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Goodness-of-fit tests based on Kullback-Leibler divergence for bladder cancer survival analysis: Applications to exponentiated exponential distribution

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Abstract – Bladder cancer is among the ten most common types of cancer worldwide, with approximately 550,000 new cases occurring each year. It accounts for comprehensively compared to 3% of all newly diagnosed cancer cases and contributes to 2.1% of cancer-related deaths globally. This article introduces goodness-of-fit tests that aim to fit the exponentialized exponential distribution. These tests are based on the Kullback-Leibler difference and have been applied to censored and complete samples of Bladder Cancer Patients. We calculated critical values and statistical power measurements, considering the best and worst bandwidth scenarios. We then comprehensively compared essential values and power across various parameters, accounting for optimal and suboptimal bandwidth choices derived from the Kullback–Leibler difference. In the final phase of our study, we used a dataset of individuals diagnosed with bladder cancer to demonstrate the practical applicability of our proposed research. Finally, this modeling type can benefit researchers and healthcare professionals through time-to-event analysis (survival analysis), investigation of events, medical decision-making, and risk prediction.

Subject Classification (2020): 62N02, 62N03

1. Introduction

Bladder cancer is one of the ten most prevalent types of cancer worldwide, representing a significant global health concern. Each year, approximately 550,000 new cases are diagnosed, making bladder cancer a substantial contributor to the worldwide cancer burden. This type of cancer accounts for about 3% of all newly diagnosed cancer cases, reflecting its widespread impact. Furthermore, bladder cancer is responsible for approximately 2.1% of all cancer-related deaths, highlighting its severity and the challenges associated with its treatment and management. Various risk factors, including smoking, exposure to certain industrial chemicals, chronic bladder inflammation, and a history of schistosomiasis in certain regions, influence the incidence of bladder cancer. Despite advances in medical research and treatment options, bladder cancer remains a formidable disease, with significant implications for patient quality of life and overall public health. Ongoing research and clinical efforts focus on improving early detection, treatment outcomes, and survival rates for individuals diagnosed with bladder cancer [1].

An exponentiated exponential distribution is a statistical model describing a measured variable's distribution. For example, a patient's survival times. This model determines the probability of an event, e.g.,

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death, occurring within a given time and calculates this probability over the distribution of the observed variable, e.g., survival time of patients. The exponential distribution is primarily used to model how events occur over time, and this distribution is particularly suitable for rare or irregularly occurring events. Thus, using this distribution allows us to understand specific patterns and trends in the data better, such as the survival time of patients. Modeling the mortality rates of bladder cancer patients with an exponentiated exponential distribution can be used to examine the probability of a death event over a given time.

This type of modeling can provide researchers and healthcare professionals with the following benefits:

i. Event Duration Analysis (Survival Analysis): Exponentialized exponential distribution is a frequently used model for event duration analysis. This analysis evaluates how long patients survive until a specific event (e.g., death) by examining the time for a particular event.
ii. Examining Events: Modeling mortality rates of bladder cancer patients can be used to understand what factors influence the risk of death in a particular population. These factors may include genetic characteristics, treatment methods, age, and gender.
iii. Medical Decision Making: Modeling results can help determine treatment processes and strategies for patients with bladder cancer. For example, it can be used to evaluate how a particular treatment method or medication affects patients' survival.
iv. Risk Forecasting: Exponentialized exponential distribution can be used to estimate the probability of an event occurring within a given time. This could be useful for predicting patients' future survival and adapting treatment plans accordingly.

Moreover, states, public institutions, organizations, and businesses use decision-making tools when planning for the future. The most crucial step in this observation-based decision-making process is modeling the population from which the observations are obtained. Naturally, the model's accuracy will be a critical consideration for the decision-making process. Inaccurate findings from a model that isn't determined appropriately could be irreversible. A probability distribution is a model representing this population, and goodness of fit tests are performed to determine whether a given probability distribution is appropriate for that population. Since the beginning, statisticians have begun their analysis by distributing the observed data. Then, they verified that their chosen distribution was suitable for the observed data. As a result, various test processes have been developed over time, and the study of these procedures is known as goodness of fit [2]. Pearson [3] invented the chi-square test in 1900, which helped to pioneer appropriateness tests. Since then, numerous more tests have been devised, each reflecting the subject's relevance and demands.

