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The Gaussian Sequence 3th Order Mod m

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Abstract
The work addresses the study of third-order recurrent sequences for mod m cases. Thus, some definitions aim to transform infinite sequences into finite ones. In this regard, the Fourier transform is used as a visualization technique, explored in Google Colab. The mathematical theorems presented are established to examine the patterns of these sequences and their corresponding cycles. As a future perspective, it is intended to investigate other mathematical theorems to generalize the sequences into finite groups.

Keywords: Finite groups, Gaussian, Sequences 3th order
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1. Introduction

Recent research examines Gaussian numbers in third-order recurrent sequences connected to finite groups in Abstract Algebra [1]-[5].

The study of recurrent sequences has been gaining relevance in mathematical literature, standing out as an area of research and promoting connections with a variety of other mathematical contents [6]-[8]. Building upon studies on the Fibonacci sequence, [9], we have third-order sequences, namely Padovan, Perrin, Narayana, and Leonardo, preliminarily presented and discussed in the following paragraphs.

The Padovan sequence is denoted in this work as P_n and has a recurrence relation given by: $P_n = P_{n-2} + P_{n-3}$, with $n \geq 3$ and initial values $P_0 = P_1 = P_2 = 1$ [10, 11].

The Perrin sequence, in turn, has mathematical properties similar to those of the Padovan numbers. Indeed, the Perrin numbers form a recurrent sequence, denoted by R_n and with recurrence relation: $R_n = R_{n-2} + R_{n-3}$, with $n \geq 3$ and initial values $R_0 = 3, R_1 = 0, R_2 = 2$ [12, 13]. It is noticeable that these numbers exhibit the same recurrence relation as the Padovan numbers, with their respective initial values altered. Similarly, this also occurs with the Fibonacci and Lucas numbers.

The Narayana sequence is also a recurrent sequence, denoted by N_n and present recurrence relation $N_n = N_{n-1} + N_{n-3}$, with $n \geq 3$ and initial values $N_0 = 0, N_1 = N_2 = 1$ [14, 15].

Finally, we have the Leonardo sequence, denoted by L_n, with recurrence relation denoted by: $L_n = L_{n-1} + L_{n-2} + 1$, with $n \geq 2$ and initial values $L_0 = L_1 = 1$. Building upon other studies on this sequence, Catarino and Borges [16] performed an algebraic operation, resulting in the recurrence $L_n = 2L_{n-1} - L_{n-3}$ with $n \geq 3$, making it third-order, with initial values

$L_0 = L_1 = 1, L_2 = 3$ [17].

The Table 1 shows a brief summary of the third-order recurrent sequences to be addressed in this work.

Table 1. Sequências recorrentes de terceira ordem. Fonte: Elaborado pelos autores.

Sequence	Recurrence	Initial values
Padovan	$P_n = P_{n-2} + P_{n-3}$	$P_0 = P_1 = P_2 = 1$
Perrin	$R_n = R_{n-2} + R_{n-3}$	$R_0 = 3, R_1 = 0, R_2 = 1$
Narayana	$N_n = N_{n-1} + N_{n-3}$	$N_0 = 0, N_1, N_2 = 1$
Leonardo	$L_n = L_{n-1} + L_{n-2} + 1$ ou $L_n = 2L_{n-1} - L_{n-3}$	$L_0 = 1, L_1 = 1, L_2 = 3$

Given the growing investigation around these numerical sequences, as well as the development of theorems and properties, there are several possibilities for integrating these relationships with other mathematical concepts. Studies conducted by [18]-[21] have opened new perspectives by extending the research to Gaussian numbers of third-order sequences in conjunction with finite groups. This approach has allowed the transition from infinite to finite sequences, resulting in the formulation of mathematical theorems and definitions for these numbers, observation, and analysis of cyclic patterns.

For a dynamic visualization, Google Colab [22] was used in this work to enable a graphical analysis of some specific examples presented in the next section, which were based on the definitions and theorems established.

Finite groups play a crucial role in group theory, and exploring them in conjunction with recurrent sequences reveals a valuable connection that expands the scope and depth of mathematical study, as well as the possibility of developing new properties and theorems [23]. This integration provides a more comprehensive understanding by exploring the relationships between finite groups and the properties of sequences, thereby enriching the perspective of students and researchers and fostering an interdisciplinary approach among the subjects.

