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Research Article

Existence and stability analysis for Φ-Caputo generalized proportional fractional differential Langevin equation

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Abstract

This paper aims to investigate the existence, uniqueness and stability results for a new class of Φ-Caputo generalized proportional fractional (GPF) differential Langevin equation. We present and discuss some of the characteristics of the generalized proportional fractional derivative which can be considered as generalization and modification of the fractional conformable derivative by generating Φ-Caputo generalized proportional fractional derivatives involving exponential functions in it's kernel also this kind of fractional derivative generalize the well-known fractional derivatives, for different values of function Φ. Utilizing Krasnoselskii's fixed point theorem and the Banach contraction principle, we established results on existence and uniqueness, we also examine various types of stability, including Ulam-Hyers stability and generalized Ulam-Hyers stability. As an application, we provide an example to illustrate our theoretical result.

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1. Introduction

Fractional differential equations (FDEs) have recently captured the attention of many mathematicians, because it can effectively represent a variety of scientific phenomena, and has been proven to be effective in physics, mechanics, biology, chemistry, and control theory , and other domains for example, see [7,9,11,14,15,23–31,33,35–37,40–44,46,49–51,54,55].

In 1908, Paul Langevin introduced the Langevin equation in the form $m\frac{d^2w}{d\tau^2} = -\nu\frac{dw}{d\tau} + \eta(\tau)$ where, $\frac{dw}{d\tau}$ denotes the particle's velocity, m is its mass, and $\eta(\tau)$ represents a noise term accounting for collisions with fluid molecules [13]. To eliminate the noise term, mathematicians employed fractional-order differential equations, making it crucial to study Langevin equations through fractional derivatives. In this context, Kubo (1966) introduced the generalized Langevin equation by incorporating a fractional memory kernel to

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represent memory and fractal characteristics [39]. With the rapid advancement of fractional calculus in the early 1990s, Mainardi et al. introduced the fractional Langevin equation [47,48] ; see also [17,18] for additional details.

There are several methods for defining fractional integrals and derivatives, with the Riemann-Liouville and Caputo definitions being the most widely recognized. In [6], Almeida introduced a generalization known as the Φ-Caputo fractional derivative. For further details on Φ-Caputo and Caputo fractional derivatives, readers may refer to [3,5,6,10,12,34,45,53].

In [19], Jarad et al. as the modification of the conformable derivatives [2,22], the authors introduced a new type of fractional derivative, known as the generalized proportional fractional GPF derivative. Anderson et al. [8] were able to handle with the fact that the fractional conformable derivative does not tends to the original function where the order χ tends to 0, by defining the proportional derivative of order χ by

$$D_\eta^\chi \mathfrak{h}(\eta) = \xi_1(\chi,\eta)\mathfrak{h}(\eta) + \xi_2(\chi,\eta)\mathfrak{h}'(\eta),$$

where $\xi_1, \xi_2 : [0,1]\times\mathbb{R} \longrightarrow [0,\infty)$ are continuous functions such that, for all $\eta \in \mathbb{R}$,

$$\lim_{\chi\to 0^+} \xi_1(\chi,\eta) = 1,\ \lim_{\chi\to 0^+} \xi_2(\chi,\eta) = 0,\ \lim_{\chi\to 1^-} \xi_1(\chi,\eta) = 0,\ \lim_{\chi\to 1^-} \xi_2(\chi,\eta) = 1,$$

and $\xi_1(\chi,\eta) \neq 0, \xi_2(\chi,\eta) \neq 0, \chi \in [0,1]$, by this modifications, the new proportional derivative tends to the initial function when χ tends to 0.
In the previous few decades, authors are interested to study this new fractional derivative, in [4], Ahmed et al. examined the existence and uniqueness of solutions for the fractional Langevin equation involving generalized Liouville-Caputo fractional derivatives. By applying Krasnoselskiis fixed point theorem and the Banach fixed point theorem, they established the intended results.

Inspired by the aforementioned works, we build on their ideas in this paper to examine the existence results for problems of this form

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi;\Phi}\Big({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi;\Phi} w(\tau) + \nu w(\tau)\Big) = \mathfrak{h}(\tau, w(\tau)),\ \tau \in \Lambda := [\gamma,\delta], \\ w(\gamma) = 0\,,\ \ w(\delta) = \displaystyle\sum_{i=1}^{n} \iota_i \mathfrak{I}_{\gamma^+}^{\beta_i,\chi;\Phi} w(\kappa_i). \end{cases} \tag{1.1}$$

Where ${}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi;\Phi}$ and ${}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi;\Phi}$ are the Φ-Caputo GPF derivative of order σ, $0<\sigma\le 1$ and ϖ, $0<\varpi\le 1$, respectively. $\mathfrak{I}_{\gamma^+}^{\beta_i,\chi;\Phi}$ is the GPF integral of order $\beta_i > 0$, $\chi \in (0,1]$, $\gamma \geq 0$, $\nu, \iota_i \in \mathbb{R}$, $i = 1,...,n$, $\gamma < \kappa_1 < ... < \kappa_n < \delta$ and $\mathfrak{h} \in \mathfrak{C}(\Lambda \times \mathbb{R}, \mathbb{R})$.

The originality of this work lies in examining a novel and challenging type of fractional derivative, known as the Φ-Caputo GPF derivative. This derivative generalizes several well-known fractional derivatives for various choices of the function Φ, such as

- When $\Phi(\tau) = \tau$, problem (1.1) simplifies to the Caputo GPF derivative.
- When $\Phi(\tau) = log(\tau)$, problem (1.1) simplifies to Caputo-Hadamard GPF derivative.
- When $\Phi(\tau) = \tau^\rho$, problem (1.1) simplifies to Caputo-Katugampola GPF derivative.

The structure of this paper is as follows: Section 2 reviews key notations, definitions, and lemmas from fractional calculus that are essential to our study. In Section 3, we establish existence results for problem (1.1) using Krasnoselskiis fixed-point theorem, and we address the uniqueness result via Banachs contraction principle. Section 4 examines the UlamHyers stability and generalized UlamHyers stability of solutions for problem (1.1). Finally, an example is presented to illustrate the main results.

