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Research Article

Zero Intersection Graph of Annihilator Ideals of Modules

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Abstract

This paper aims to associate a new graph to nonzero unital modules over commutative rings. Let R be a commutative ring having a nonzero identity and M be a nonzero unital R-module. The zero intersection graph of annihilator ideals of R-module M, denoted by $\mathfrak{C}_R(M)$, is a simple (undirected) graph whose vertex set $M^\star = M - \{0\}$, and two distinct vertices m and m' are adjacent if $ann_R(m) \cap ann_R(m') = (0)$. We investigate the conditions under which $\mathfrak{C}_R(M)$ is a star graph, bipartite graph, complete graph, edgeless graph. Furthermore, we characterize certain classes of modules and rings such as torsion-free modules, torsion modules, semisimple modules, quasi-regular rings, and modules satisfying Property T in terms of their graphical properties.

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1. Introduction

Throughout the paper, all rings under consideration are assumed to be commutative with nonzero identity and all modules are nonzero unital. In particular, R will always denote such a ring and M will denote such an R-module. Graph theory is a crucial branch of mathematics studying graphs that are mathematical structures used to model pairwise relations between discrete objects and has many applications in other areas such as game theory, commutative algebra, designs of networks, chemistry, medicine, etc. For more information on applications of graph theory, the reader may consult [40]. First, in 1988, I. Beck initiated the study of graphs on commutative rings. In his paper [10], the author considered the coloring of a given commutative ring R. Afterwards, Anderson and Livingston introduced the zero divisor graph $\Gamma(R)$ of R and studied the connections between graphical properties of $\Gamma(R)$ and algebraic properties of R. The *zero divisor graph* $\Gamma(R)$ of R is a simple (undirected) graph whose vertex set is $zd^*(R)$, the set of nonzero

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zero divisors of R, where any two distinct vertices x and y are adjacent if $xy = 0$. For many years, many researchers associated certain graphs to commutative rings/modules and they characterized certain algebraic properties of these algebraic structures in terms of some combinatorial properties of the given graphs. See, for example, [1], [3], [8], [11–13], [19] [34–39] and [43]. The purpose of the paper is to associate a new graph $\mathfrak{C}_R(M)$ to a unital R-module M and investigate the connections between the graphical properties of $\mathfrak{C}_R(M)$ and the algebraic properties of M. For the sake of completeness, we will give some notions and notations which will be followed in the sequel.

Let L be a submodule of M, K be a nonempty subset of M and I be a nonempty subset of R. The residuals of L by K and I are defined as follows:

$$(L :_R K) := \{x \in R : xK \subseteq L\}$$
$$(L :_M I) := \{m \in M : Im \subseteq L\}.$$

In particular, if $L = (0)$ is the zero submodule and $K = \{m\}$ $(I = \{a\})$ is the singleton, where $m \in M$ $(a \in R)$, then we use $ann_R(m)$ $(ann_M(a))$ to denote $((0) :_R \{m\})$ $(((0) :_M \{a\}))$. Recall from [41] that an R-module M is said to be a *faithful module* if $ann_R(M) = (0 : M)$ is the zero ideal (0). An R-module M is said to be a *multiplication module* if every submodule N of M has the form $N = ann_R(M/N)M$, where $ann_R(M/N) = (N :_R M)$ [9]. For more details on multiplication modules, we refer [2], [18] to the reader. An element $m \in M$ $(a \in R)$ is said to be a *torsion element* (*zero divisor* on M) if $ann_R(m) \neq (0)$ $((ann_M(a) \neq (0))$. The set of all torsion elements (zero divisors on M) is denoted by $T(M)$ $(z(M))$. Here, we note that if we consider the ring R as an R-module, then $T(R) = z(R) = zd(R)$. If $T(M) = 0$, then M is said to be a *torsion-free module [41]*. Also, M is called a *torsion module* if $T(M) = M$. Otherwise, we say that M is a *non-torsion module*, that is, there exists an element $m \in M$ such that $ann_R(m) = (0)$. An R-module M is called a *torsionable module* if $T(M)$ is a submodule of M [4]. Note that all torsion-free modules are torsionable but the converse is not true in general. For instance, $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}_n$ is not a torsion-free, while it is a torsionable (non-torsion) module, where $n \geq 2$.

A commutative ring R is said to be a *von Neumann regular ring* if its every principal ideal (a) of R is generated by an idempotent element $e \in R$ [42]. The notion of von Neumann regular ring and its generalizations have drawn considerable interest and studied by many authors. See, for example, [17], [21–26] and [29]. Also, a ring R is said to be a *quasi-regular ring* if its total quotient ring $q(R)$ is a von neumann regular ring. It is well known that a ring R is a quasi-regular ring if and only if R is reduced and for each $a \in R$, $ann_R(ann_R(a)) = ann_R(b)$ for some $b \in R$ [17]. A commutative ring R is said to satisfy *Property A*, if for each finitely generated ideal I of R contained in $zd(R)$, then there exists $0 \neq x \in R$ such that $xI = 0$ [20]. We know that (by [20]) every quasi-regular ring also satisfies Property A but the converse is not true in general. The class of rings satisfying Property A is quite wide including integral domains, quasi-regular rings, polynomial rings, etc. (See, [20]). Property A has been extended to modules in two different ways and studied in several papers. See, for example, [5], [6], [15] and [30]. Recall from [5] that an R-module M is said to satisfy Property T (Strong Property n-T) if for each finitely generated submodule N of M (each subset B of M with $|B| \leq n$) contained in $T(M)$, then there exists $0 \neq x \in R$ such that $xN = (0)$ $(xB = (0))$.