Moreover, it is important to consider the sequence of finite groups, which transforms infinite recurrent sequences into finite ones through group theory. By applying group theory to these sequences and transforming them into finite sequences, it is possible to understand the properties of these sequences more clearly and adjustably. Thus, a sequence generating a finite group G (of length n) consists of a finite sequence $(g_1, \ldots, g_n)$ of elements G that generate G, [24]. This implies that this sequence contains essential information about the structure and properties of the finite group, facilitating its analysis and application in various mathematical contexts.

2. The Gaussian Padovan Sequence $\pmod{m}$

The Gaussian Padovan sequence as defined by Tusci [25] with the following formula: $GP_n = P_n + iP_{n-1}$, in which P_n is the n-th term of the Padovan sequence and GP_n the n-th term of the Gaussian Padovan sequence.

Definition 2.1. *Therefore, the recurrence relation of the Gaussian Padovan sequence is given by:*

$$GP_n = GP_{n-2} + GP_{n-3},$$

with initial values $GP_0 = 1, GP_1 = 1 + i, GP_2 = 1 + i$ and $n \geq 3$.

In this regard, the study of the modular sequence is based on the work of Tas and Karaduman [21]. Therefore, a sequence is said to be simply periodic, with period k, when its first k different elements in the sequence forming a new sequence (or subsequence) repeated. With this, it is denoted $GP_i \pmod{m}$ for $\left\{GP_i^{(m)}\right\}$, in which:

$$\left\{GP_i^{(m)}\right\} = \left\{GP_0^{(m)}, GP_1^{(m)}, GP_2^{(m)}, GP_3^{(m)}, \ldots, GP_n^{(m)}\right\}.$$

Indeed, it can be noted that the sequence preserves the recurrence relation of the Gaussian Padovan numbers $(GP_n = GP_{n-2} + GP_{n-3})$.

Theorem 2.2. $\left\{GP_n^{(m)}\right\}$ *is the simply periodic form of the Gaussian Padovan sequence.*

Proof. The sequence has only a finite number given by m^3 possible triplets of terms, where the repetition of the triples is nothing more than the iteration of all subsequent terms.

Thus, based on Definition 2.1, we have:

$$GP_{i+2}^m = GP_{j+2}^m,$$
$$GP_{i+1}^m = GP_{j+1}^m,$$
$$GP_i^m = GP_j^m.$$

Therefore:

$$GP^{m}_{i-j+2} = GP^{m}_{2},$$
$$GP^{m}_{i-j+1} = GP^{m}_{1},$$
$$GP^{m}_{i-j} = GP^{m}_{0}.$$

Implying that the sequence $\left\{GP_n^{(m)}\right\}$ is simply periodic, as required. $\square$

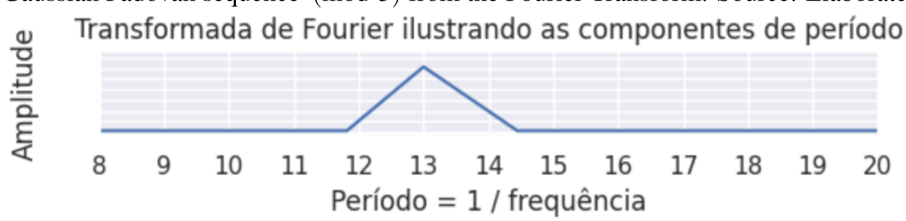
Thus, one can denote $kGP(m)$ as being the smallest period of $\left\{GP_n^{(m)}\right\}$, called the period of the Gaussian Padovan sequence modulo m.

Example 2.3. *The Gaussian Padovan sequence* $\pmod 3$ *with its first terms calculated from* GP_1 *and* $kGP(3) = 13$*:*

$$\left\{GP_n^{(3)}\right\} = \{1+i, 1+i, 2+i, 2+2i, 2i, 1, 2+i, 1+2i, i, 0, 1, i, 1, 1+i, 1+i, 2+i, \ldots\}.$$