2. Preliminaries

$C(\Lambda,\mathbb{R})$ denote the Banach space of all continuous functions from Λ into $\mathbb{R}$ with the norm defined by $\|\mathfrak{h}\| = sup_{\tau\in\Lambda}\{|\mathfrak{h}(\tau)|\}$. We denote by $C^n(\Lambda,\mathbb{R})$ the n-times absolutely continuous functions given by $C^n(\Lambda,\mathbb{R}) = \left\{\mathfrak{h} : \Lambda \longrightarrow \mathbb{R}\ ;\ \mathfrak{h}^{(n-1)} \in C(\Lambda,\mathbb{R})\right\}$. $\mathcal{B}_\rho$ denote the closed ball centered at 0 with radius ρ. We denote by $L^1(\Lambda,\mathbb{R})$ the space of Lebesgue integrable real-valued functions on Λ equipped with the norm $\|\mathfrak{h}\|_{L^1} = \int_\Lambda |\mathfrak{h}(\tau)|d\tau$.

Definition 2.1 ([1, 21]). For $\chi \in (0,1]$. Let $\xi_1,\xi_2 : [0,1]\times\mathbb{R} \longrightarrow [0,\infty)$ be continuous functions such that, for all $\tau \in \mathbb{R}$,

$$\lim_{\chi\to0^+}\xi_1(\chi,\tau) = 1,\ \lim_{\chi\to0^+}\xi_2(\chi,\tau) = 0,\ \lim_{\chi\to1^-}\xi_1(\chi,\tau) = 0,\ \lim_{\chi\to1^-}\xi_2(\chi,\tau) = 1,$$

and $\xi_1(\chi,\tau) \neq 0, \xi_2(\chi,\tau) \neq 0, \chi \in [0,1]$. Let Φ be a strictly increasing continuous function, then the proportional derivative of order χ of $\mathfrak{h}$ with respect to Φ is defined as follows:

$$D^{\chi;\Phi}\mathfrak{h}(\tau) = \xi_1(\chi,\tau)\mathfrak{h}(\tau) + \xi_2(\chi,\tau)\frac{\mathfrak{h}'(\tau)}{\Phi'(\tau)}, \tag{2.1}$$

by setting $\xi_1(\chi,\tau) = \chi - 1$ and $\xi_2(\chi,\tau) = \chi$, (2.1) becomes

$$D^{\chi;\Phi}\mathfrak{h}(\tau) = (1-\chi)\mathfrak{h}(\tau) + \chi\frac{\mathfrak{h}'(\tau)}{\Phi'(\tau)}. \tag{2.2}$$

Definition 2.2 ([1, 20, 21]). For $\chi \in (0,1]$, $\sigma \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$ and $\Phi \in C([\gamma,\delta],\mathbb{R})$ with $\Phi' > 0$. The $\Phi-$ GPF integral of order σ of a function $\mathfrak{h} \in L^1([\gamma,\delta],\mathbb{R})$ with respect to Φ is defined as follows:

$$\mathfrak{I}^{\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{h}(\tau) = \frac{1}{\chi^\sigma\Gamma(\sigma)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\sigma-1}\mathfrak{h}(s)ds, \tag{2.3}$$

where $\Gamma(.)$ is the gamma function.

Definition 2.3 ([20, 21]). For $\chi \in (0,1]$, $\sigma \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$ and $\Phi \in C^n([\gamma,\delta],\mathbb{R})$ with $\Phi'(\tau) > 0$. The Φ-Caputo GPF derivative of order σ of a function $\mathfrak{h} \in C^n([\gamma,\delta],\mathbb{R})$ with respect to Φ is defined as follows:

$$^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{h}(\tau) = \frac{1}{\chi^{n-\sigma}\Gamma(n-\sigma)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{n-\sigma-1}D^{n,\chi;\Phi}\mathfrak{h}(s)ds, \tag{2.4}$$

where $n-1 < \sigma < n$, $n = [\mathcal{R}e(\sigma)] + 1$ where $[\mathcal{R}e(\sigma)]$ is the integer part of $\mathcal{R}e(\sigma)$, and $(D^{n,\chi;\Phi}\mathfrak{h})(\tau) = (D^{\chi;\Phi}\mathfrak{h}(\tau))^n$ with $D^{\chi;\Phi}\mathfrak{h}(\tau) = (1-\chi)\mathfrak{h}(\tau) + \chi\frac{\mathfrak{h}'(\tau)}{\Phi'(\tau)}$.

Lemma 2.4 ([20, 21]). *For $\chi \in (0,1]$, $\sigma,\varpi \in \mathbb{C}$ with $\mathcal{R}e(\sigma) > 0$ and $\mathcal{R}e(\varpi) > 0$. If $\mathfrak{h} \in C([\gamma,\delta],\mathbb{R})$ then we have*

$$\mathfrak{I}^{\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{I}^{\varpi,\chi;\Phi}_{\gamma^+}\mathfrak{h}(\tau) = \mathfrak{I}^{\sigma+\varpi,\chi;\Phi}_{\gamma^+}\mathfrak{h}(\tau), \tau > \gamma. \tag{2.5}$$

Lemma 2.5 ([20, 21]). *Let $\chi \in (0,1]$, $\sigma > 0$, $\upsilon > 0$ and $\tau \in \Lambda$. Then*

$$(i)\left(\mathfrak{I}^{\sigma,\chi;\Phi}_{\gamma^+}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}(\Phi(\tau)-\Phi(\gamma))^{\upsilon-1}\right)(\tau) = \frac{\Gamma(\upsilon)}{\chi^\sigma\Gamma(\upsilon+\sigma)}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}(\Phi(\tau)-\Phi(\gamma))^{\upsilon+\sigma-1}.$$

$$(ii)\left({}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi;\Phi}_{\gamma^+}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}(\Phi(\tau)-\Phi(\gamma))^{\upsilon-1}\right)(\tau) = \frac{\chi^\sigma\Gamma(\upsilon)}{\Gamma(\upsilon-\sigma)}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}(\Phi(\tau)-\Phi(\gamma))^{\upsilon-\sigma-1}.$$

Lemma 2.6 ([20,21]). *Let $\chi \in (0,1]$, $n \in \mathbb{N}$, $\mathfrak{h} \in L^1([\gamma,\delta],\mathbb{R})$ and $\mathfrak{I}^{\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{h} \in AC^n([\gamma,\delta],\mathbb{R})$. Then*

$$\mathfrak{I}^{\sigma,\chi;\Phi}_{\gamma^+}({}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{h})(\tau) = \mathfrak{h}(\tau) - e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}\sum_{k=0}^{n-1}(\mathfrak{I}^{k-\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{h})(\gamma)\frac{(\Phi(\tau)-\Phi(\gamma))^{\sigma-k}}{\chi^{\sigma-k}\Gamma(\sigma-k+1)}. \quad (2.6)$$

Theorem 2.7 (Krasnoselskiis fixed point theorem, [38]). *Let $\mathcal{B}$ be a nonempty bounded closed convex subset of a Banach space $\mathcal{X}$.*
Let $\mathcal{K}_1$, $\mathcal{K}_2 : \mathcal{B} \longrightarrow \mathcal{X}$, be two continuous operators satisfying:

(i)- *$\mathcal{K}_1 v + \mathcal{K}_2 z \in \mathcal{B}$, whenever $v, z \in \mathcal{B}$,*
(ii)- *$\mathcal{K}_1$ is compact and continuous,*
(iii)- *$\mathcal{K}_2$ is a contraction mapping.*

Then, there exists a fixed point $w \in \mathcal{B}$ such that $\mathcal{K}_1 w + \mathcal{K}_2 w = w$.