Let G be a graph and $V(G)$ be its set of vertices. For every two distinct vertices u and v, by $u - v$, we mean u and v are adjacent. A subgraph G' is said to be an *induced subgraph* if G' contains all the edges $x - y$ in G for each $x, y \in V(G')$ [16]. Let $n \geq 0$ be an integer. A graph G is said to be an *edgeless (n-empty)* graph if G has some vertices (n-vertices) but no edges. In particular, the 0-empty graph is called an *empty graph.* A vertex $u \in V(G)$ that is adjacent to every other vertex is called a *universal vertex of G*. A graph G is said to be a *star graph* if there is a universal vertex $u \in V(G)$ and any other two vertices different from u are not adjacent. A star graph with n vertices is denoted by S_n. Let G be a graph and $x_1, x_2, \dots, x_{n+1} \in V(G)$ be distinct vertices. Then $x_1 - x_2 - \cdots - x_{n+1}$ is said to be a *path* from x_1 to x_{n+1} of length n if $x_i - x_{i+1}$ for each $i = 1, 2, \dots, n$. A graph G is said to be a connected graph if, for every pair of vertices x and y, there is a path between x and y. A path $x_1 - x_2 - \cdots - x_n$ is said to be a *cycle* if all vertices are distinct except $x_1 = x_n$. If every two distinct vertices x and y are adjacent in G, then G is called a *complete graph [16]*. Also, a complete graph with n vertices is denoted by K_n. An isolated point of a graph G is a vertex $x \in V(G)$ such that there is no edge between x and y for every $y \in V(G)$. Also, for any $x \in V(G)$, $\deg(x)$ denotes the degree of x. $\Delta(G)$ is the maximum degree of G, that is, $\Delta(G) = \sup\{\deg(x) : x \in V(G)\}$, and similarly, $\delta(G)$ is the minimum degree of G. Note that G has no isolated point if and only if $\delta(G) \geq 1$. Let $x, y \in V(G)$ and $d(x, y)$ denote the length of the shortest path (if there exists) from x to y. If there is no path between x and y, we say that $d(x, y) = \infty$. The *diameter* of a graph G is defined as $diam(G) = \sup\{d(x, y) : x, y \in V(G)\}$. Also, the *girth of a graph* G, denoted by $girth(G)$, is the length of the shortest cycle (if there exists) in G. If there is no cycle in G, then we assume that $girth(G) = \infty$. A graph G with the vertex set $V(G)$ is called a *bipartite graph* if the vertex set $V(G)$ can be partitioned into two disjoint subsets V_1, V_2 of $V(G)$ such that every edge has endpoints from different sets V_1 and V_2. Note that a nontrivial graph G is bipartite if and only if it has no odd cycle [14, Theorem 1.12].

In this paper, we associate a graph $\mathfrak{C}_R(M)$ to an R-module M which we call the zero intersection graph of annihilator ideals of M, and study its induced subgraph $\mathfrak{C}_R(T(M))$. The zero intersection graph of annihilator ideals $\mathfrak{C}_R(M)$ is a simple (undirected) graph whose vertex set $M^\star = M - \{0\}$, where two distinct vertices m and m' are adjacent if $ann_R(m) \cap ann_R(m') = (0)$. Also, $\mathfrak{C}_R(T(M))$ is an induced subgraph of $\mathfrak{C}_R(M)$ with the vertex set $V(\mathfrak{C}_R(T(M))) = T(M) - \{0\}$. Among other things in this paper, we show that $\mathfrak{C}_R(M)$ and $\mathfrak{C}_R(T(M))$ have different graphical properties since $\mathfrak{C}_R(M)$ is always connected with $diam(\mathfrak{C}_R(M)) \leq 2$, where M is a non-torsion module, while $\mathfrak{C}_R(T(M))$ may be an edgeless graph (See, Proposition 2.1 and Example 2.2). In Theorem 2.4, we showed that either $girth(\mathfrak{C}_R(M)) = \infty$ or $girth(\mathfrak{C}_R(M)) = 3$. Also, we determine the conditions under which $\mathfrak{C}_R(M)$ is a bipartite graph (See, Theorem 2.6). In Theorem 2.7, we give a test to detect universal vertices in $\mathfrak{C}_R(M)$. Furthermore, we characterize certain classes of rings/modules such as quasi-regular rings, torsion-free modules, torsion modules, semisimple modules, modules satisfying Property T in terms of the conditions under which $\mathfrak{C}_R(M)$ and $\mathfrak{C}_R(T(M))$ is an edgeless graph, star graph and complete graph (See, Proposition 2.1, Theorem 2.3, Theorem 2.9, Theorem 2.11, Proposition 2.13).

2. Zero intersection graph of annihilator ideals

Recall that an ideal P of R is said to be an *irreducible ideal* if whenever $P = I \cap J$ for some ideals I and J of R, then $P = I$ or $P = J$ [41]. A ring R is said to be a *valuation ring* if the lattice $\mathcal{L}(R)$ of all ideals of R is totally ordered by inclusion, that is, $I \subseteq J$ or $J \subseteq I$ for each $I, J \in \mathcal{L}(R)$ [27]. Note that in a valuation ring, every ideal is irreducible. Darani and Hojjat extended the concept of irreducible ideals to 2-irreducible ideals. An ideal I of R is called *2-irreducible* if whenever $I = J \cap K \cap L$ for some ideals J, K, L of R, then either $I = K \cap L$ or $I = J \cap K$ or $I = J \cap L$ [33]. Note that every irreducible ideal is a 2-irreducible ideal but the converse is not true in general. For instance, $I = 6\mathbb{Z}$ is a 2-irreducible ideal which is not irreducible in the ring $\mathbb{Z}$ of integers.

Now, we investigate the conditions under which $\mathfrak{C}_R(T(M))$ or $\mathfrak{C}_R(M)$ is an edgeless graph.

Proposition 2.1. *Let M be an R-module and $0 \neq m, m' \in M$. Then the following statements are satisfied.*

(a) m and m' are adjacent in $\mathfrak{C}_R(M)$ if and only if rm and rm' are adjacent in $\mathfrak{C}_R(M)$ for each $r \in R - z(M)$.

(b) If $ann_R(m) \subseteq ann_R(m')$ or $ann_R(m') \subseteq ann_R(m)$ for some $m, m' \in T(M)$, then m and m' can not be adjacent in $\mathfrak{C}_R(T(M))$. In particular, if R is a valuation (chain) ring, then $\mathfrak{C}_R(T(M))$ is an edgeless graph.

(c) If zero ideal is an irreducible ideal, then $\mathfrak{C}_R(T(M))$ is an edgeless graph. In particular, if R is an integral domain, then $\mathfrak{C}_R(T(M))$ is an edgeless graph.

(d) Assume that $|V(\mathfrak{C}_R(M))| \geq 2$. If $\mathfrak{C}_R(M)$ is an edgeless graph, then M is a torsion module, that is, $M = T(M)$.

(e) If M is not a faithful module, then $\mathfrak{C}_R(M)$ is an edgeless graph.

(f) M is a torsion-free module if and only if $\mathfrak{C}_R(T(M))$ is an empty graph.

(g) $\mathfrak{C}_R(T(M))$ is an edgeless graph if and only if M satisfies strong Property 2-T.

(h) Suppose that M is a non torsion module. Then $\mathfrak{C}_R(M)$ is a connected graph with $diam(\mathfrak{C}_R(M)) \leq 2$.