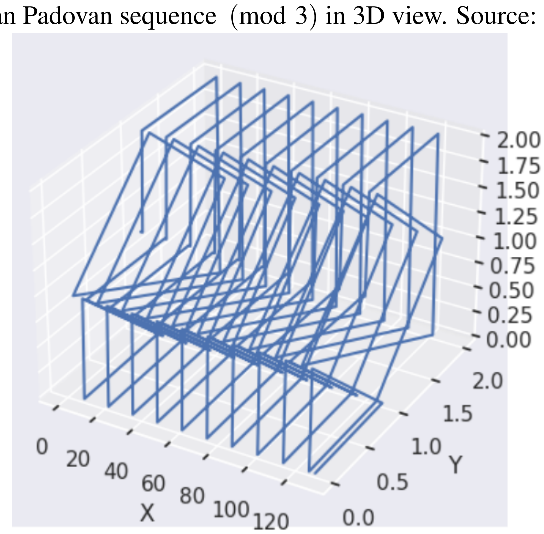
With this, we have conducted a brief analysis of the periodicity given in Example 2.3 from the Fourier Transform, which allows visualization of the period components (see Figure 2.1) of the amplitude versus period, where it is possible to observe the period 13 of the finite sequence. In the analysis of the periodicity of the sequence, one can highlight the presence of a peak at the point 13, revealed by the Fourier transform. This finding validates the previous mathematical calculation, identifying the recurrence of the sequence exactly at the point 13, being the period determined by the frequency of occurrence of terms in the sequence. Due to the proximity of terms, especially 0, 1 and 2, some additional peaks are noted in other periods. However, the primary period is clearly defined in 13, marking the beginning of the sequence repetition.

Figure 2.1. The Gaussian Padovan sequence $\pmod 3$ from the Fourier Transform. Source: Elaborated by the authors



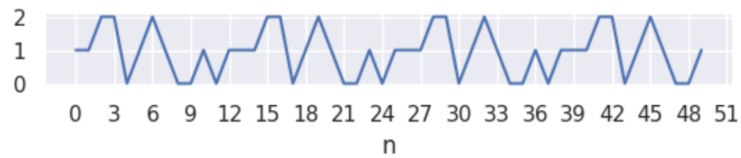
The Figure 2.2 presents a visualization of the terms of the sequence in a 3D perspective, where the construction of the Gaussian Padovan spiral can be observed $\pmod 3$ along the axes n, complex variable and real variable.

Figure 2.2. The Gaussian Padovan sequence $\pmod 3$ in 3D view. Source: Elaborated by the authors



In the Figure 2.3 one can observe the cyclic pattern that occurs with the Gaussian Padovan sequence $\pmod 3$, presenting a cycle within the interval 0 to 12, i.e., period equals 13 as mentioned earlier.

Figure 2.3. The Gaussian Padovan sequence $\pmod 3$ - Pattern. Source: Elaborated by the authors



Based on the matrix form of the Gaussian Padovan numbers, studied by Tasci [25] and Vieira [26], it is possible to recall that:

$$QP = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, v_P = \begin{bmatrix} 1+i & 1+i & 1 \\ 1+i & 1 & i \\ 1 & i & 1 \end{bmatrix}.$$

Thus, we have:

$$QP^n v_P = \begin{bmatrix} GP_{n+2} & GP_{n+1} & GP_n \\ GP_{n+1} & GP_n & GP_{n-1} \\ GP_n & GP_{n-1} & GP_{n-2} \end{bmatrix}.$$

For the matrix $M = [a_{ij}]_{(K+1)\times(k+1)}$ with a_{ij} integer, $M \pmod m$ means that all entries of M are reduced modulo m, i.e., $M \pmod m = (a_{ij} \pmod m)$. Let $\langle QP \rangle_{p^\alpha} = \left\{QP^i \pmod{p^\alpha} | i \geq 0\right\}$ a cyclic group and $|\langle QP \rangle_{p^\alpha}|$ denotes the order of $\langle QP \rangle_{p^\alpha}$. From the Gaussian matrix form of the Padovan sequence, we have that $kGP(p^\alpha) = |\langle QP \rangle_{p^\alpha}|$.

Theorem 2.4. *Let t a positive integer such that $kGP(p) = kGP(p^t)$. Therefore, we have that $kGP(p^\alpha) = r^{\alpha-t}kGP(p)$ for $\alpha \geq t$. Particularly, if $kGP(p) \neq kGP(r^2)$, then: $kGP(p^\alpha) = p^{\alpha-t}kGP(p)$ com $\alpha > 1$.*

Proof. Using a demonstration similar to the work of [27], we have:

Let q a positive integer. Being $QP^{kGP(p^{q+1})} \equiv I \pmod{r^{q+1}}$ and $QP^{kGP(p^{q+1})} \equiv I \pmod{r^q}$, we have that $kGP(p^q)$ divides $kGP(p^{q+1})$, in which I is the identity matrix and v_P the vector whose contains the initial values of the Gaussian Padovan sequence. Therefore, we have:

$$QP^{kGP(p^q)} v_P = I + \left(a_{ij}^{(q)} p^q\right),$$

So, we can write:

$$QP^{kGP(p^q)p} v_P = I + \left(a_{ij}^{(q)} p^q\right)^p = \sum_{i=0}^{p} \binom{p}{i} \left(a_i j^{(q)} p^q\right) \equiv I \pmod{p^{q+1}} v_P$$

$$QP^{kGP(p^q)p} \equiv I \pmod{p^{q+1}},$$

Resulting in $kGP(p^{q+1})$ divides $kGP(p)p$. Therefore, we obtain:

$kGP(p^{q+1}) = kGP(p^q)$ or $kGP(p^{q+1}) = kGP(p^q)p$ and this last one is valid if, and only if, there exists a $a_{ij}^{(q)}$ that is not divisible by p, as long as $kGP(p^t) \neq kR(p^{t+1})$ exists a $a_{ij}^{(t+1)}$ which is not divisible by p, and so, $kGP(p^{t+1}) \neq kGP(p^{t+2})$. The proof is concluded by induction on t. $\square$

Example 2.5. *For $p = 3$ and $q = 1$, $kGP(9) \equiv QP^{39} \equiv I \pmod 9$, then:*

$$QP^{39} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{39} = \begin{bmatrix} 23833 & 31572 & 17991 \\ 17991 & 23833 & 13581 \\ 13581 & 17991 & 10252 \end{bmatrix}_{\pmod 9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. The Gaussian Perrin Sequence $\pmod m$

The Gaussian Perrin sequence, studied by Kartal [28] alters the recurrence definition of Gaussian numbers. Thus, the present research makes a correction in its formula, allowing it to follow the pattern established by Jordan [29], Tusci [25], among others. Thus, the Gaussian Perrin numbers have their recurrence as: $GR_n = R_n + iR_{n-1}$, in which R_n is the $n-$th term of the Perrin sequence and GR_n the $n-$th term of the Gaussian Perrin sequence.

Definition 3.1. *The recurrence relation of Gaussian Perrin sequence is defined by:*

$$GR_n = GR_{n-2} + GR_{n-3},$$

with the initial values iniciais $GR_0 = 3 - i, GR_1 = 3i, GR_2 = 2$ *and* $n \geq 3$.

Thus, a connection can be established between the Gaussian Perrin sequence and its modular form, denoted by $GR_i \pmod{m}$ for $\left\{GR_i^{(m)}\right\}$, in which:

$$\left\{GR_i^{(m)}\right\} = \left\{GR_0^{(m)}, GR_1^{(m)}, GR_2^{(m)}, GR_3^{(m)}, \ldots, GR_n^{(m)}\right\}.$$

And it is worth noting that the recurrence relation of Gaussian Perrin numbers is preserved.

Theorem 3.2. $\left\{GR_n^{(m)}\right\}$ *is the simply periodic form of the Gaussian Perrin sequence.*

Proof. The proof follows analogously to Theorem 2.2. □

Example 3.3. *The Gaussian Perrin sequence* $\pmod{3}$ *with its first terms calculated from* GR_1 *and* $kGR(3) = 13$*:*

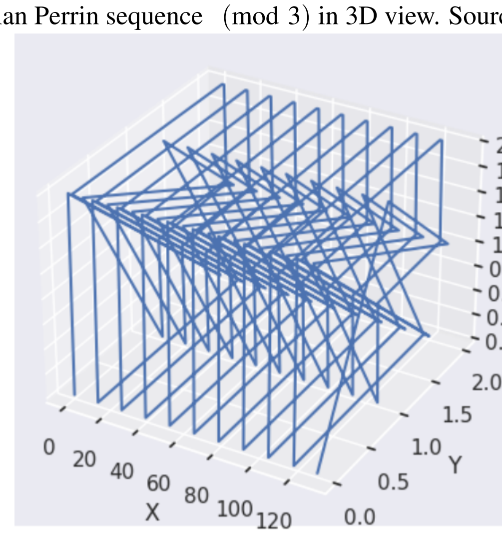
$$\left\{GR_n^{(3)}\right\} = \{0, 2, 2i, 2, 2+2, 2+2i, 1+2i, 1+i, i, 2, 1+2i, 2+i, 2i, 0, 2, 2i, \ldots\}.$$

The Figure 3.1 details the period of the sequence in the Example 3.3, allowing to see the period 13.