Furthermore, it is impossible to dispute the significance of statistical or probabilistic modeling in the modern world. High-speed computers have enabled the development and using complicated models for crucial operations. These models facilitate Effective decision-making and associated statistical analyses in various domains, including marketing, medicine, management, politics, military systems, and food science. Evaluating the validity of models with statistical distributions is known as "goodness of fit."

It is a fundamental and occasionally overlooked part of modeling work. Nonetheless, a wide range of statistical or probabilistic distribution models have found widespread use in engineering, science, economics, and medicine. It's critical to assess these models' applicability or determine how well the data fits the suggested distribution model. To ensure that the selected model accurately represents the underlying population, various distribution families have been proposed for goodness of fit tests, particularly in complete and censored samples. These families are chosen because they offer a flexible framework for capturing the characteristics of different types of data, and the Kullback-Leibler divergence measure is utilized to quantify the fit between the data and the proposed distribution. This approach is critical as it allows for a more precise assessment of the model's suitability across various applications. Here are a few of these studies. Arizona and Ohta [4] presented a normal distribution appropriateness test for a complete

sample based on the Kullback-Leibler divergence measure, the extended version of entropy. To determine which test was the strongest compared to other tests using normal distributions, they compared it to the Durbin version of the K-S test, Cramer Von Mises, weighted Cramer Von Mises, and Chi-Square tests. Şenoğlu and Sürücü [5] used the Kullback-Leibler divergence measure, Shapiro Wilk, Tiku's test, and sample correlation test for Normal, Exponential, and Uniform distributions to compare tests for various distributions (skewed, long, and short-tailed symmetric). Based on the test results, it was observed that the test with the Kullback-Leibler divergence measure generally has more power in distributions with short tails than in distributions with long tails. Choi et al. [6] put out an exponential test that relied on the divergence metric of Kullback-Leibler. Van Es and Correa's entropy was utilized as an estimator of Shannon entropy. Its more significant power than other tests has been determined by comparison with different goodness of fit tests. Park [7] proposed an exponentiality test specifically tailored for type-2 censored data, leveraging Kullback-Leibler insights. When scrutinizing the test's statistical power across alternative distributions such as Gamma, Weibull, and Chi-square, it was observed that the suggested statistical metric exhibited greater sensitivity when applied to distributions with hazard functions that show a consistent upward trend.

In a related study, Lim and Park [8] compared statistical power comprehensively, focusing on partial Kullback-Leibler divergence within the context of type-2 censored samples. This comparison encompassed distributions characterized by monotone decreasing, increasing, and non-monotone hazard functions, particularly for the Exponential and Normal distributions. Notably, the Tukey test exhibited superior power in the normality test. At the same time, distributions marked by monotonically increasing hazard functions displayed greater power compared to other tests in the context of the exponentiality test.

Expanding on the theme of goodness-of-fit tests, Balakrishnan et al. [9] proposed an exponentiality goodness-of-fit test grounded in Kullback-Leibler principles, specifically for progressive type-2 censored data. This test demonstrated strength, particularly in scenarios involving alternatives with non-monotonic hazard functions, as revealed through comparisons with various options.

Lim and Park [10] aim to develop a Kullback-Leibler Divergence-based information measure and goodness of fit test for working with censored datasets. In particular, they focus on how this test can be applied to cases of Type II censoring. Rad et al. [11] proposed a goodness-of-fit test for progressive type-2 censored data based on Kullback-Leibler information. For model parameters related to Pareto, Log-normal, and Weibull distributions, their analysis considered both maximum and approximate maximum likelihood estimators, evaluating the test's effectiveness over various choices and sample sizes. Gurevich and Davidson [12] show how statistical tests based on the Kullback-Leibler Divergence can be standardized to test their suitability for particular distributions.