Theorem 2.8 (Banachs fixed point theorem, [16]). *Let $\mathcal{X}$ be a Banach space, C a closed subset of $\mathcal{X}$. Then any contraction mapping $\mathcal{K}$ from C into itself has a unique fixed point.*

3. Main result

Lemma 3.1. *Let $\gamma \geq 0$, $0 < \sigma \leq 1$, $0 < \varpi \leq 1$, $\mathfrak{f} \in C(\Lambda,\mathbb{R})$ and $\nu \in \mathbb{R}$. Then the function w is a solution of the following boundary value problem:*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\sigma,\chi;\Phi}_{\gamma^+}\Big({}^{\mathfrak{C}}\mathfrak{D}^{\varpi,\chi;\Phi}_{\gamma^+}w(\tau)+\nu w(\tau)\Big) = \mathfrak{f}(\tau), \quad \tau \in \Lambda := [\gamma,\delta], \\ w(\gamma) = 0\,, \;\; w(\delta) = \displaystyle\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\beta_i,\chi;\Phi}_{\gamma^+}w(\kappa_i), \; \gamma < \kappa_i < \delta, \end{cases} \quad (3.1)$$

if and only if

$$\begin{aligned} w(\tau) &= \mathfrak{I}^{\varpi+\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{f}(\tau) - \nu\mathfrak{I}^{\varpi,\chi;\Phi}_{\gamma^+}w(\tau) + \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\varpi+\sigma+\beta_i,\chi;\Phi}_{\gamma^+}\mathfrak{f}(\kappa_i) - \mathfrak{I}^{\varpi+\sigma,\chi;\Phi}_{\gamma^+}\mathfrak{f}(\delta) + \nu\mathfrak{I}^{\varpi,\chi;\Phi}_{\gamma^+}w(\delta) - \nu\sum_{i=1}^{n}\iota_i\mathfrak{I}^{\varpi+\beta_i,\chi;\Phi}_{\gamma^+}w(\kappa_i)\right], \\ &= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{f}(s)ds \\ &- \nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}w(s)ds \\ &+ \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\ &\times\left[\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}\mathfrak{f}(s)ds\right. \\ &- \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{f}(s)ds \\ &+ \nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}w(s)ds \qquad (3.2) \\ &\left.- \nu\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}w(s)ds\right], \end{aligned}$$

where

$$\Theta = \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(\gamma))}}{\chi^{\varpi}\Gamma(\varpi+1)} - \sum_{i=1}^{n}\iota_i\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(\gamma))}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} \neq 0. \quad (3.3)$$

Proof. Firstly, applying the $\Phi-\mathsf{GPF}$ integral of order σ to both sides of (3.1) we obtain by using Lemma 2.6

$$
{}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi;\Phi}w(\tau)+\nu w(\tau)=\mathfrak{I}_{\gamma^+}^{\sigma,\chi;\Phi}\mathfrak{f}(\tau)+d_0e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}, \tag{3.4}
$$

where d_0 is constant. Secondly, applying the $\Phi-\mathsf{GPF}$ integral of order ϖ to both sides of (3.4) we obtain by using Lemma 2.6.

$$
w(\tau)=\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{f}(\tau)-\nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\tau)+d_0\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\chi^{\varpi}\Gamma(\varpi+1)}+d_1e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}, \tag{3.5}
$$

where d_1 is constant. Next, by using the boundary condition $w(\gamma)=0$ in (3.5) we obtain $d_1=0$ then

$$
w(\tau)=\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{f}(\tau)-\nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\tau)+d_0\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\chi^{\varpi}\Gamma(\varpi+1)}, \tag{3.6}
$$

then, by using the boundary condition $w(\delta)=\sum\limits_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\beta_i,\chi;\Phi}w(\kappa_i)$ in (3.6) we obtain

$$
d_0=\frac{1}{\Theta}\left[\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}\mathfrak{f}(\kappa_i)-\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{f}(\delta)+\nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\delta)-\nu\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}w(\kappa_i)\right], \tag{3.7}
$$

where Θ is given by (3.3). Substituting the value of d_0 in (3.6) we obtain

$$
\begin{aligned}
w(\tau)={}&\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{f}(\tau)-\nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\tau)+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times\left[\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}\mathfrak{f}(\kappa_i)-\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{f}(\delta)+\nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\delta)-\nu\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}w(\kappa_i)\right].
\end{aligned} \tag{3.8}
$$

The converse follows by direct computation that the solution $w(\tau)$ given by (3.2) satisfies problem (3.1) under the given boundary conditions. □

4. Existence and uniqueness results for problem (1.1)

In this section, we present the results on existence and uniqueness for problem (1.1).

In view of Lemma 3.1 we define the operator $\mathcal{K} : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\begin{aligned}
(\mathcal{K}w)(\tau) &= \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{h}(\tau, w(\tau)) - \nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\tau) + \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \Bigg[\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}\mathfrak{h}(\kappa_i, w(\kappa_i)) - \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{h}(\delta, w(\delta)) + \nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\delta) \\
&- \nu\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}w(\kappa_i)\Bigg], \\
&= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s, w(s))ds \\
&- \nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}w(s)ds \\
&+ \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \Bigg[\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s) \\
&\times (\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}\mathfrak{h}(s, w(s))ds \\
&- \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s, w(s))ds \\
&+ \nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}w(s)ds \qquad (4.1) \\
&- \nu\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}w(s)ds\Bigg],
\end{aligned}$$

where $\mathcal{C} = C([\gamma, \delta], \mathbb{R})$ denotes the Banach space of all continuous functions from $[\gamma, \delta]$ into $\mathbb{R}$ with the norm $\|w\| := sup\{|w(\tau)|; \tau \in [\gamma, \delta]\}$.