Figure 3.1. The Gaussian Perrin sequence $\pmod{3}$ from the Fourier Transform. Source: Elaborated by the authors

The spiral presented in Figure 3.2 provides a visualization of the terms of the sequence:

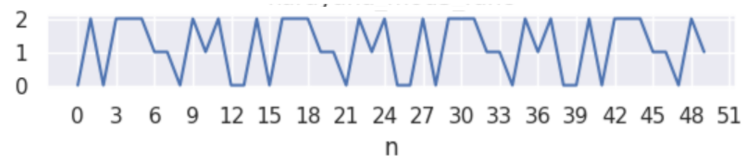
Figure 3.2. The Gaussian Perrin sequence $\pmod{3}$ in 3D view. Source: Elaborated by the authors



In Figure 3.3 we can observe the cyclic pattern that occurs with the Gaussian Perrin sequence $\pmod{3}$, presenting a cycle within the interval 0 to 12, i.e., period equals 13 as seen earlier.

Based on the matrix form of Gaussian Perrin numbers, it can be verified that:

Figure 3.3. The Gaussian Perrin sequence $\pmod 3$ - Pattern. Source: Elaborated by the authors



$$QR = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, v_R = \begin{bmatrix} GR_2 & GR_1 & GR_0 \\ GR_1 & GR_0 & GR_{-1} \\ GR_0 & GR_{-1} & GR_{-2} \end{bmatrix}.$$

Thus, we have:

$$QR^n v_R = \begin{bmatrix} GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+1} & GR_n & GR_{n-1} \\ GR_n & GR_{n-1} & GR_{n-2} \end{bmatrix}.$$

For the matrix $M = [a_{ij}]_{(K+1)\times(k+1)}$ with a_{ij} integer, $M \pmod m$ means that all entries of M are reduced modulo m, i.e., $M \pmod m = (a_{ij} \pmod m)$. Let $\langle QR \rangle_{p^\alpha} = \left\{ QR^i \pmod{p^\alpha} | i \geq 0 \right\}$ a cyclic group and $|\langle QR \rangle_{p^\alpha}|$ denotes the order of $\langle QR \rangle_{p^\alpha}$. From the Gaussian matrix form of the Perrin sequence, we have that $kGR(p^\alpha) = |\langle QR \rangle_{p^\alpha}|$.

Theorem 3.4. *Let t a positive integer such that $kGR(p) = kGR(p^t)$. So, we have $kGR(p^\alpha) = r^{\alpha-t} kGR(p)$ for $\alpha \geq t$. Particularly, if $kGR(p) \neq kGR(r^2)$, then: $kGR(p^\alpha) = p^{\alpha-t} kGR(p)$ com $\alpha > 1$.*

Proof. The proof follows analogously to the Theorem 2.4. □

Example 3.5. *For $p = 3$ and $q = 1$, $kGR(9) \equiv QR^{39} \equiv I \pmod 9$, then:*

$$QR^{39} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{39} = \begin{bmatrix} 23833 & 31572 & 17991 \\ 17991 & 23833 & 13581 \\ 13581 & 17991 & 10252 \end{bmatrix}_{\pmod 9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. The Gaussian Nayarana Sequence $\pmod m$

Starting the study around the Gaussian Narayana sequence, investigated by Ozkan and Kuloglu [30], its formula is given by: $GN_n = N_n + iN_{n-1}$, in which N_n is the $n-$th term of Narayana sequence and GN_n the $n-$th term of the Gaussian Narayana sequence.

Definition 4.1. *The recurrence relation of the Gaussian Narayana sequence is given by:*

$$GN_n = GN_{n-1} + GN_{n-3},$$

with the initial values $GN_0 = 0, GN_1 = 1, GN_2 = 1 + i$ and $n \geq 3$.

Establishing a connection between the Gaussian Narayana sequence and its modular form, denoting $GN_i \pmod m$ with $\left\{ GN_i^{(m)} \right\}$, in which [20]:

$$\left\{ GN_i^{(m)} \right\} = \left\{ GN_0^{(m)}, GN_1^{(m)}, GN_2^{(m)}, GN_3^{(m)}, \ldots, GN_n^{(m)} \right\}.$$

Indeed, the recurrence relation of Gaussian Narayana numbers is preserved.

Theorem 4.2. $\left\{ GN_n^{(m)} \right\}$ *is the simply periodic form of the Gaussian Narayana sequence.*

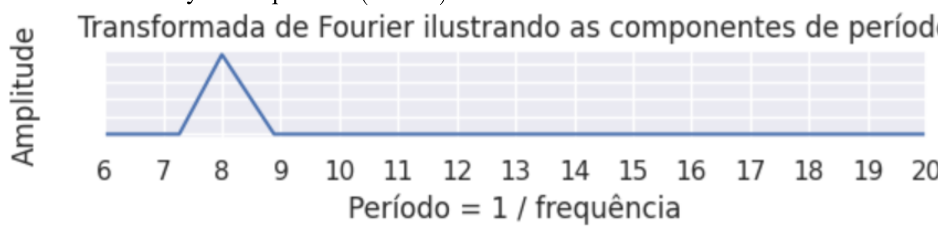
Proof. The proof follows analogously to the Theorem 2.2. □