Furthermore, Park and Pakyari [13] presented Kullback-Leibler data and conducted a comparative analysis of goodness-of-fit test results, focusing on progressive type-2 censored data. Meanwhile, Elsherpieny et al. [14] delved into the challenge of discriminating between gamma and log-logistic distributions in the context of progressive type-censored samples. They employed the minimized Kullback-Leibler divergence ratio method and the maximum likelihood ratio approach to differentiate between these two distributions. Simulation experiments were conducted to identify optimal choices, especially in cases with limited sample sizes. Additionally, asymptotic findings and selection probabilities were estimated to determine the minimal sample size required for effective discrimination.

Lastly, Bitaraf et al. [15] proposed a novel Kullback-Leibler distance test based on Verma's entropy, adding to the statistical methods used in similar research. The results regarding mean square error, critical values, and powers were examined against a few alternatives for conformity with the normal and exponential distributions. The differential entropy $H(f)$ of the random variable X with distribution function $F$and continuous density function $f$is defined as follows:

$$H(f) = -\int_{-\infty}^{\infty} f(x)\log f(x)\,dx$$

Using a novel Kullback-Leibler knowledge under the type 2 censored sample that advanced in his paper, Noughabi [16] created a general goodness of fit test. He contrasted the test's robustness under various censorship models for the exponential distribution. While Kullback-Leibler divergence has been used in the literature to build goodness of fit tests for numerous distribution families in complete sampling, this number is notably inadequate when considering both complete and censored samples. The progressive type of censored samples is the most widely used among the censored samples. This study is crucial to calculate critical values, obtain power comparisons under various alternatives, and handle goodness of fit tests for various continuous distribution families based on Kullback-Leibler divergence under complete and progressive type censored samples. Among the estimators used, the Vasicek estimator, Van Es's estimator, and Correa estimator are given by

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{m}\log\left\{\frac{n}{2m}X_{(i+m)} - X_{(i-m)}\right\}$$

$$HVE_{mn} = \frac{1}{n-m}\sum_{i=1}^{n-m}\log\left(\frac{n+1}{m}\left(X_{(i+m)} - X_{(i)}\right)\right) + \sum_{k=m}^{n}\frac{1}{k} + \log(m) - \log(n+1)$$

and

$$HC_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum_{j=i-m}^{i+m}\left(X_{(j)} - \bar{X}_{(i)}\right)(j-i)}{n\sum_{j=i-m}^{i+m}\left(X_{(j)} - \bar{X}_{(i)}\right)^2}\right)$$

Where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$, $i > n$, and $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ the order statistics are based on a random sample of size n. The main topic of this article is Kullback-Leibler information-based appropriateness tests for exponential and exponentiated exponential Poisson distributions. Some new goodness-of-fit tests are offered for given distributions using various entropy estimates. Next, the critical values of the suggested test statistics for different sample sizes were found using a Monte Carlo simulation. Under the best and worst bandwidth, essential values and powers are produced. Furthermore, power values have been contrasted with other options.

2. The Proposed Tests

2.1 Goodness of Fit Test for Exponentiated Exponential Distribution

The Exponentiated Exponential distribution introduced by Gupta and Kundu [17] has attracted much attention with the generalization of the Exponential distribution. Here, the exponential distribution is obtained when $\lambda = 1$ in Figure 1 and scale parameter distribution. With the exponential distribution $x > 0$, $\beta > 0$, and $\lambda > 0$, the probability density and distribution functions are as follows:

$$f(x) = \lambda\beta e^{-x\beta}\left(1 - e^{-x\beta}\right)^{\lambda-1}$$

and

$$F(x) = \left(1 - e^{-x\beta}\right)^{\lambda}$$

For a random variable with an Exponentiated Exponential distribution with λ and β parameters, its representation will be used (X∼Exponentiated Exponential(λ, β)). A random variable's expected value and variance expressed in this way are as follows:

$$E(X) = \{\psi(\lambda + 1) + C\}/\beta$$

and

$$Var(X) = \{\pi^2 - 6\psi'(\lambda + 1)/6\beta^2\}$$

2.1.1. Complete Sample Status

Let $X_1, X_2, \dots, X_n$ be independent random variables having Exponentiated Exponential(λ, β) distribution with λ and β parameters. Then, the log-likelihood function is provided by,

$$\ln L\,(\lambda, \beta) = \sum_{i=1}^{n} \ln f\,(x_i) \cong n \ln \lambda + n \ln \beta - \beta \sum_{i=1}^{n} x_i + (\lambda - 1) \sum_{i=1}^{n} \ln\left(1 - e^{-x_i \beta}\right) \qquad (2.1)$$