Proof. (a) : Let $m - m'$ in $\mathfrak{C}_R(M)$. It is clear that rm and rm' are two distinct nonzero elements of M for each $r \in R - z(M)$. Since m and m' are adjacent, $ann_R(m) \cap ann_R(m') = (0)$. As $r \in R - z(M)$, we have $ann_R(rm) \cap ann_R(rm') = ann_R(m) \cap ann_R(m') = (0)$. Then rm and rm' are adjacent in $\mathfrak{C}_R(M)$. Suppose that rm and rm' are adjacent in $\mathfrak{C}_R(M)$. Then $ann_R(rm) \cap ann_R(rm') = (0)$. As $ann_R(m) \cap ann_R(m') \subseteq ann_R(rm) \cap ann_R(rm')$, we get m and m' are adjacent in $\mathfrak{C}_R(M)$.

(b) : Without loss of generality, we may assume that $ann_R(m) \subseteq ann_R(m')$ for some nonzero elements $m, m' \in T(M)$. If m and m' are adjacent in $\mathfrak{C}_R(T(M))$, we have $ann_R(m) \cap ann_R(m') = ann_R(m) = (0)$, which implies that $m \notin T(M)$, a contradiction. Thus m and m' are not adjacent in $\mathfrak{C}_R(T(M))$. The rest is clear.

(c) : Suppose that zero ideal is an irreducible ideal of R. Assume that m and m' are adjacent in $\mathfrak{C}_R(T(M))$. Then we have $ann_R(m) \cap ann_R(m') = (0)$. Since zero ideal is an irreducible ideal, we get either $ann_R(m) = (0)$ or $ann_R(m') = (0)$, which implies either $m \notin T(M)$ or $m' \notin T(M)$. This is a contradiction. Therefore, $\mathfrak{C}_R(T(M))$ is an edgeless graph. The rest follows from the fact that every prime ideal is irreducible.

(d) : Let $\mathfrak{C}_R(M)$ be an edgeless graph. Suppose to the contrary that M is not a torsion module. Then there exists $m \in M$ such that $ann_R(m) = (0)$. Since $|V(\mathfrak{C}_R(M))| \geq 2$, choose an element $m' \in M$ such that $m' \in M - \{0, m\}$. Then we have $ann_R(m) \cap ann_R(m') = (0)$, which implies that m and $m'$are adjacent in $\mathfrak{C}_R(M)$. This is a contradiction. Thus we have $M = T(M)$.

(e) : Suppose that M is not a faithful module. Then there exists a nonzero element $x \in R$ such that $xM = 0$. Let m and m' be two distinct nonzero elements of M. Then $xm = xm' = 0$ and so $x \in ann_R(m) \cap ann_R(m')$. Hence, m and m' are not adjacent in $\mathfrak{C}_R(M)$.

(f) : Suppose that M is a torsion-free module. Then we have $T(M) - \{0\} = \emptyset$ and so $\mathfrak{C}_R(T(M))$ is an empty graph. Let $\mathfrak{C}_R(T(M))$ be an empty graph. Then we have $T(M) - \{0\} = \emptyset$, which implies that $T(M) = \{0\}$. Thus, M is a torsion-free module.

(g) : Suppose that $\mathfrak{C}_R(T(M))$ is an edgeless graph. Let $B = \{m, m'\} \subseteq T(M)$ with $|B| = 2$. We may assume that m and m' are nonzero. Since $\mathfrak{C}_R(T(M))$ is an edgeless graph, we have $ann_R(m) \cap ann_R(m') \neq (0)$, which implies that $ann_R(B) \neq (0)$. Therefore, M satisfies strong Property 2-T. Let M satisfy strong Property 2-T. Assume that m and m' are adjacent in $\mathfrak{C}_R(T(M))$. Then $m, m' \in T(M)$ and pick $B = \{m, m'\} \subseteq T(M)$. Since M satisfies strong Property 2-T, we have $ann_R(B) = ann_R(m) \cap ann_R(m') \neq (0)$, which implies that m and m' are not adjacent in $\mathfrak{C}_R(T(M))$. Therefore, $\mathfrak{C}_R(T(M))$ is an edgeless graph.

(h) : Suppose that M is a non-torsion module. Let $m \notin T(M)$. Since $ann_R(m) = (0)$, m is a universal vertex. If M is a torsion-free module, then it is clear that $diam(\mathfrak{C}_R(M)) = 1$. Now, choose $m', m'' \in T(M)$. Note that $m' - m - m''$ is a path so we have $d(m', m'') \leq 2$, which completes the proof. □

Note that $\mathfrak{C}_R(T(M))$ and $\mathfrak{C}_R(M)$ have different graphical properties. The following example illustrates this.

Example 2.2. Consider the $R = \mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_3$. Then note that $T(M) = \{(0, \overline{0}), (0, \overline{1}), (0, \overline{2})\}$ and $M - T(M) = \{(x, \overline{k}) : x \neq 0 \text{ and } k = 0, 1, 2\}$. Since M is non torsion module, $\mathfrak{C}_R(M)$ is a connected graph with $diam(\mathfrak{C}_R(M)) \leq 2$. On the other hand, note that $ann_R((0, \overline{1})) = ann_R((0, \overline{2})) = 3\mathbb{Z}$, it is clear that $(0, \overline{1})$ and $(0, \overline{2})$ are not adjacent. Therefore, $\mathfrak{C}_R(T(M))$ is an edgeless graph.

The following first figure illustrates Example 2.2. Note that in the following figure $(z, \overline{k})$ and $(x, \overline{y})$ denote the arbitrary infinite elements of $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}_3$.

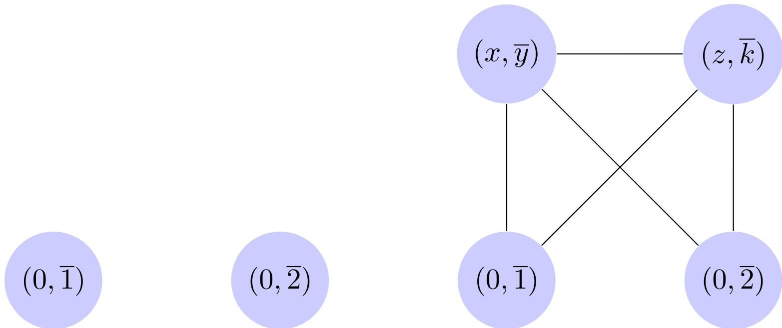

Figure 1. $\mathfrak{C}_{\mathbb{Z}}(T(\mathbb{Z} \times \mathbb{Z}_3))$ vs $\mathfrak{C}_{\mathbb{Z}}(\mathbb{Z} \times \mathbb{Z}_3)$

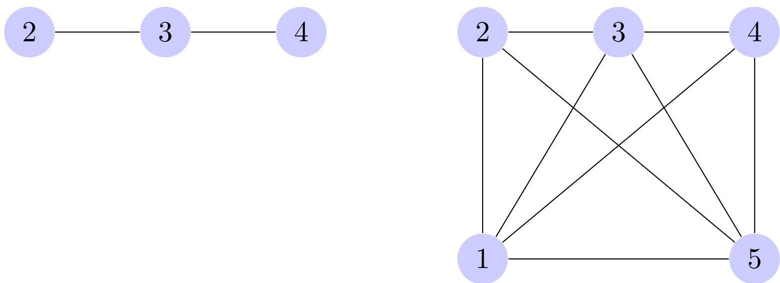

Figure 2. $\mathfrak{C}_{\mathbb{Z}_6}(T(\mathbb{Z}_6))$ vs $\mathfrak{C}_{\mathbb{Z}_6}(\mathbb{Z}_6)$

Theorem 2.3. *Suppose that zero ideal is a 2-irreducible ideal of R. Then the following statements are equivalent.*

(i) M is an R-torsionable module satisfying property (T).