Example 4.3. *The Gaussian Narayana sequence* $\pmod 3$ *with its first terms calculated from* GN_1 *and* $kGN(3) = 8$*:*

$$\left\{GN_n^{(3)}\right\} = \{1, 1+i, 1+i, 2+i, 2i, 1, i, 0, 1, 1+i, 1+i, 2+i, 2i, \ldots\}.$$

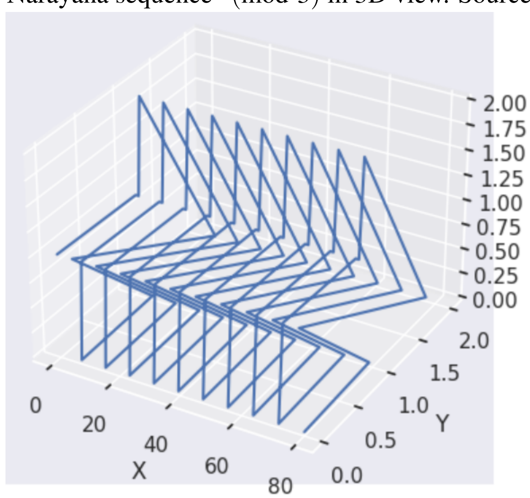
The Figure 4.1 detais the period of the sequence in the Example4.3, allowing to see the period 8.

Figure 4.1. The Gaussian Narayana sequence $\pmod 3$ from the Fourier Transform. Source: Elaborated by the authors



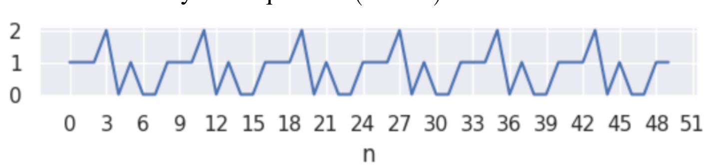
The spiral presented in Figure 4.2 provides a visualization of the terms of the sequence.

Figure 4.2. The Gaussian Narayana sequence $\pmod 3$ in 3D view. Source: Elaborated by the authors



In Figure 4.3 we can observe the cyclic pattern that occurs with the Gaussian Narayana sequence $\pmod 3$, presenting a cycle within the interval 0 to 7, i.e., period equals to 8 as seen earlier.

Figure 4.3. The Gaussian Narayana sequence $\pmod 3$ - Pattern. Source: Elaborated by the authors



Based on the matrix form of Gaussian Narayana numbers, it can be verified that [31]:

$$QN = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, v_N = \begin{bmatrix} 1 & 0 & i \\ i & 1-i & 0 \\ 0 & i & 1-i \end{bmatrix}.$$

So, we have:

$$QN^n v_R = \begin{bmatrix} GN_{n+1} & GN_{n-1} & GN_n \\ GN_n & GN_{n-2} & GN_{n-1} \\ GN_{n-1} & GN_{n-3} & GN_{n-2} \end{bmatrix}.$$

For the matrix $M = [a_{ij}]_{(K+1)\times(k+1)}$ with a_{ij} integer, $M \pmod m$ means that all entries of M are reduced modulo m, i.e., $M \pmod m = (a_{ij} \pmod m)$. Let $\langle QN \rangle_{p^\alpha} = \left\{QN^i \pmod{p^\alpha} | i \geq 0\right\}$ a cyclic group and $|\langle QN \rangle_{p^\alpha}|$ denotes the order of $\langle QN \rangle_{p^\alpha}$. From the Gaussian matrix form of the Narayana sequence, we have that $kGN(p^\alpha) = |\langle QN \rangle_{p^\alpha}|$.

Theorem 4.4. *Let t a positive integer such that $kGN(p) = kGN(p^t)$. Therefore, we have that $kGN(p^\alpha) = r^{\alpha-t}kGN(p)$ for $\alpha \geq t$. Particularly, if $kGN(p) \neq kGN(r^2)$, then: $kGN(p^\alpha) = p^{\alpha-t}kGN(p)$ com $\alpha > 1$.*

Proof. The proof follows analogously to the Theorem 2.4. □

Example 4.5. *For $p = 3$ and $q = 1$, $kGN(9) \equiv QN^{24} \equiv I \pmod 9$, then:*

$$QN^{24} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{24} = \begin{bmatrix} 5896 & 2745 & 4023 \\ 4023 & 1873 & 2745 \\ 2745 & 1278 & 1873 \end{bmatrix}_{\pmod 9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. The Gaussian Leonardo Sequence $\pmod m$

The Gaussian Leonardo sequence, investigated by Tasci [32], has its formula given by: $GL_n = L_n + iL_{n-1}$, in which L_n is the $n-$th term of Leonardo sequence and GL_n the $n-$th term of the Gaussian Leonardo sequence.