To address the existence and uniqueness results for problem (1.1), we introduce the following notations to streamline the computations.

$$\begin{aligned}
\mathfrak{A} &= \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\right], \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B} &= |\nu|\Bigg\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \left[\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right]\Bigg\}. \qquad (4.3)
\end{aligned}$$

We reveal the principal results under the following hypotheses.

(H_1): $|\mathfrak{h}(\tau, v) - \mathfrak{h}(\tau, w)| \le \mathcal{L}|v - w|$; $\mathcal{L} > 0$, for each $\tau \in [\gamma, \delta]$ and $v, w \in \mathbb{R}$.

(H_2): $\mathfrak{h} : [\gamma, \delta] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and there exist non-negative continuous function ψ, such that $|\mathfrak{h}(\tau, w)| \leq \psi(\tau)$, $(\tau, w) \in [\gamma, \delta] \times \mathbb{R}$, with $\|\psi\| = sup_{\tau\in[\gamma,\delta]}|\psi(\tau)|$.

(H_3): $\mathfrak{B} < 1$, where $\mathfrak{B}$ is given by (4.3).

4.1. Existence result based on Krasnoselskii fixed point theorem

Theorem 4.1. *Assume that* (H_2) *and* (H_3) *are satisfied. Then, there exists at least one solution for the problem* (1.1) *on* $[\gamma, \delta]$.

Proof. Let $sup_{\tau\in[\gamma,\delta]}|\psi(\tau)| = \|\psi\|$ and $\mathcal{B}_r = \{w \in \mathcal{C}; \|w\| \leq r\}$, where $r \geq \dfrac{\|\psi\|\mathfrak{A}}{1-\mathfrak{B}}$, We will demonstrate that the operator $\mathcal{K}$ defined by (4.1) satisfies the conditions of Krasnoselskii's fixed point theorem. To do this, we decompose the operator $\mathcal{K}$ into the sum of two operators, $\mathcal{K}_1$ and $\mathcal{K}_2$, defined on $\mathcal{B}_r$, as follows:

$$
\begin{aligned}
&(\mathcal{K}_1 w)(\tau)\\
&= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s,w(s))ds\\
&+ \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times \left[\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\int_\gamma^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)\right.\\
&\times (\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}\mathfrak{h}(s,w(s))ds\\
&- \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s,w(s))ds, \qquad (4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathcal{K}_2 w)(\tau)\\
&= -\nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}w(s)ds\\
&+ \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\\
&\times \left[\nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}w(s)ds\right. \qquad (4.5)\\
&\left.- \nu\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_\gamma^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}w(s)ds\right].
\end{aligned}
$$

For any $v, w \in \mathcal{B}_r$ we have

$$
\begin{aligned}
&\big|(\mathcal{K}_1 v)(\tau)+(\mathcal{K}_2 w)(\tau)\big| \\
&\leq \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds \\
&+|\nu|\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}|w(s)|ds \\
&+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\Bigg[\sum_{i=1}^n|\iota_i|\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \\
&\times\int_\gamma^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,w(s))|ds \\
&+\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds \\
&+|\nu|\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}|w(s)|ds \\
&+|\nu|\sum_{i=1}^n|\iota_i|\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_\gamma^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}|w(s)|ds\Bigg],
\end{aligned}
$$

using (H_1) and the property $e^{\frac{\chi-1}{\chi}(\Phi(t)-\Phi(s))}\leq 1$ for $0\leq\gamma<s<t<\tau\leq\delta$ we obtain

$$
\begin{aligned}
\big|(\mathcal{K}_1 v)(\tau)+(\mathcal{K}_2 w)(\tau)\big| &\leq \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\|\psi\|+|\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}|w| \\
&+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^n|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\|\psi\| \\
&+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\|\psi\|+|\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}|w| \\
&+\sum_{i=1}^n|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}|w|\Bigg], \\
&\leq\|\psi\|\Bigg\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\Bigg[\sum_{i=1}^n|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\Bigg]\Bigg\} \\
&+|w||\nu|\Bigg\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} \\
&+\sum_{i=1}^n|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\Bigg]\Bigg\}, \\
&\leq\|\psi\|\mathfrak{A}+|w|\mathfrak{B}, \\
&\leq\|\psi\|\mathfrak{A}+r\mathfrak{B}, \\
&\leq r,
\end{aligned}
$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are given by (4.2) and (4.3), then $\|\mathcal{K}_1 v + \mathcal{K}_2 w\| \leq r$, which leads to $\mathcal{K}_1 v + \mathcal{K}_2 w \in \mathcal{B}_r$.
Since $\mathfrak{h}$ is continuous, then the operator $\mathcal{K}_1$ is continuous and it is uniformly bounded on $\mathcal{B}_r$ as :

$$\begin{aligned}
&|(\mathcal{K}_1 w)(\tau)| \\
&\quad \leq \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_\gamma^\tau e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))} \Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1} |\mathfrak{h}(s,w(s))| ds \\
&\quad + \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\quad \times \left[\sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \right. \\
&\quad \times \int_\gamma^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))} \Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1} |\mathfrak{h}(s,w(s))| ds \\
&\quad \left. + \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_\gamma^\delta e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))} \Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1} |\mathfrak{h}(s,w(s))| ds \right], \\
&\quad \leq \|\psi\| \left\{ \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \right. \\
&\quad \left. \times \left[\sum_{i=1}^{n} |\iota_i| \frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} \right] \right\},
\end{aligned}$$

then we obtain

$$\|\mathcal{K}_1 w\| \leq \mathfrak{A}\|\psi\|, \tag{4.6}$$

where $\mathfrak{A}$ is given by (4.2), by (4.6) the operator $\mathcal{K}_1$ is uniformly bounded on $\mathcal{B}_r$.