(ii) $\mathfrak{C}_R(T(M))$ *is an edgeless graph.*

Proof. $(i) \Rightarrow (ii)$: Assume that M is an R-torsionable module satisfying property (T). Let $m_1, m_2 \in T(M)$. Since M is R-torsionable, $T(M)$ is a submodule and so $N = Rm_1 + Rm_2 \subseteq T(M)$. By Propoerty (T), we get $ann_R(N) = ann_R(m_1) \cap ann_R(m_2) \neq (0)$ and so $\mathfrak{C}_R(T(M))$ is an edgeless graph.

$(ii) \Rightarrow (i)$: Suppose that $\mathfrak{C}_R(T(M))$ is an edgeless graph. Let $m \in T(M)$. Then for each $a \in R$, we have $am \in T(M)$. Now, take $0 \neq m, m' \in T(M)$. Since $\mathfrak{C}_R(T(M))$ is an edgeless graph, we have $ann_R(m) \cap ann_R(m') \neq (0)$, which implies that $m + m' \in T(M)$. Thus, M is an R-torsionable module. Now, assume that N is a finitely generated submodule with $N \subseteq T(M)$. Then there exists $m_1, m_2, ..., m_n \in M$ such that $N = \sum_{i=1}^{n} Rm_i \subseteq T(M)$. Since $\mathfrak{C}_R(T(M))$ is edgeless, we get $ann(m_i) \cap ann(m_j) \neq (0)$ for all $1 \leq i, j \leq n$. Assume that $ann_R(N) = (0)$. Then we have $\bigcap_{i=1}^{n} ann_R(m_i) = (0)$. As (0) is 2-irreducible, we conclude that there exist distinct $i_1, i_2 \in \{1, 2, ..., n\}$ such that $ann_R(m_{i_1}) \cap ann_R(m_{i_2}) = (0)$, a contradiction. So that $ann_R(N) \neq (0)$ and thus M satisfies property (T). □

Theorem 2.4. *Let M be an R-module with* $|V(\mathfrak{C}_R(M))| > 2$*. The following statements are satisfied.*

(i) If M is a non torsion module, then $girth(\mathfrak{C}_R(M)) = 3$*.*

(ii) If M is a torsion module, then $girth(\mathfrak{C}_R(M)) = \infty$ *or* $girth(\mathfrak{C}_R(M)) = 3$*.*

Proof. (i) : Suppose that M is a non torsion module. Then there exists $m \in M$ such that $ann(m) = 0$. Now, we have two cases. **Case 1:** Let $char(R) \neq 2$. This gives $m \neq -m$ since $ann(m) = 0$. As $|V(\mathfrak{C}_R(M))| > 2$, choose another vertex $m^* \in M - \{m, -m\}$. Then we conclude that $m - (-m) - m^* - m$ is a triangle. **Case 2:** Let $char(R) = 2$. Then $m^* = -m^*$ for all $m^* \in M$. Choose $0 \neq m^* \in M - \{m\}$. Thus we conclude that $m - m^* - (m + m^*) - m$ is a triangle. Hence, we have $girth(\mathfrak{C}_R(M)) = 3$.

(ii) : Suppose that M is a torsion module. We may assume that $girth(\mathfrak{C}_R(M)) \neq \infty$. Then $\mathfrak{C}_R(M)$ can not be edgeless graph. Then we can choose an edge $m - m^*$ in $\mathfrak{C}_R(M)$. If $m + m^* = 0$, then $ann(m^*) = ann(-m) = ann(m)$. This gives $ann(m) = ann(m) \cap ann(m^*) = 0$ and thus M is a non torsion module, a contradiction. Then we have $m + m^* \neq 0$. Also one can easily check that $m - m^* - (m + m^*) - m$ is a triangle. In this case, $girth(\mathfrak{C}_R(M)) = 3$. □

Corollary 2.5. *Let M be an R-module with* $|V(\mathfrak{C}_R(M))| > 2$*. Then* $girth(\mathfrak{C}_R(M)) = \infty$ *or* $girth(\mathfrak{C}_R(M)) = 3$*.*

Proof. Follows from Theorem 2.4. □

Theorem 2.6. *Let M be an R-module with* $|V(\mathfrak{C}_R(M))| > 1$*. The following statements are equivalent.*

(i) $\mathfrak{C}_R(M)$ *is a bipartite graph.*

(ii) $\mathfrak{C}_R(M)$ *is an edgeless graph or* $\mathfrak{C}_R(M) \cong K_2$*.*

(iii) $girth(\mathfrak{C}_R(M)) = \infty$*.*

Proof. $(i) \Rightarrow (ii)$: Let $\mathfrak{C}_R(M)$ be a bipartite graph. Now, we will show that $\mathfrak{C}_R(M)$ is an edgeless graph or $\mathfrak{C}_R(M) \cong K_2$. If $|V(\mathfrak{C}_R(M))| = 2$, then the claim is trivial. So we assume that $|V(\mathfrak{C}_R(M))| > 2$. Since $\mathfrak{C}_R(M)$ is bipartite, it can not contain a triangle. Then by the proof of Theorem 2.4, $\mathfrak{C}_R(M)$ is an edgeless graph.

$(ii) \Rightarrow (iii)$: It is clear.

$(iii) \Rightarrow (i)$: By the assumption, there is no odd cycle in $\mathfrak{C}_R(M)$. The rest follows from [14, Theorem 1.12]. □

Now, we are ready to give a test to detect universal vertices in $\mathfrak{C}_R(M)$.

Theorem 2.7. *Let M be an R-module. An element $0 \neq m \in M$ is a universal vertex in $\mathfrak{C}_R(M)$ if and only if one of the following conditions holds.*
(a) $ann_R(m) = 0$.
(b) $T(M) = Rm = \{0, m\}$.
(c) There exists an idempotent element $0, 1 \neq e \in R$ such that $ann_R(m) = (1 - e)R$, $ann_R(m') = eR = \{0, e\}$ and $m + m' \notin T(M)$ for every $m' \in T(M) - \{0, m\}$.

