=t+1}^{r} \binom{s_k^{(n)}}{i} \\ &= \beta_i(\mathbf{A_n}) + \sum_{k=t+1}^{r} \binom{s_k^{(n-2)} + 1}{i} \quad \text{by using (3.3)} \\ &= \beta_i(\mathbf{A_n}) + \sum_{k=t+1}^{r} \left[\binom{s_k^{(n-2)}}{i} + \binom{s_k^{(n-2)}}{i-1}\right] \\ &= \beta_i(\mathbf{A_n}) + \beta_i(I_{n-2}) + \beta_{i-1}(I_{n-2}).\end{aligned}$$

Therefore

$$\beta_i(\mathbf{A_n}) = \beta_i(I_n) - \beta_i(I_{n-2}) - \beta_{i-1}(I_{n-2}). \tag{4.1}$$

On the other hand, for $n > 4$, using $A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4})$, and the equality (3.1) in Theorem 3.5, we obtain

$$\begin{aligned}\beta_i(\mathbf{A_n}) &= \beta_i(I_{n-3}) + \sum_{k=p+1}^{t} \binom{a_k^{(n-4)} + 2}{i} \\ &= \beta_i(I_{n-3}) + \sum_{k=p+1}^{t} \binom{a_k^{(n-4)}}{i-2} + 2\sum_{k=p+1}^{t} \binom{a_k^{(n-4)}}{i-1} + \sum_{k=p+1}^{t} \binom{a_k^{(n-4)}}{i}\end{aligned}$$

where $p = |G(I_{n-3})|$, and we denote the size of k-th colon of $\mathbf{A_n}$ by $a_k^{(n)}$. This gives

$$\beta_i(\mathbf{A_n}) = \beta_i(I_{n-3}) + \beta_{i-2}(\mathbf{A_{n-4}}) + 2\beta_{i-1}(\mathbf{A_{n-4}}) + \beta_i(\mathbf{A_{n-4}}). \tag{4.2}$$

For $n > 6$, combining (4.1) together with (4.2) gives us the required recursive formula of total Betti numbers of I_n. $\square$

Next we give a recursive formula to compute graded Betti numbers of $DI(P_n)$, for $n > 6$.

Theorem 4.4. *Let $I_n = DI(P_n)$ and $n > 6$. Then*

$$\begin{aligned}\beta_{i,i+j}(I_n) = {} & \beta_{i,i+j-1}(I_{n-3}) \\ & + \beta_{i,i+j-1}(I_{n-2}) + \beta_{i-1,i+j-2}(I_{n-2}) \\ & + \beta_{i,i+j-2}(I_{n-4}) + 2\beta_{i-1,i+j-3}(I_{n-4}) + \beta_{i-2,i+j-4}(I_{n-4}) \\ & - \beta_{i,i+j-3}(I_{n-6}) - 3\beta_{i-1,i+j-4}(I_{n-6}) - 3\beta_{i-2,i+j-5}(I_{n-6}) - \beta_{i-3,i+j-6}(I_{n-6}).\end{aligned}$$

Proof. We proceed as in the case of total Betti numbers and follow the same notations given in Theorem 4.3. Let $G(I_n) = \{u_1, \dots, u_r\}$ and $|A_n| = t$. Theorem 4.1 together with the linear quotient order given in Remark 3.3 gives

$$\begin{aligned}\beta_{i,i+j}(I_n) &= \beta_{i,i+j}(\mathbf{A_n}) + \sum_{\substack{k=t+1 \\ \deg u_k = j}}^{r} \binom{s_k^{(n)}}{i} \\ &= \beta_{i,i+j}(\mathbf{A_n}) + \sum_{\substack{k=t+1 \\ \deg u_k = j-1}}^{r} \binom{s_k^{(n-2)} + 1}{i} \quad \text{by using (3.3) and Theorem 3.2} \\ &= \beta_{i,i+j}(\mathbf{A_n}) + \sum_{\substack{k=t+1 \\ \deg u_k = j-1}}^{r} \left[\binom{s_k^{(n-2)}}{i} + \binom{s_k^{(n-2)}}{i-1}\right] \\ &= \beta_{i,i+j}(\mathbf{A_n}) + \beta_{i,i+j-1}(I_{n-2}) + \beta_{i-1,i+j-2}(I_{n-2}).\end{aligned}$$

Therefore

$$\beta_{i,i+j}(\mathbf{A_n}) = \beta_{i,i+j}(I_n) - \beta_{i,i+j-1}(I_{n-2}) - \beta_{i-1,i-1+j-1}(I_{n-2}). \tag{4.3}$$

On the other hand, for $n > 4$, using $A_n = x_{n-1}G(I_{n-3}) \cup x_{n-1}(x_{n-2}A_{n-4})$ and the equality (3.1) in Theorem 3.5, we obtain

$$\begin{aligned}\beta_{i,i+j}(\mathbf{A_n}) &= \beta_{i,i+j-1}(I_{n-3}) + \sum_{\substack{k=p+1 \\ \deg u_k = j-2}}^{t} \binom{a_k^{(n-4)} + 2}{i} \\ &= \beta_{i,i+j-1}(I_{n-3}) + \sum_{\substack{k=p+1 \\ \deg u_k = j-2}}^{t} \left[\binom{a_k^{(n-4)}}{i-2} + 2\binom{a_k^{(n-4)}}{i-1} + \binom{a_k^{(n-4)}}{i}\right]\end{aligned}$$

where $p = |G(I_{n-3})|$. This gives

$$\begin{aligned}\beta_{i,i+j}(\mathbf{A_n}) = {} & \beta_{i,i+j-1}(I_{n-3}) + \beta_{i-2,i+j-4}(\mathbf{A_{n-4}}) + 2\beta_{i-1,i+j-3}(\mathbf{A_{n-4}}) \\ & + \beta_{i,i+j-2}(\mathbf{A_{n-4}}).\end{aligned}$$

For $n > 6$, combining (4.3) together with the above equality gives us the required recursive formula of the total Betti numbers of dominating ideals of path graphs. □

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