Next, we prove that the operator $\mathcal{K}_1$ is compact, for that let $\tau_1, \tau_2 \in [\gamma, \delta]$ such that $\tau_1 < \tau_2$, then we obtain

$$
\begin{aligned}
&|(\mathcal{K}_1 w)(\tau_2) - (\mathcal{K}_1 w)(\tau_1)| \\
&\quad \leq \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\tau_1} \Phi'(s)\big[(\Phi(\tau_2)-\Phi(s))^{\varpi+\sigma-1} - (\Phi(\tau_1)-\Phi(s))^{\varpi+\sigma-1}\big] \\
&\quad \times |\mathfrak{h}(s,w(s))| ds \\
&\quad + \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\tau_1}^{\tau_2} \Phi'(s)(\Phi(\tau_2)-\Phi(s))^{\varpi+\sigma-1} |\mathfrak{h}(s,w(s))| ds \\
&\quad + \frac{\Big|(\Phi(\tau_2)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau_2)-\Phi(\gamma))} - (\Phi(\tau_1)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau_1)-\Phi(\gamma))}\Big|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\quad \times \left[\sum_{i=1}^{n} \iota_i \frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \int_{\gamma}^{\kappa_i} \Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1} |\mathfrak{h}(s,w(s))| ds \right. \\
&\quad \left. + \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\delta} \Phi'^{\varpi+\sigma-1} |\mathfrak{h}(s,w(s))| ds \right], \\
&\quad \leq \frac{\|\psi\|}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\tau_1} \Phi'(s)\big[(\Phi(\tau_2)-\Phi(s))^{\varpi+\sigma-1} - (\Phi(\tau_1)-\Phi(s))^{\varpi+\sigma-1}\big] ds \\
&\quad + \frac{\|\psi\|}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\tau_1}^{\tau_2} \Phi'(s)(\Phi(\tau_2)-\Phi(s))^{\varpi+\sigma-1} ds \\
&\quad + \frac{\Big|(\Phi(\tau_2)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau_2)-\Phi(\gamma))} - (\Phi(\tau_1)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau_1)-\Phi(\gamma))}\Big|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\quad \times \left[\sum_{i=1}^{n} \iota_i \frac{\|\psi\|}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \int_{\gamma}^{\kappa_i} \Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1} ds \right. \\
&\quad \left. + \frac{\|\psi\|}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\delta} \Phi'^{\varpi+\sigma-1} ds \right], \\
&\quad \leq \frac{\|\psi\|}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\gamma}^{\tau_1} \Phi'(s)\big[(\Phi(\tau_2)-\Phi(s))^{\varpi+\sigma-1} - (\Phi(\tau_1)-\Phi(s))^{\varpi+\sigma-1}\big] ds \\
&\quad + \frac{\|\psi\|}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)} \int_{\tau_1}^{\tau_2} \Phi'(s)(\Phi(\tau_2)-\Phi(s))^{\varpi+\sigma-1} ds \\
&\quad + \frac{\Big|(\Phi(\tau_2)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau_2)-\Phi(\gamma))} - (\Phi(\tau_1)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau_1)-\Phi(\gamma))}\Big|}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\quad \times \left[\sum_{i=1}^{n} \iota_i \frac{\|\psi\|}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)} (\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i} \right. \\
&\quad \left. + \frac{\|\psi\|}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)} (\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma} \right],
\end{aligned}
$$

the right hand side tends to zero as $\tau_2 \longrightarrow \tau_1$, independently of $w \in \mathcal{B}_r$ which leads to $|(\mathcal{K}_1 w)(\tau_2) - (\mathcal{K}_1 w)(\tau_1)| \longrightarrow 0$ as $\tau_2 \longrightarrow \tau_1$. Therefore, $\mathcal{K}_1$ is equicontinuous and consequently, $\mathcal{K}_1$ is relatively compact on $\mathcal{B}_r$. By the ArzelàAscoli theorem, it follows that $\mathcal{K}_1$ is compact on $\mathcal{B}_r$.

In the final step, we will show that $\mathcal{K}_2$ is a contraction, for that let $w, v \in \mathcal{C}$, and for $\tau \in [\gamma, \delta]$ we have

$$
\begin{aligned}
&\left|(\mathcal{K}_2 v)(\tau) + (\mathcal{K}_2 w)(\tau)\right| \\
&\leq |\nu| \frac{1}{\chi^{\varpi}\Gamma(\varpi)} \int_{\gamma}^{\tau} e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))} \Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1} |v(s)-w(s)| ds \\
&+ \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi} e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times \Bigg[|\nu| \frac{1}{\chi^{\varpi}\Gamma(\varpi)} \int_{\gamma}^{\delta} e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))} \Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1} |v(s)-w(s)| ds \\
&+ |\nu| \sum_{i=1}^{n} |\iota_i| \frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)} \\
&\times \int_{\gamma}^{\kappa_i} e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))} \Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1} |v(s)-w(s)| ds \Bigg], \\
&\leq |x-y||\nu| \Bigg\{ \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \Bigg[\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} \\
&+ \sum_{i=1}^{n} |\iota_i| \frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)} \Bigg] \Bigg\}, \\
&\leq \mathfrak{B}\|v-w\|.
\end{aligned}
$$

Which implies $\|\mathcal{K}_2 v - \mathcal{K}_2 w\| \leq \mathfrak{B}\|v-w\|$, where $\mathfrak{B}$ is given by (4.3), it follows by using (H_3) that $\mathcal{K}_2$ is a contraction mapping. Finally by Krasnoselskii fixed point theorem, we deduce that the problem (1.1) has at least one solution on $[\gamma, \delta]$. □

4.2. Uniqueness result based on Banach fixed point theorem

The second result on existence and uniqueness will be derived using Banach's fixed point theorem.

Theorem 4.2. *Assume that (H_1) is verified. If $\mathcal{L}\mathfrak{A} + \mathfrak{B} < 1$, where $\mathfrak{A}$ and $\mathfrak{B}$ are respectively given by* (4.2) *and* (4.3)*, then the problem* (1.1) *has a unique solution on* $[\gamma, \delta]$.

Proof. Consider the operator $\mathcal{K}$ defined in (4.1). The problem (1.1) is then can be transformed into a fixed point problem $w = \mathcal{K}w$. By using Banach contraction principle we will show that $\mathcal{K}$ has a unique fixed point.
We set $sup_{\tau\in[\gamma,\delta]} = |\mathfrak{h}(\tau, 0)| = \mathcal{M} < \infty$, and choose $\rho > 0$ such that

$$\rho \geq \frac{\mathcal{M}\mathfrak{A}}{1 - \mathcal{L}\mathfrak{A} - \mathfrak{B}}, \tag{4.7}$$