Proof. $(\Rightarrow)$: Suppose that $0 \neq m$ is a universal vertex in $\mathfrak{C}_R(M)$ and $ann(m) \neq 0$. Let $a \notin ann_R(m)$. Then, $am \neq 0$. If $am \neq m$, then we have $am - m$ in $\mathfrak{C}_R(M)$. This implies that $ann_R(m) \cap ann_R(am) = ann_R(m) = (0)$, which is a contradiction. Thus, $am = m$, which implies that $Rm = \{0, m\}$. If $T(M) = Rm$, then we are done. So assume that $T(M) \neq Rm$. Now, choose an element $m' \in T(M) - \{0, m\}$. Then we conclude that $ann_R(m) \cap ann_R(m') = 0$. Also, it is easy to see that $R/ann_R(m) \cong \mathbb{Z}_2$ so that $ann_R(m)$ is a maximal ideal of R. This implies that $ann_R(m) + ann_R(m') = R$ since $ann_R(m') \nsubseteq ann_R(m)$. Then, by the Chinese Remainder Theorem, we get $R \cong R/ann_R(m) \times R/ann_R(m')$. Now, we will show that $ann_R(m') = eR = \{0, e\}$ for some idempotent element $0, 1 \neq e \in R$. Let $0 \neq e \in ann_R(m')$. Then $e \notin ann_R(m)$. This implies that $em = m$ and so $(1 - e)m = 0$. Then we have $e(1-e) \in ann_R(m) \cap ann_R(m') = (0)$, which implies that $e = e^2$. Now, take $y \in ann_R(m')$. Then $y(1 - e) \in ann_R(m) \cap ann_R(m') = 0$ implying that $y = ey \in eR$ and so $ann_R(m') = eR$. Also, note that $(1 - e)R \subseteq ann_R(m)$. Let $0 \neq t \in ann_R(m)$. Then we have $et \in ann_R(m) \cap ann_R(m') = (0)$. Thus we conclude that $t = t - et = t(1 - e) \in (1 - e)R$. Hence, we get $ann_R(m) = (1-e)R$. As $R/ann_R(m) = R/(1-e)R \cong eR \cong \mathbb{Z}_2$, we have $eR = ann_R(m') = \{0, e\}$. Now, we will show that $m + m' \notin T(M)$. Suppose that $m + m' \in T(M)$. Then, $m + m'$ is nonzero and $m + m' \notin \{m, m'\}$. As $ann_R(m)$ is a maximal ideal and $ann_R(m + m') \nsubseteq ann_R(m)$, we get $ann_R(m + m') + ann_R(m) = R$. Then we conclude that $ann_R(m) + [ann_R(m') \cap ann_R(m + m')] = R$. Since $ann_R(m') \cap ann_R(m + m') \subseteq ann_R(m)$, we have $ann_R(m) = R$, which implies that $m = 0$, a contradiction. Therefore, $m + m' \notin T(M)$. Now, choose $m'' \in T(M) - \{0, m\}$. A similar argument shows that there exists an idempotent element $0, 1 \neq e' \in R$ such that $ann_R(m'') = e'R = \{0, e'\}$ and $ann_R(m) = (1-e')R = (1-e)R$. Since $m - m''$ in $\mathfrak{C}_R(M)$, we have $ann_R(m) \cap ann_R(m'') = (1 - e)R \cap e'R = (1 - e)e'R = (0)$, which implies that $e' = ee' \in eR = \{0, e\}$. Since $e' \neq 0$, we have $e' = e$, which completes the proof.

$(\Leftarrow)$: **First Case:** Suppose that $ann_R(m) = 0$. Then clearly, m is a universal vertex in $\mathfrak{C}_R(M)$.

Second Case: Suppose that $T(M) = Rm = \{0, m\}$. Take an element $m' \in M^\star - \{m\}$. Then we have $ann_R(m') = 0$ and so $ann_R(m) \cap ann_R(m') = (0)$, which implies that $m - m'$ in $\mathfrak{C}_R(M)$.

Third Case: Suppose that (c) holds. Let $m' \in M^\star - \{m\}$. If $m' \in T(M)$, then by (c), we have $ann(m') = eR$ and $ann_R(m) = (1 - e)R$ for some idempotent element $0, 1 \neq e \in R$. Then we have $ann_R(m) \cap ann_R(m') = e(1 - e)R = (0)$, which implies that $m - m'$ in $\mathfrak{C}_R(M)$. If $m' \notin T(M)$, then clearly we have $m - m'$ in $\mathfrak{C}_R(M)$. □

Recall that a ring R is said to be a *decomposable ring* if $R \cong R_1 \times R_2$ for some nontrivial rings R_1 and R_2. Otherwise, we say that R is *indecomposable*. It is clear that R is indecomposable if and only if all idempotents are the only $0, 1$. Recall from [41] that a commutative ring R is said to be a *quasi-local* if it has a unique maximal ideal. Note that all quasi-local rings are indecomposable. But the converse is not true in general. For instance, let k be a field. Then $k[X]$ is indecomposable but not a quasi-local ring. As a consequence of Theorem 2.7, we have the following explicit result.

Corollary 2.8. *Let M be an R-module. The following statements are satisfied.*

(i) Suppose that R is an indecomposable ring. Then $0 \neq m \in M$ is a universal vertex in $\mathfrak{C}_R(M)$ if and only if $ann_R(m) = (0)$ or $T(M) = Rm = \{0, m\}$.

(ii) Suppose that R is a quasi-local ring. Then $0 \neq m \in M$ is a universal vertex in $\mathfrak{C}_R(M)$ if and only if $ann_R(m) = (0)$ or $T(M) = Rm = \{0, m\}$.

(iii) Suppose that M is a torsionable R-module. Then $0 \neq m \in M$ is a universal vertex in $\mathfrak{C}_R(M)$ if and only if $ann_R(m) = (0)$ or $T(M) = Rm = \{0, m\}$.

(iv) Suppose that R is an integral domain. Then $0 \neq m \in M$ is a universal vertex in $\mathfrak{C}_R(M)$ if and only if $ann_R(m) = (0)$ or $T(M) = Rm = \{0, m\}$.

Now, we are ready to determine when $\mathfrak{C}_R(M)$ is a star graph.

Theorem 2.9. *Let M be an R-module. Then the following statements are equivalent.*

(i) $\mathfrak{C}_R(M)$ is a star graph.

(ii) $\mathfrak{C}_R(M) \cong S_1$ or $\mathfrak{C}_R(M) \cong S_2$.