The hypothesis to be tested here,

H_0: The population probability distribution is Exponentiated Exponential
H_1: is not

or is provided by

$$H_0: F_0(x) = \left(1 - e^{-x\beta}\right)^{\lambda}$$

$$H_1: F_0(x) \neq \left(1 - e^{-x\beta}\right)^{\lambda}$$

The statistics to be utilized for testing the mentioned hypothesis, based on the log-likelihood function provided in (2.1), are as follows:

i. Vasicek's test (TV)

ii. VanEs' test (TVE)

iii. Correa's test (TC)

These statistics are commonly employed in the context of the hypothesis being discussed [18-20]. We reject H_0 large values TV_{mn} [1].

$$\begin{aligned} TV_{mn} &= -HV_{mn} - \frac{1}{n}\left(n \ln \lambda + n \ln \beta - \beta \sum_{i=1}^{n} x_i + (\lambda - 1) \sum_{i=1}^{n} ln\left(1 - e^{-x_i \beta}\right)\right) \\ &= -HV_{mn} - ln\,\lambda - ln\,\beta - \frac{\beta}{n} - \frac{(\lambda - 1)}{n} \sum_{i=1}^{n} ln\left(1 - e^{-x_i \beta}\right) \\ TVE_{mn} &= -HVE_{mn} - \frac{1}{n}\left(n \ln \lambda + n \ln \beta - n \sum_{i=1}^{n} x_i + (\lambda - 1) \sum_{i=1}^{n} ln\left(1 - e^{-x_i \beta}\right)\right) \\ &= -HVE_{mn} - ln\,\lambda - ln\,\beta - \frac{\beta}{n} - \frac{(\lambda - 1)}{n} \sum_{i=1}^{n} ln\left(1 - e^{-x_i \beta}\right) \\ TC_{mn} &= -HC_{mn} - \frac{1}{n}\left(n \ln \lambda + n \ln \beta - n \sum_{i=1}^{n} x_i + (\lambda - 1) \sum_{i=1}^{n} ln\left(1 - e^{-x_i \beta}\right)\right) \\ &= -HC_{mn} - ln\,\lambda - ln\,\beta - \frac{\beta}{n} - \frac{(\lambda - 1)}{n} \sum_{i=1}^{n} ln\left(1 - e^{-x_i \beta}\right) \end{aligned}$$

2.1.2 Progressively Type-II Censored Status

Let $X_{1:m:n}^{\boldsymbol{R}}, X_{2:m:n}^{\boldsymbol{R}}, \ldots, X_{m:m:n}^{\boldsymbol{R}}$ be independent random variables having Exponentiated Exponential (λ, β) distribution with λ and β parameters. Then, the log-likelihood function is given by

$$\ln L(\lambda,\beta) \propto m\ln\lambda + m\ln\beta - \beta\sum_{i=1}^{m} x_i + (\lambda-1)\sum_{i=1}^{m}\ln\left(1-e^{-x_i\beta}\right) + \sum_{i=1}^{m} R_i \ln\left(1-\left(1-e^{-x_i\beta}\right)^{\lambda}\right)$$

Maximum likelihood estimators of the λ and β parameters are obtained from the solution of the likelihood equations concerning $\hat{\lambda}$ and $\hat{\beta}$.