$\mathcal{B}_\rho = \{w \in \mathcal{C}([\gamma, \delta], \mathbb{R}); \|w\| \leq \rho)\}$, where $\mathfrak{A}, \mathfrak{B}$ are respectively given by (4.2) and (4.3).
Step 1: We show that $\mathcal{K}\mathcal{B}_\rho \subset \mathcal{B}_\rho$.
For any $w \in \mathcal{B}_\rho$ we have

$$
\begin{aligned}
|\mathfrak{h}(\tau, w(\tau))| &\leq |\mathfrak{h}(\tau, w(\tau)) - \mathfrak{h}(\tau, 0)| + |\mathfrak{h}(\tau, 0)|, \\
&\leq \mathcal{L}|w(\tau)| + \mathcal{M}, \\
&\leq \mathcal{L}|w| + \mathcal{M},
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\big|(\mathcal{K}w)(\tau)\big| \\
&\leq \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}|\mathfrak{h}(\tau,w(\tau))| + |\nu|\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}|w(\tau)| + \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\Bigg[\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}|\mathfrak{h}(\kappa_i,w(\kappa_i))| + \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}|\mathfrak{h}(\delta,w(\delta))| + |\nu|\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}|w(\delta)| \\
&+|\nu|\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}|w(\kappa_i)|\Bigg], \\
&\leq \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds \\
&+|\nu|\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}|w(s)|ds \\
&+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^{n}|\iota_i|\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \\
&\times\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}|\mathfrak{h}(s,w(s))|ds \\
&+\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}|\mathfrak{h}(s,w(s))|ds \\
&+|\nu|\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}|w(s)|ds \\
&+|\nu|\sum_{i=1}^{n}|\iota_i|\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}|w(s)|ds\Bigg],
\end{aligned}
$$

using (H_1) and the property $e^{\frac{\chi-1}{\chi}(\Phi(t)-\Phi(s))}\leq 1$ for $0\leq\gamma<s<t<\tau\leq\delta$ it leads to

$$\begin{aligned}
\left|(\mathcal{K}w)(\tau)\right| \leq\ & \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\big(\mathcal{L}|w|+\mathcal{M}\big)+|\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}|w| \\
&+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\big(\mathcal{L}|w|+\mathcal{M}\big)\right. \\
&+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\big(\mathcal{L}|w|+\mathcal{M}\big)+|\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}|w| \\
&\left.+\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}|w|\right], \\
&\leq \big(\mathcal{L}|w|+\mathcal{M}\big)\left\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\right. \\
&\left.\times\left[\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\right]\right\} \\
&+|w||\nu|\left\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\right.\right. \\
&\left.\left.+\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right]\right\}, \\
&\leq \big(\mathcal{L}|w|+\mathcal{M}\big)\mathfrak{A}+|w|\mathfrak{B}, \\
&\leq \big(\mathcal{L}|w|+\mathcal{M}\big)\mathfrak{A}+\rho\mathfrak{B}, \\
&\leq \rho,
\end{aligned}$$

which implies that $\mathcal{K}\mathcal{B}_\rho \subset \mathcal{B}_\rho$.

Step 2: We show that the operator $\mathcal{K}$ is a contraction.
For any $v, w \in \mathcal{C}$, and for $\tau \in [\gamma, \delta]$, we have

$$\begin{aligned}
&\left|(\mathcal{K}v)(\tau)-(\mathcal{K}w)(\tau)\right| \\
&\leq \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}|\mathfrak{h}(\tau,v(\tau))-\mathfrak{h}(\tau,w(\tau))|+|\nu|\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}|v(\tau)-w(\tau)| \\
&+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}|\mathfrak{h}(\kappa_i,v(\kappa_i))-\mathfrak{h}(\kappa_i,w(\kappa_i))|\right. \\
&+\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}|\mathfrak{h}(\delta,v(\delta))-\mathfrak{h}(\delta,w(\delta))| \\
&\left.+|\nu|\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}|v(\delta)-w(\delta)|+|\nu|\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}|v(\kappa_i)-w(\kappa_i)|\right], \\
&\leq (\mathcal{L}|v-w|)\left\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\right. \\
&\left.\times\left[\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\right]\right\} \\
&+|v-w||\nu|\left\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}+\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\right.\right. \\
&\left.\left.+\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right]\right\}, \\
&\leq (\mathcal{L}\mathfrak{A}+\mathfrak{B})|v-w|,
\end{aligned}$$

which implies, $\|\mathcal{K}v-\mathcal{K}w\| \leq (\mathcal{L}\mathfrak{A}+\mathfrak{B})\|v-w\|$. As $\mathcal{L}\mathfrak{A}+\mathfrak{B} < 1$, then $\mathcal{K}$ is a contraction. Therefore, by Banach fixed-point theorem, the operator $\mathcal{K}$ has a unique fixed point which is indeed the unique solution of problem (1.1). □

5. Ulam-Hyers stability analysis

In this section, we focus on studying the UlamHyers (U-H) and generalized Ulam-Hyers (G-U-H) stability for problem (1.1).

Let $\varepsilon > 0$, we consider the following inequality

$$\left| {}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi;\Phi}\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi;\Phi}\tilde{w}(\tau) + \nu\tilde{w}(\tau)\right) - \mathfrak{h}(\tau, \tilde{w}(\tau))\right| \leq \varepsilon, \quad \tau \in \Lambda := [\gamma, \delta], \tag{5.1}$$

Definition 5.1 ([32, 52]). The problem (1.1) is U-H stable if there exists $\lambda > 0$, such that for each $\varepsilon > 0$ and for each solution $\tilde{w} \in \mathcal{C}$ of inequality (5.1), there exists $w, \in \mathcal{C}$ solution of the problem (1.1) complying with

$$\|\tilde{w} - w\| \leq \lambda\varepsilon. \tag{5.2}$$

Definition 5.2 ([32, 52]). The problem (1.1) is G-U-H stable if there exists $\varphi \in \mathcal{C}$ with $\varphi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $\tilde{w} \in \mathcal{C}$ of inequality (5.1), there exists $w \in \mathcal{C}$ solution of the problem (1.1) complying with.

$$\|\tilde{w} - w\| \leq \varphi(\varepsilon). \tag{5.3}$$

Remark 5.3 ([32, 52]). A function $\tilde{w} \in \mathcal{C}$ is a solution of inequalities (5.1) if and only if there exists a function $\mathfrak{f} \in \mathcal{C}$ such that

i- $|\mathfrak{g}(\tau)| \leq \varepsilon$,

ii- for $\tau \in [\gamma, \delta]$:

$${}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi;\Phi}\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi;\Phi}\tilde{w}(\tau) + \nu\tilde{w}(\tau)\right) = \mathfrak{h}(\tau, \tilde{w}(\tau)) + \mathfrak{g}(\tau). \tag{5.4}$$