(iii) $M = Rm = \{0, m\}$ or $M = \{0, m, -m\}$ with $ann(m) = 0$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $\mathfrak{C}_R(M)$ is a star graph. Then there exists $0 \neq m \in M$ is a universal vertex in $\mathfrak{C}_R(M)$. Assume that $ann(m) = 0$. First, we will show that $|V(\mathfrak{C}_R(M))| \leq 2$. Let $|V(\mathfrak{C}_R(M))| \geq 3$. Let $m_1, m_2 \in V(\mathfrak{C}_R(M)) - \{m\}$. If $ann_R(m_1) = (0)$ or $ann_R(m_2) = (0)$, then note that $m - m_1 - m_2 - m$ is a triangle, which is a contradiction. Thus $m_1, m_2 \in T(M) - \{m\}$. Choose $x \in R$. Assume that $x \neq 0, 1$. Then note that $xm \neq m$ and $(1 - x)m \neq m$ since $ann_R(m) = (0)$. Let $y \in ann_R(xm) \cap ann_R((1 - x)m)$. Then we have $yx = 0 = y(1 - x)$ since $ann_R(m) = (0)$. This implies that $y = yx = 0$ so that $ann_R(xm) \cap ann_R((1 - x)m) = (0)$. If $xm \neq (1 - x)m$, then $m - xm - (1 - x)m - m$ is a triangle, which is a contradiction. Thus, we have $xm = (1 - x)m$, which implies that $2x = 1$, that is, 2 is a unit of R. Now, take a unit element $a \in R$. If $am \neq m$, then $m - am - m_1 - m$ is a triangle, which is again a contradiction. Thus $am = m$ and so $a = 1$. Since 2 is a unit of R, we conclude that $2 = 1$, again a contradiction. Therefore, $x = 0$ or $x = 1$, that is, $R = \mathbb{Z}_2$. However, this implies that $ann_R(m_1) = R = ann_R(m_2)$, again a contradiction. Therefore, $|V(\mathfrak{C}_R(M))| \leq 2$. In this case, $\mathfrak{C}_R(M) \cong S_1$ or $\mathfrak{C}_R(M) \cong S_2$.

Now, assume that $ann(m) \neq 0$. Since m is a universal vertex in $\mathfrak{C}_R(M)$, by Theorem 2.7, we have two cases. **Case 1:** $T(M) = Rm = \{0, m\}$. We will show that $|M - T(M)| \leq 1$. Take two distinct elements $m_1, m_2 \in M - T(M)$. Then note that $m_1 - m - m_2 - m_1$ is a triangle, which is a contradiction. Thus either $M - T(M) = \emptyset$ or $M - T(M) = \{m'\}$ for $m' \in M$. If $M - T(M) = \emptyset$, then $M = \{0, m\} = Rm = T(M)$ and so $\mathfrak{C}_R(M) \cong S_1$. So assume that $M - T(M) = \{m'\}$ for $m' \in M$. In this case, we have $m' = -m$ which implies that $ann(m) = ann(m') = 0$ which is a contradiction. Thus, $M = \{0, m\} = Rm = T(M)$ and $\mathfrak{C}_R(M) \cong S_1$.

Case 2: Assume that there exists an element $m' \in T(M) - \{0, m\}$. In this case, by Theorem 2.7, there exists an idempotent element $0, 1 \neq e \in R$ such that $ann_R(m) = (1 - e)R$, $ann_R(m^*) = eR = \{0, e\}$ and $m + m^* \notin T(M)$ for every $m^* \in T(M) - \{0, m\}$. Then m and m' are adjacent. In this case, $M - T(M) = \emptyset$. Otherwise, we would have a triangle. Indeed, if $y \in M - T(M)$, $y - m - m' - y$ is a triangle, which is a contradiction. Thus $M - T(M) = \emptyset$, that is, M is a torsion module. On the other hand, by Theorem 2.7, $m + m' \notin T(M)$, that is, $M - T(M) \neq \emptyset$, again a contradiction. Thus the second case is impossible, that is $\mathfrak{C}_R(M) \cong S_1$.

$(ii) \Leftrightarrow (iii)$: It is clear.

$(ii) \Rightarrow (i)$: It is straightforward. □

Let M be an R-module. Recall from [7] that M is said to be a *simple module* if its only proper submodule is the zero submodule. Also, M is said to be a *semisimple module* if it is a direct sum of simple submodules. In [31], the authors introduced and studied the concept of torsion graph for modules as follows. The torsion graph $\Gamma_R(M)$ of M is a simple graph whose vertices are non-zero torsion elements of M and two different elements x, y are adjacent if and only if $ann_R(x) \cap ann_R(y) \neq 0$. Clearly, the torsion graph $\Gamma_R(M)$ of M is the complement graph of $C_R(T(M))$. By using this fact and some results in [31], [32], we obtain the following proposition.

Proposition 2.10. *Let M be a multiplication R-module. The following statements are satisfied.*

(1) If $C_R(T(M))$ has a universal vertex, then $M = M_1 \oplus M_2$ is a faithful R-module where M_1 and M_2 are two submodules of M such that M_1 has only two elements. Especially, if M is finite then M_2 is simple.

(2) $C_R(T(M))$ is a complete graph if and only if $M \simeq M_1 \oplus M_2$ with $|M_1| \leq 2$ and $|M_2| \leq 2$.

Proof. (1) Follows from [31, Theorem 2.6].

(2) Follows from [32, Corollary 2.3]. □

Now, we determine the conditions under which $\mathfrak{C}_R(M)$ is a complete graph (even if M is multiplication or not).

Theorem 2.11. *Let M be an R-module. Then $\mathfrak{C}_R(M)$ is a complete graph if and only of one of the following conditions holds.*

(i) M is a torsion-free module.

(ii) $T(M) = \{0, m\} = Rm$ for some $0 \neq m \in M$. In this case, M is an R-torsionable module.

(iii) There exist $m, m' \in T(M)$ and a non trivial idempotent element $e \in R$ such that $ann_R(m) = (1-e)R$, $ann_R(m') = eR$ and $T(M) = \{0, m, m'\}$, $M = Rm \oplus Rm'$, where $Rm = \{0, m\}$ and $Rm' = \{0, m'\}$. In this case, M is semisimple and $\mathfrak{C}_R(M)$ is a triangle.