Definition 5.1. *The recurrence relation of the Gaussian Leonardo sequence is given by:*

$$GL_n = GL_{n-1} + GL_{n-2} + (1+i),$$

with the initial values $GL_0 = 1 - i$, $GN_1 = 1 + i$ and $n \geq 2$.

Lemma 5.2. *For $n \geq 1$, we have:*

$$GL_n = 2GL_{n-1} - GL_{n-3}.$$

Proof. Using the recurrence of the Leonardo sequence ($L_n = 2L_{n-1} - L_{n-3}$) and the Gaussian recurrence ($GL_n = L_n + iL_{n-1}$), we have:

$$\begin{aligned} GL_n &= L_n + iL_{n-1} \\ &= 2L_{n-1} - L_{n-3} + i(2L_{n-2} - L_{n-4}) \\ &= 2(L_{n-1} + iL_{n-2}) - (L_{n-3} + iL_{n-4}) \\ &= 2GL_{n-1} - GL_{n-3}. \end{aligned}$$

□

Therefore, we can establish a connection between the Gaussian Leonardo sequence and its modular form, denoting $GL_i \pmod m$ with $\left\{GL_i^{(m)}\right\}$, in which:

$$\left\{GL_i^{(m)}\right\} = \left\{GL_0^{(m)}, GL_1^{(m)}, GL_2^{(m)}, GL_3^{(m)}, \ldots, GL_n^{(m)}\right\}.$$

Indeed, the recurrence relation of Gaussian Leonardo numbers is preserved.

Theorem 5.3. $\left\{GL_n^{(m)}\right\}$ *is the simply periodic form of the Gaussian Leonardo sequence.*

Proof. The proof follows analogously to the Theorem 2.2. □

Figure 5.1. The Gaussian Leonardo sequence $\pmod 3$ from the Fourier Transform. Source: Elaborated by the authors

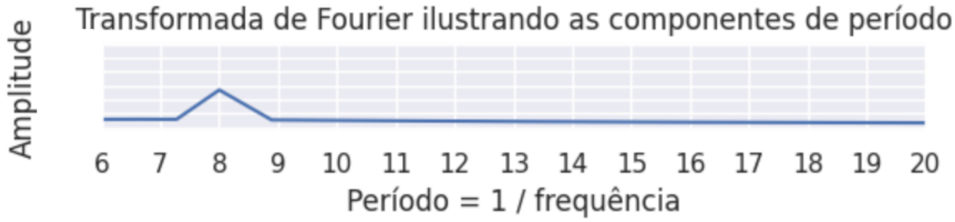
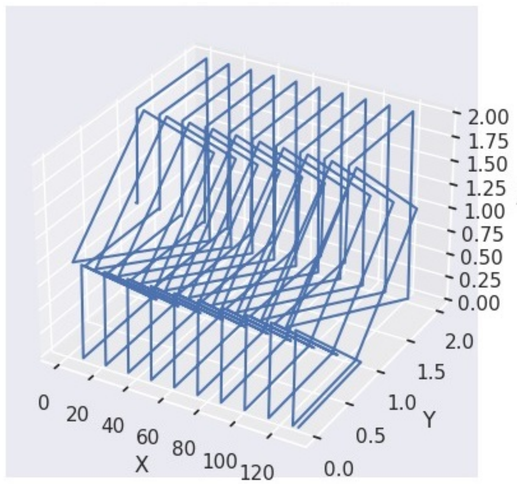


Figure 5.2. The Gaussian Leonardo sequence $\pmod 3$ in 3D view. Source: Elaborated by the authors



Example 5.4. *The Gaussian Leonardo sequence* $\pmod 3$ *with its first terms calculated from* GL_1 *and* $kGL(3) = 8$*:*

$$\left\{GL_n^{(3)}\right\} = \{1+i, i, 2, 2i, 0, 1, 2+i, 1+2i, 1+i, i, 2, 2i, 0, 1, 2+i, 1+2i, 1+i, \ldots\}.$$

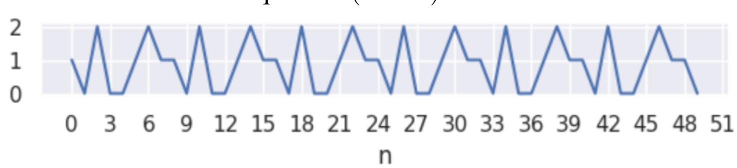
The Figure 5.1 details the period of the sequence in the Example 5.4, allowing to see the period 8.