To streamline the computations, we introduce the following notations:

$$\begin{aligned}
\Omega_1 = {} & \frac{(\Phi(\delta) - \Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L} + |\nu|\frac{(\Phi(\delta) - \Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} \\
& + \frac{(\Phi(\delta) - \Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n} |\iota_i| \frac{(\Phi(\kappa_i) - \Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\mathcal{L}\right. \\
& + \frac{(\Phi(\delta) - \Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L} + |\nu|\frac{(\Phi(\delta) - \Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} \\
& \left. + |\nu|\sum_{i=1}^{n} |\iota_i| \frac{(\Phi(\kappa_i) - \Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right],
\end{aligned} \tag{5.5}$$

$$\Omega_2 = \frac{(\Phi(\delta) - \Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}, \tag{5.6}$$

Theorem 5.4. *Assume that (H_1) hold, if $\Omega_1 < 1$ then the problem (1.1) is UlamHyers stable on $[\gamma, \delta]$ and consequently is generalized UlamHyers stable, where Ω_1 is given by (5.5).*

Proof. Let $\varepsilon > 0$, and $\tilde{w} \in \mathcal{C}$ satisfies inequality (5.1), and $w \in \mathcal{C}$ be the unique solution of the problem (1.1) with the conditions $\tilde{w}(\gamma) = w(\gamma)$, $\tilde{w}(\delta) = w(\delta)$, then by Lemma 2.6, we obtain

$$
\begin{aligned}
w(\tau) &= \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{h}(\tau, w(\tau)) - \nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\tau) + \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\left[\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}\mathfrak{h}(\kappa_i, w(\kappa_i)) - \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{h}(\delta, w(\delta)) + \nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}w(\delta)\right. \\
&\left.-\nu\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}w(\kappa_i)\right], \\
&= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s, w(s))ds \\
&-\nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}w(s)ds \\
&+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)}\right. \\
&\times\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}\mathfrak{h}(s, w(s))ds \\
&-\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s, w(s))ds \\
&+\nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}w(s)ds \\
&\left.-\nu\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}w(s)ds\right],
\end{aligned} \tag{5.7}
$$

Since, $\tilde{w} \in \mathcal{C}$ satisfies inequality (5.1) by using Remark 5.3 we have

$$
\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\sigma,\chi;\Phi}\left({}^{\mathfrak{C}}\mathfrak{D}_{\gamma^+}^{\varpi,\chi;\Phi}\tilde{w}(\tau) + \nu\tilde{w}(\tau)\right) = \mathfrak{h}(\tau, \tilde{w}(\tau)) + \mathfrak{g}(\tau),\ \tau \in \Lambda := [\gamma, \delta], \\
\tilde{w}(\gamma) = w(\gamma) \quad , \quad \tilde{w}(\delta) = w(\delta),
\end{cases} \tag{5.8}
$$

then by Lemma 2.6, we obtain

$$
\begin{aligned}
\tilde{w}(\tau) &= \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{h}(\tau,\tilde{w}(\tau)) - \nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}\tilde{w}(\tau) + \frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)} \\
&\times\Bigg[\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}\mathfrak{h}(\kappa_i,\tilde{w}(\kappa_i)) - \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{h}(\delta,\tilde{w}(\delta)) + \nu\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}\tilde{w}(\delta) \\
&-\nu\sum_{i=1}^{n}\iota_i\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}\tilde{w}(\kappa_i)\Bigg] + \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\mathfrak{g}(\tau), \\
&= \frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s,\tilde{w}(s))ds \\
&-\nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi-1}\tilde{w}(s)ds \\
&+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{\Theta\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i)} \\
&\times\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\sigma+\beta_i-1}\mathfrak{h}(s,\tilde{w}(s))ds \\
&-\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{h}(s,\tilde{w}(s))ds \\
&+\nu\frac{1}{\chi^{\varpi}\Gamma(\varpi)}\int_{\gamma}^{\delta}e^{\frac{\chi-1}{\chi}(\Phi(\delta)-\Phi(s))}\Phi'(s)(\Phi(\delta)-\Phi(s))^{\varpi-1}\tilde{w}(s)ds \\
&-\nu\sum_{i=1}^{n}\iota_i\frac{1}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i)}\int_{\gamma}^{\kappa_i}e^{\frac{\chi-1}{\chi}(\Phi(\kappa_i)-\Phi(s))}\Phi'(s)(\Phi(\kappa_i)-\Phi(s))^{\varpi+\beta_i-1}\tilde{w}(s)ds\Bigg], \\
&+\frac{1}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma)}\int_{\gamma}^{\tau}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(s))}\Phi'(s)(\Phi(\tau)-\Phi(s))^{\varpi+\sigma-1}\mathfrak{g}(s)ds, \qquad (5.9)
\end{aligned}
$$

for each $\tau\in[\gamma,\delta]$, we have

$$
\begin{aligned}
&\left|\tilde{w}(\tau)-w(\tau)\right| \\
&\leq \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\left|\mathfrak{h}(\tau,w(\tau))-\mathfrak{h}(\tau,\tilde{w}(\tau))\right| + |\nu|\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}\left|w(\tau)-\tilde{w}(\tau)\right| \\
&+\frac{(\Phi(\tau)-\Phi(\gamma))^{\varpi}e^{\frac{\chi-1}{\chi}(\Phi(\tau)-\Phi(\gamma))}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\Bigg[\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\sigma+\beta_i,\chi;\Phi}\left|\mathfrak{h}(\kappa_i,w(\kappa_i))-\mathfrak{h}(\tau,\tilde{w}(\tau))\right| \\
&+\mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}\left|\mathfrak{h}(\delta,w(\delta))-\mathfrak{h}(\tau,\tilde{w}(\tau))\right| + |\nu|\mathfrak{I}_{\gamma^+}^{\varpi,\chi;\Phi}\left|w(\delta)-\tilde{w}(\delta)\right| \\
&+|\nu|\sum_{i=1}^{n}|\iota_i|\mathfrak{I}_{\gamma^+}^{\varpi+\beta_i,\chi;\Phi}\left|w(\kappa_i)-\tilde{w}(\kappa_i)\right|\Bigg] + \mathfrak{I}_{\gamma^+}^{\varpi+\sigma,\chi;\Phi}|\mathfrak{g}(\tau)|,
\end{aligned}
$$

using (H_1), the property $e^{\frac{\chi-1}{\chi}(\Phi(t)-\Phi(s))}\leq 1$ for $0\leq\gamma<s<t<\tau\leq\delta$ and Remark 5.3 leads to