Proof. $(\Rightarrow)$: Suppose that $\mathfrak{C}_R(M)$ is a complete graph. Assume that M is not a torsion-free module. **First case:** Assume that $T(M)$ has one nonzero element, that is, $T(M) = \{0, m\}$. Let $x \in R$. Then $xm \in T(M)$, which implies that $xm = 0$ or $xm = m$, that is, $Rm = \{0, m\} = T(M)$. **Second Case:** Assume that $T(M)$ has at least two nonzero element. Choose $m, m' \in T(M) - \{0\}$. Since m is a universal vertex, by Theorem 2.7, we have $ann_R(m) = (1-e)R$, $ann_R(m') = eR$ and $m + m' \notin T(M)$ for some nontrivial idempotent $e \in R$. Now, choose another element $m^\star \notin T(M)$. Since $ann_R(em^\star) = (1-e)R$, we have $em^\star = m$. Otherwise, $em^\star$ and m are not adjacent, which is a contradiction. Similarly, we have $(1-e)m^\star = m'$. This implies that $m^\star = em^\star + (1-e)m^\star = m + m'$, that is, $M - T(M) = \{m + m'\}$. Now, choose $m'' \in T(M) - \{0, m\}$. Since m is universal vertex, by Theorem 2.7, we have $m + m'' \notin T(M)$, which implies that $m + m'' = m + m'$, that is, $m'' = m'$. Thus, we have $T(M) = \{0, m, m'\}$ and so $M = \{0, m, m', m + m'\}$. As m, m' are universal vertices, one can easily show that $Rm = \{0, m\}$ and $Rm' = \{0, m'\}$, which implies that $M = Rm \oplus Rm'$. Therefore, M is semisimple and $\mathfrak{C}_R(M)$ is a triangle with the cycle $m - (m + m') - m' - m$.

$(\Leftarrow)$: **First Case:** Suppose that M is a torsion-free module. Then for each $0 \neq m \in M$, we have $ann_R(m) = (0)$. In this case, m is a universal vertex so that $\mathfrak{C}_R(M)$ is a complete graph. **Second Case:** Now, assume that $T(M) = \{0, m\} = Rm$ for some $0 \neq m \in M$. Choose, $0 \neq m' \in M - \{m\}$. Then by assumption, $ann_R(m') = (0)$, which implies that $m - m'$ in $\mathfrak{C}_R(M)$. Therefore, $\mathfrak{C}_R(M)$ is a complete graph. **Third Case:** Assume that (iii) holds. Then $m - (m + m') - m' - m$ is a cycle, that is, $\mathfrak{C}_R(M)$ is a triangle. □

Theorem 2.12. *(i) Let M be a module over an indecomposable ring R. Then $\mathfrak{C}_R(M)$ is a complete graph if and only if M is a torsion-free module or M is a torsionable module with $T(M) = \{0, m\} = Rm$ for some $0 \neq m \in M$.*

(ii) Let M be a module over a quasi-local ring. Then $\mathfrak{C}_R(M)$ is a complete graph if and only if M is a torsion-free module or M is a torsionable module with $T(M) = \{0, m\} = Rm$ for some $0 \neq m \in M$.

Proof. (i) : Follows from Theorem 2.11.
(ii) : Follows from (i). □

Recall from [17, Proposition 2.5] that a ring R is a quasi-regular ring if and only if for every $x \in R$, there exists $y \in R$ such that $xy = 0$ and $x + y$ is a regular element (non zero divisor) of R. Recently, Jayaram et al. extended quasi-regular rings to modules in their paper [22]. An R-module M is called a *weak quasi regular* if for each $m \in M$, there exists $r \in R$ such that $ann_M(ann_R(m)) = ann_M(r)$. Also, recall from [28] that an R-module M is said to be a reduced module, if whenever $a^2m = 0$ for some $a \in R$ and $m \in M$, then $am = 0$.

Proposition 2.13. *Let M be a non-torsion reduced R-module. If $\mathfrak{C}_R(T(M))$ is a complete graph and R satisfies property (A), then R is a quasi regular ring.*

Proof. We will show that for any $a \in R$, there exists $b \in R$ such that $ab = 0$ and $a+b$ is a regular element. Assume that $0 \neq a$ is a zero divisor of R. Then there exists $0 \neq b \in R$ such that $ab = 0$. Since M is non-torsion, there exists $m \in M$ such that $ann_R(m) = (0)$. Assume that $am = bm$. Then $a^2m = abm = 0$, by the fact that M is reduced module, we have $am = 0$ so that $a = 0$, a contradiction. Thus we have $am \neq bm$. Since $ann_R(m) = (0)$, it is clear that $ann_R(am) = ann_R(a) \neq (0)$ and $ann_R(bm) = ann_R(b) \neq (0)$ so that $am, bm \in T(M) - \{0\}$. Since $\mathfrak{C}_R(T(M))$ is a complete graph, we deduce $ann_R(am) \cap ann_R(bm) = ann_R(a) \cap ann_R(b) = (0)$. This implies that $ann_R(Ra + Rb) = (0)$. Since R satisfies Property (A), $Ra + Rb$ has a regular element so that $xa + yb$ is a regular element for some $x, y \in R$. Now, we will show that $a + yb$ is regular. Suppose not. There exists $0 \neq t \in R$ such that $t(a + yb) = 0$ and so $ta = -tyb$. Since M is reduced non-torsion module, R is reduced ring. As R is reduced ring and $ta^2 = -tyab = 0$ we have $ta = 0$ and this yields $tyb = 0$. Then we have $t(xa + yb) = x(ta) + tyb = 0$, a contradiction. So that $a + yb$ is a regular element and also $a(yb) = y(ab) = 0$. Thus R is a quasi regular ring. □

If M is a non-torsion module, then it is clear that $\mathfrak{C}_R(M)$ is a connected graph, so $\mathfrak{C}_R(M)$ has no isolated point. However, in this case, there may be an isolated point in $\mathfrak{C}_R(T(M))$ (See, Example 2.2). Now, we investigate the condition under which $\mathfrak{C}_R(T(M))$ has no isolated point.

Proposition 2.14. *Let M be a reduced multiplication non-torsion module. If M is a weak quasi regular module, then $\mathfrak{C}_R(T(M))$ has no isolated point.*

Proof. Suppose that M is a weak quasi regular module and $0 \neq m \in T(M)$. Then there exists $a \in R$ such that $ann_M(ann_R(m)) = ann_M(a)$. Then $ann_R(m) = ann_R(ann_M(a))$. Take $m^\star \in M - T(M)$ and put $m' = am^\star$. Note that $ann_R(m') = ann_R(a)$. If $ann_R(a) = (0)$, then $(ann_M(a) : M) = ann_R(a) = (0)$ and so $ann_M(a) = (ann_M(a) : M)M = (0)$ implying that $ann_R(m) = R$, a contradiction. Thus we get $ann_R(m') \neq (0)$, i.e., $0 \neq m' \in T(M)$. If $m = am^\star$, then $am = a^2m^\star = 0$ and so $am^\star = m = 0$, a contradiction. Thus $m \neq am^\star$. Now we will show that $ann_R(m) \cap ann_R(m') = (0)$. First note that $ann_R(m) \cap ann_R(m') = ann_R(ann_M(a)) \cap ann_R(a)$. Take $x \in ann_R(ann_M(a)) \cap ann_R(a)$. Then we have $xa = 0$ and also $xann_M(a) = (0)$. Since M is multiplication, we have $xann_M(a) = x(ann_M(a) : M)M = xann_R(a)M = (0)$. This implies that $x^2M = (0)$. Since $x^2m^\star = 0$, we have $xm^\star = 0$ and so $x = 0$. Therefore, m and m' are adjacent. □