The spiral presented in the Figure 5.2 provides a visualization of the terms of the sequence.

It is observed that 3D figures have better visibility of periodicity, as they have an extra dimension so that the sides can be visualized, allowing information to be found in a clearer and less overloaded way than in the case of 2D. It is important to realize that when viewing the figures in 3D, the sides allow for better interpretation and identification of the period of the sequences.

In Figure 5.3 we can observe the cyclic pattern that occurs with the Gaussian Leonardo sequence $\pmod 3$, presenting a cycle within the interval 0 to 7, i.e., period equals to 8 as seen earlier.

Figure 5.3. The Gaussian Leonardo sequence $\pmod 3$ - Pattern. Source: Elaborated by the authors



Based on the matrix form of Gaussian Leonardo numbers, it can be verified that [32]:

$$QL = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, v_L = \begin{bmatrix} GL_3 & GL_2 & GL_1 \\ GL_2 & GL_1 & GL_0 \\ GL_1 & GL_0 & GL_{-1} \end{bmatrix}.$$

Thus, we have that:

$$v_L QL^n = \begin{bmatrix} GL_{n+3} & GL_{n+2} & GL_{n+1} \\ GL_{n+2} & GL_{n+1} & GL_n \\ GL_{n+1} & GL_n & GL_{n-1} \end{bmatrix}.$$

For the matrix $M = [a_{ij}]_{(K+1)\times(k+1)}$ with a_{ij} integer, $M \pmod m$ means that all entries of M are reduced modulo m, i.e., $M \pmod m = (a_{ij} \pmod m)$. Let $\langle QL\rangle_{p^\alpha} = \left\{QL^i \pmod{p^\alpha} | i \geq 0\right\}$ a cyclic group and $|\langle QL\rangle_{p^\alpha}|$ denotes the order of $\langle QL\rangle_{p^\alpha}$. From the Gaussian matrix form of the Leonardo sequence, we have that $kGL(p^\alpha) = |\langle QL\rangle_{p^\alpha}|$.

Theorem 5.5. *Let t a positive integer such that $kGL(p) = kG(p^t)$. Therefore, we have that $kGL(p^\alpha) = r^{\alpha-t}kGL(p)$ for $\alpha \geq t$. Particularly, if $kGL(p) \neq kGL(r^2)$, then $kGL(p^\alpha) = p^{\alpha-t}kGL(p)$ com $\alpha > 1$.*

Proof. The proof follows analogously to the Theorem 2.4. □

Example 5.6. *For $p = 3$ and $q = 1$, $kGL(9) \equiv QL^{24} \equiv I \pmod 9$, then:*

$$QL^{48} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{24} = \begin{bmatrix} 196417 & 121392 & 75024 \\ -75024 & -46367 & -28656 \\ -121392 & -75024 & -46367 \end{bmatrix}_{\pmod 9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Final Considerations

The current research provides an analysis of Gaussian numbers in third-order recurrent sequences, in association with finite groups. During this study, definitions were introduced that enabled the analysis and transformation of infinite sequences into finite sequences. For a better understanding, visualization techniques were explored, including the use of Fourier transform and other graphics, developed in the Google Colab environment using available libraries and programming.

Transforming infinite recurrent sequences into finite ones through the study of group theory is a relevant approach to mathematics, as it allows for a more accessible and structured analysis of these sequences. Cyclic groups play an essential role in this process, as they are groups that can be generated by a single element, simplifying the representation and understanding of these sequences. Additionally, the visualization of sequences can be enhanced with the use of the Fourier Transform, a tool that decomposes a periodic function into a sum of sine and cosine functions.

Therefore, by combining concepts from group theory, especially cyclical groups, with the Fourier transform, we have shown in this article that it is possible to transform infinite recurrent sequences into finite ones, as well as to better understand their underlying properties and patterns. This approach enables the mathematical analysis of sequences, providing the opportunity to identify practical applications in areas such as signal processing, communications, and data analysis, where understanding the properties of sequences is essential. As a result, mathematical theorems were established that examined the patterns of these sequences and their corresponding cycles.

As a perspective for future research, it is aimed to investigate other mathematical theorems, aiming for a generalization of sequences into finite groups.

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