$$\frac{\partial \ln L(\lambda,\beta)}{\partial\lambda} = \frac{m}{\lambda} + \sum_{i=1}^{m}\ln\left(1-e^{-x_i\beta}\right) + \sum_{i=1}^{m} R_i \frac{\left(1-e^{-x_i\beta}\right)^{\lambda}\ln\left(1-e^{-x_i\beta}\right)}{\left(1-(1-e^{-x_i\beta})^{\lambda}\right)} = 0$$

$$\frac{\partial \ln L(\lambda,\beta)}{\partial\beta} = \frac{m}{\beta} - \sum_{i=1}^{m} x_i + (\lambda-1)\sum_{i=1}^{m}\frac{x_i e^{-x_i\beta}}{\left(1-e^{-x_i\beta}\right)} - \lambda\sum_{i=1}^{m} R_i \frac{\left(1-e^{-x_i\beta}\right)^{\lambda} x_i e^{-x_i\beta}}{\left(1-(1-e^{-x_i\beta})^{\lambda}\right)} = 0$$

Since these equations have no analytical solutions, they are only solved through numerical techniques and maximum likelihood estimations of their parameters. The following hypotheses are put out in this context to verify conformance to the Exponentiated Exponential distribution.

$$H_0: F_0 = \left(1-e^{-x\beta}\right)^{\lambda}$$

$$H_A: F_0 \neq \left(1-e^{-x\beta}\right)^{\lambda}$$

The suitability of the Exponentiated Exponential distribution for progressive type-censored samples is assessed by calculating the Kullback-Leibler information, which is obtained as follows:

$$I_{1\ldots m:m:n}(f:f^0) = -n\bar{H}_{1\ldots m:m:n} - \left(\begin{matrix} m\,ln\,\lambda + m\,ln\,\beta - \beta\sum_{i=1}^{m} x_i + (\lambda-1)\sum_{i=1}^{m} ln\left(1-e^{-x_i\beta}\right) \\ + \sum_{i=1}^{m} R_i\, ln\left(1-\left(1-e^{-x_i\beta}\right)^{\lambda}\right) \end{matrix}\right)$$

$$= -n\bar{H}_{1\ldots m:m:n} - m\,ln\,\lambda - m\,ln\,\beta + \beta\sum_{i=1}^{m} x_i - (\lambda-1)\sum_{i=1}^{m} ln\left(1-e^{-x_i\beta}\right)$$

$$-\sum_{i=1}^{m} R_i\, ln\left(1-\left(1-e^{-x_i\beta}\right)^{\lambda}\right)$$

The estimators are the above-mentioned by taking derivatives from the following parameters λ and β the Kullback-Leibler the aforesaid data. The test statistic below is generated by substituting the maximum likelihood estimators for the λ and β parameters. The following formula is used to get the Kullback-Leibler information test statistics for the progressive type of censored sample.

$$\mathrm{TA}(w,n,m) = -\frac{1}{n}\sum_{i=1}^{m}\log m\sum\left\{\frac{G\left(X_{(i+w:m:n)};\hat{\theta}\right) - G\left(X_{(i-w:m:n)};\hat{\theta}\right)}{X_{i+w:m:n} - X_{i-w:m:n}}\right\} + \frac{1}{n}\sum_{i=1}^{m} R_i \log\sum\left\{\frac{1-m/n}{1-G\left(X_{(i:m:n)};\hat{\theta}\right)}\right\}$$

3. Simulation Study

This section discusses the goodness of fit tests based on the Kullback-Leibler mismatch under complete and progressively censored samples for the Exponentially Expanded Distribution. TV, TVE, and TC test statistics are given in the complete sample case, and critical values with 0.05 significance levels for these tests were obtained with the simulation study. In addition, the powers of this goodness of fit test for the relevant distribution under different alternatives are obtained and compared with the famous test K-S test. Critical values are obtained for the distributions considered under progressively censored samples, and complete sample cases and power analysis are performed using different alternatives. All simulations mentioned here are performed over 10000 repetitions, considering various sample sizes. Inf values in calculations indicate cases where the critical value is numerically infinite, while NA values indicate cases where calculation is impossible. These cases reflect testing limitations encountered at specific sample sizes or bandwidths. Since the distribution of test statistics is complex for analytical evaluation, critical values are calculated using Monte Carlo simulation. The following steps are followed to determine the critical values of the proposed test statistics.

i. Create a sample of size n from the exponentially exponential distribution,
ii. Calculate the proposed statistics according to the created samples.

3.1. Critical Values for Exponentiated Exponential Distribution

Tables 1-4 present the critical values for the Exponentiated Exponential Distribution obtained through simulation.