$$
\begin{aligned}
\|\tilde{w}-w\| \leq & \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L}\|\tilde{w}-w\| + |\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\|\tilde{w}-w\| \\
& + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\mathcal{L}\|\tilde{w}-w\|\right. \\
& + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L}\|\tilde{w}-w\| + |\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\|\tilde{w}-w\| \\
& \left. + |\nu|\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\|\tilde{w}-w\|\right] + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\varepsilon, \\
\leq & \|\tilde{w}-w\|\left\{\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L} + |\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)}\right. \\
& + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{|\Theta|\chi^{\varpi}\Gamma(\varpi+1)}\left[\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\sigma+\beta_i}}{\chi^{\varpi+\sigma+\beta_i}\Gamma(\varpi+\sigma+\beta_i+1)}\mathcal{L}\right. \\
& + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\mathcal{L} + |\nu|\frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi}}{\chi^{\varpi}\Gamma(\varpi+1)} \\
& \left.\left. + |\nu|\sum_{i=1}^{n}|\iota_i|\frac{(\Phi(\kappa_i)-\Phi(\gamma))^{\varpi+\beta_i}}{\chi^{\varpi+\beta_i}\Gamma(\varpi+\beta_i+1)}\right]\right\} + \frac{(\Phi(\delta)-\Phi(\gamma))^{\varpi+\sigma}}{\chi^{\varpi+\sigma}\Gamma(\varpi+\sigma+1)}\varepsilon, \\
\leq & \|\tilde{w}-w\|\Omega_1 + \Omega_2\varepsilon, \\
\leq & \Omega_1\|\tilde{w}-w\| + \Omega_2\varepsilon, \\
\leq & \frac{\Omega_2}{1-\Omega_1}\varepsilon,
\end{aligned}
$$

which implies,

$$\|\tilde{w}-w\| \leq \frac{\Omega_2}{1-\Omega_1}\varepsilon. \tag{5.10}$$

By setting $\lambda = \dfrac{\Omega_2}{1-\Omega_1}$, where Ω_1 and Ω_2 are given by (5.5) and (5.6), we obtain

$$\|\tilde{w}-w\| \leq \lambda\varepsilon. \tag{5.11}$$

This proves that the problem (1.1), is U-H stable.
consequently, by setting $\varphi(\varepsilon) = \lambda\varepsilon$ with $\varphi(0) = 0$ we get

$$\|\tilde{w}-w\| \leq \varphi(\varepsilon). \tag{5.12}$$

This shows that the problem (1.1) is G-H-U stable. □

6. Example

Consider the following problem

$$
\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{3}{8},\frac{1}{3};\frac{e^\tau}{3}}\left({}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\frac{5}{8},\frac{1}{3};\frac{e^\tau}{3}}w(\tau) + \frac{2}{3}w(\tau)\right) = \frac{e^{-\tau}}{4+e^{\tau}}\left(\frac{|w(\tau)|}{1+|w(\tau)|}\right),\ \tau \in \Lambda := [0,1], \\
w(0) = 0\ ,\ \ w(1) = \frac{3}{7}\mathfrak{I}^{\frac{2}{3},\frac{1}{3}}w(\frac{1}{4}) + \frac{5}{7}\mathfrak{I}^{\frac{4}{3},\frac{1}{3}}w(\frac{3}{4}).
\end{cases}
\tag{6.1}
$$

Where $\sigma = \dfrac{3}{8}$, $\varpi = \dfrac{5}{8}$, $\chi = \dfrac{1}{3}$, $\gamma = 0$, $\delta = 1$, $\Lambda = [0,1]$, $\nu = \dfrac{3}{2}$, $n = 2$, $\iota_1 = \dfrac{3}{7}$, $\iota_2 = \dfrac{5}{7}$, $\beta_1 = \dfrac{2}{3}$, $\beta_2 = \dfrac{4}{3}$, $\kappa_1 = \dfrac{1}{4}$, $\kappa_2 = \dfrac{3}{4}$ and $\Phi(\tau) = \dfrac{e^\tau}{3}$.

For $(\tau, w) \in [0,1] \times \mathbb{R}_+$, we define $\mathfrak{h}(\tau, w) = \frac{e^{-\tau}}{4+e^{\tau}}\Big(\frac{w}{1+w}\Big)$. $\mathfrak{h}$ is a continuous function, furthermore for every $\tau \in [0,1]$ and $v, w \in \mathbb{R}_+$we have

$$|\mathfrak{h}(\tau, v) - \mathfrak{h}(\tau, w)| \leq \Big|\frac{1}{4+e^{\tau}}\Big|\Big|\frac{v-w}{(1+v)(1+w)}\Big|, \\ \leq \frac{1}{5}|v-w|.$$

By setting $\mathcal{L} = \frac{1}{5} > 0$ the hypotheses (H_1) holds. Next by using the data given above, we get : $|\Theta| = 0.884357$, $\mathfrak{A} = 0.824175$, $\mathfrak{B} = 0.076381$.
Then

$$\mathcal{L}\mathfrak{A} + \mathfrak{B} = 0.2 \times 0.824175 + 0.076381 \simeq 0.241216 < 1.$$

The problem (6.1) satisfies all the hypothesis of Theorem 4.2, thus, the problem (6.1) has a unique solution on $[0,1]$. Additionally, $\Omega_1 = 0.785426 < 1$. Hence, using Theorem 5.4, the problem (6.1) is both UlamHyers and also generalized UlamHyers stable on $[0,1]$.

7. Conclusion

In this paper, we have explored the existence, uniqueness, and stability results for a new class of Φ-Caputo generalized proportional fractional differential Langevin equations. The novelty of the problem lies in its investigation under the Φ-Caputo generalized proportional fractional derivative, which is more general than the traditional fractional derivatives such as the Caputo fractional derivative, Caputo-Hadamard fractional derivative, and Caputo-Katugampola fractional derivative, for various values of the function Φ. We established the existence and uniqueness results for problem (1.1) using standard fixed point theorems (Krasnoselskii's fixed point theorem and the Banach contraction principle). Additionally, we examined the stability of the problem using Ulam-Hyers and generalized Ulam-Hyers stability. Finally, a numerical example is provided to illustrate the obtained results. As a direction for future research, we aim to extend these results to study the Φ-Hilfer generalized proportional fractional derivative, along with graphical and numerical examples.

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Data Availability

The data used to support the finding of this study are available from the corresponding author upon request.

Conflicts Of Interest

The authors declare that they have no conflicts of interest.

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

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