Theorem 2.15. *Suppose that M is an R-module and $\mathfrak{C}_R(M)$ has no isolated point with $\Delta(\mathfrak{C}_R(M)) < \infty$. Then M satisfies the ascending chain condition on cyclic submodules.*

Proof. Assume that M is a non-torsion module, that is, $M - T(M) \neq \emptyset$. Then there exists $m \in M$ such that $ann_R(m) = (0)$, which implies that m is a universal vertex. Since $\Delta(\mathfrak{C}_R(M)) < \infty$, we have M is finite so that M is a Noetherian module, which completes the proof.

Now, assume that M is a torsion module, that is, $M = T(M)$.

Take an ascending chain of cylic submodules of M as follows:

$$Rm_1 \subseteq Rm_2 \subseteq \cdots \subseteq Rm_k \subseteq \cdots$$

This implies that $ann_R(m_1) \supseteq ann_R(m_2) \supseteq \cdots \supseteq ann_R(m_k) \supseteq \cdots$. Since $\mathfrak{C}_R(M)$ has no isolated point, there exists $m \in T(M) = M$ such that $m - m_1$ in $\mathfrak{C}_R(M)$, that is, $ann_R(m) \cap ann_R(m_1) = (0)$. This implies that $m - m_i$ since $ann_R(m_i) \subseteq ann_R(m_1)$. As $\Delta(\mathfrak{C}_R(M)) < \infty$, we have $\deg(m) < \infty$, which implies that $Rm_k = Rm_{k+1} = \cdots$ for some $k \in \mathbb{N}$ which completes the proof. □

Proposition 2.16. *Let M be an R-module and $\mathfrak{C}_R(T(M))$ be a complete graph. Then, either $T(M) = \{0, m\}$ for some $0 \neq m \in M$ or $Jac(R) = (0)$.*

Proof. Suppose that $\mathfrak{C}_R(T(M))$ is a complete graph. Let $0 \neq m \in T(M)$ and $x \in Jac(R)$. Assume that $x \notin ann_R(m)$. Then $xm \neq 0$. Since $\mathfrak{C}_R(T(M))$ is a complete graph, $m - xm$ in $\mathfrak{C}_R(T(M))$. This implies that $ann_R(m) \cap ann_R(xm) = ann_R(m) = (0)$, which is a contradiction. Thus, we have $x \in ann_R(m)$, which implies that $Jac(R) \subseteq ann_R(m)$. This implies that $Jac(R) \subseteq \bigcap_{m \in T(M)} ann_R(m)$. Suppose that $T(M)$ has at least two nonzero elements. Then $\bigcap_{m \in T(M)} ann_R(m) = (0)$, which implies that $Jac(R) = (0)$. □

In Theorem 2.11, we determine when $\mathfrak{C}_R(M)$ is a complete graph. Now, we investigate the completeness of $\mathfrak{C}_R(T(M))$.

Theorem 2.17. *Let M be an R-module. Then $\mathfrak{C}_R(T(M))$ is a complete graph if and only if one of the following conditions holds.*

(i) $T(M) = \{0, m\}$ for some $0 \neq m \in M$.

(ii) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $T(M) = \{0, m, m'\}$, $M = Rm \oplus Rm'$, where $Rm = \{0, m\}$ and $Rm' = \{0, m'\}$. In this case, M is semisimple, $\mathfrak{C}_R(T(M)) \cong K_2$ and $\mathfrak{C}_R(M) \cong K_3$.

Proof. $(\Rightarrow)$: Suppose that $\mathfrak{C}_R(T(M))$ is a complete graph. Assume that $T(M)$ has at least two nonzero elements. Choose $0 \neq m, m' \in T(M)$. Since $\mathfrak{C}_R(T(M))$ is a complete graph, $m - m'$ in $\mathfrak{C}_R(T(M))$, which implies that $ann_R(m) \cap ann_R(m') = (0)$. Also note that $R/ann(m) \cong \mathbb{Z}_2 \cong R/ann(m')$. Thus, $ann_R(m)$ and $ann_R(m')$ are maximal ideals of R. This implies that $ann_R(m) + ann_R(m') = R$, by the Chinese Remainder Theorem, $R \cong R/ann_R(m) \times R/ann_R(m') \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, one can easily see that $Rm = \{0, m\}$, $Rm' = \{0, m'\}$. Now, let $R = \{0, 1, e, 1-e\}$, where e is a nontrivial idempotent. Without loss of generality, we may assume that $ann_R(m) = eR$ and $ann_R(m') = (1-e)R$. Note that $m + m' \notin \{0, m, m'\}$. If $m + m' \in T(M)$, by similar argument in the proof of Theorem 2.7, we have $m = 0$, a contradiction. Thus $m + m' \notin T(M)$. Choose, $0 \neq m'' \in T(M)$. Note that $ann_R(m'') = eR$ or $ann_R(m'') = (1-e)R$ since all the ideals of R are $(0), eR, (1-e)R$ and R. Without loss of generality, we may assume that $ann_R(m'') = eR$. If $m'' \neq m$, m and m'' can not be adjacent, which is a contradiction. Thus $m = m''$ and so $T(M) = \{0, m, m'\}$. Now, choose an element $m^\star \in M - T(M)$. Then $ann_R(em^\star) = (1-e)R$ and $ann_R((1-e)m^\star) = eR$. Since $\mathfrak{C}_R(T(M))$ is a complete graph, we have $em^\star = m'$ and $(1-e)m^\star = m$, which implies that $m^\star = em^\star + (1-e)m^\star = m + m'$. Therefore, $M - T(M) = \{m + m'\}$ and so $M = \{0, m, m', m + m'\}$. Thus $M = Rm \oplus Rm'$. The rest is easy.

$(\Leftarrow)$: Suppose that $T(M) = \{0, m\}$. Then $\mathfrak{C}_R(T(M)) \cong K_1$. Now assume that (ii) holds. Then $\mathfrak{C}_R(T(M)) \cong K_2$ is a complete graph. □

Corollary 2.18. *Suppose that M is not a torsion-free module. Then $\mathfrak{C}_R(T(M))$ is a complete graph if and only if $\mathfrak{C}_R(M)$ is a complete graph.*

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