Table 1. Critical values for the Exponentiated Exponential test in case n = 10, 20, 30, and 40

	n	**10**			**20**			**30**			**40**		
	Tests	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**
Bandwidth (w)	**1**	NA	-2.79	-0.83	NA	-8.82	-6.75	NA	-14.90	-12.75	NA	-20.92	-18.62
	2	NA	-2.38	-0.95	NA	-8.23	-6.73	NA	-14.45	-12.82	NA	-20.43	-18.66
	3	-0.34	-2.26	-0.94	NA	-8.00	-6.71	NA	-14.17	-12.76	NA	-20.38	-18.90
	4	-0.61	-2.27	-0.99	NA	-8.09	-6.85	NA	-14.16	-12.87	NA	-20.12	-18.72
	5	0	0	0	Inf	-7.79	-6.63	NA	-14.05	-12.82	NA	-20.02	-18.72
	6	0	0	0	-6.42	-8.01	-6.89	NA	-13.74	-12.57	NA	-20.25	-18.97
	7	0	0	0	-6.55	-8.00	-6.90	NA	-13.78	-12.63	NA	-20.03	-18.84
	8	0	0	0	-6.51	-7.83	-6.73	-12.04	-13.84	-12.77	NA	-19.86	-18.71
	9	0	0	0	-6.52	-7.80	-6.69	-12.38	-13.88	-12.76	NA	-19.99	-18.88
	10	0	0	0	0	0	0	-12.36	-13.69	-12.61	Inf	-19.77	-18.70
	11	0	0	0	0	0	0	-12.36	-13.73	-12.65	-18.18	-19.84	-18.74
	12	0	0	0	0	0	0	-12.60	-13.85	-12.76	-18.47	-19.93	-18.85
	13	0	0	0	0	0	0	-12.44	-13.60	-12.58	-18.46	-19.77	-18.72
	14	0	0	0	0	0	0	-12.65	-13.72	-12.69	-18.51	-19.80	-18.78
	15	0	0	0	0	0	0	0	0	0	-18.76	-20.00	-18.97
	16	0	0	0	0	0	0	0	0	0	-18.70	-19.88	-18.83
	17	0	0	0	0	0	0	0	0	0	-18.66	-19.81	-18.79
	18	0	0	0	0	0	0	0	0	0	-18.47	-19.58	-18.54
	19	0	0	0	0	0	0	0	0	0	-18.78	-19.77	-18.79
	20 and above	0	0	0	0	0	0	0	0	0	0	0	0

Table 2. Critical values for the Exponentiated Exponential test in case n = 50, 60, and 70

	n	50			60			70		
	Tests	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**
Bandwidth (w)	**1**	NA	-26.93	-24.53	NA	-33.52	-30.98	NA	-39.88	-37.32
	2	NA	-26.79	-25.03	NA	-33.12	-31.22	NA	-39.58	-37.56
	3	NA	-26.55	-24.91	NA	-32.89	-31.30	NA	-39.11	-37.38
	4	NA	-26.26	-24.80	NA	-32.57	-31.09	NA	-38.77	-37.18
	5	NA	-26.17	-24.85	NA	-32.61	-31.14	NA	-38.84	-37.35
	6	NA	-26.33	-25.07	NA	-32.49	-31.17	NA	-39.02	-37.64
	7	NA	-26.13	-24.87	NA	-32.51	-31.20	NA	-38.66	-37.34
	8	NA	-25.98	-24.80	NA	-32.44	-31.20	NA	-38.67	-37.37
	9	NA	-25.96	-24.82	NA	-32.47	-31.24	NA	-38.58	-37.33
	10	NA	-26.24	-25.09	NA	-32.26	-31.11	NA	-38.61	-37.40
	11	NA	-25.88	-24.74	NA	-32.41	-31.25	NA	-38.61	-37.40
	12	NA	-26.02	-24.93	NA	-32.48	-31.29	NA	-38.58	-37.42
	13	-24.09	-25.92	-24.86	NA	-32.30	-31.17	NA	-38.52	-37.40
	14	-24.27	-25.79	-24.75	NA	-32.40	-31.31	NA	-38.62	-37.48
	15	-24.67	-26.07	-25.04	Inf	-32.20	-31.15	NA	-38.82	-37.75
	16	-24.57	-25.86	-24.82	-30.66	-32.31	-31.25	NA	-38.66	-37.57
	17	-24.65	-25.91	-24.89	-30.88	-32.36	-31.35	NA	-38.45	-37.39
	18	-24.49	-25.74	-24.72	-30.83	-32.18	-31.14	-36.69	-38.57	-37.56
	19	-24.69	-25.91	-24.89	-30.93	-32.21	-31.19	-36.87	-38.41	-37.38
	20	-24.65	-25.79	-24.78	-31.10	-32.32	-31.28	-37.06	-38.50	-37.48
	21	-24.99	-26.15	-25.10	-31.00	-32.22	-31.24	-36.96	-38.33	-37.30
	22	-24.81	-25.95	-24.88	-31.16	-32.39	-31.38	-37.18	-38.44	-37.45
	23	-24.72	-25.76	-24.75	-30.84	-31.96	-31.02	-37.62	-38.83	-37.83
	24	-25.00	-25.90	-24.92	-30.87	-31.97	-30.99	-37.27	-38.46	-37.52
	26	0	0	0	-30.94	-32.04	-31.06	-37.45	-38.61	-37.66
	27	0	0	0	-31.09	-32.14	-31.15	-37.20	-38.33	-37.42
	28	0	0	0	-31.27	-32.24	-31.27	-37.01	-38.16	-37.15
	29	0	0	0	-30.80	-31.78	-30.77	-37.12	-38.20	-37.21
	30	0	0	0	0	0	0	-37.48	-38.46	-37.52
	31	0	0	0	0	0	0	-37.67	-38.65	-37.71
	32	0	0	0	0	0	0	-37.64	-38.62	-37.63
	33	0	0	0	0	0	0	-37.34	-38.31	-37.32
	34	0	0	0	0	0	0	-37.49	-38.35	-37.37
	35 and above	0	0	0	0	0	0	0	0	0

Table 3. Critical values for the Exponentiated Exponential test in case n = 80, 90, and 100

	n	80			90			100		
	Tests	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**	**TV**	**TVE**	**TC**
Bandwidth (w)	**1**	NA	-46.19	-43.54	NA	-52.49	-49.79	NA	-59.27	-56.53
	2	NA	-45.69	-43.69	NA	-52.31	-50.20	NA	-58.67	-56.56
	3	NA	-45.70	-43.89	NA	-51.87	-50.05	NA	-58.41	-56.53
	4	NA	-45.41	-43.74	NA	-51.78	-50.11	NA	-58.07	-56.35
	5	NA	-45.26	-43.68	NA	-51.82	-50.25	NA	-58.32	-56.71
	6	NA	-45.43	-43.94	NA	-51.79	-50.26	NA	-57.81	-56.27
	7	NA	-45.37	-43.99	NA	-51.85	-50.37	NA	-58.22	-56.73
	8	NA	-45.12	-43.77	NA	-51.57	-50.18	NA	-58.25	-56.82
	9	NA	-44.97	-43.68	NA	-51.35	-50.02	NA	-57.80	-56.42
	10	NA	-44.77	-43.51	NA	-51.43	-50.16	NA	-57.74	-56.37
	11	NA	-45.54	-44.28	NA	-51.30	-50.04	NA	-57.79	-56.51
	12	NA	-44.91	-43.69	NA	-51.38	-50.11	NA	-57.89	-56.58
	13	NA	-45.02	-43.86	NA	-51.31	-50.09	NA	-57.92	-56.66
	14	NA	-44.78	-43.61	NA	-51.44	-50.26	NA	-57.88	-56.64
	15	NA	-45.17	-44.00	NA	-51.27	-50.07	NA	-57.73	-56.50
	16	NA	-44.81	-43.69	NA	-51.16	-50.01	NA	-58.01	-56.81
	17	NA	-44.92	-43.84	NA	-51.54	-50.42	NA	-57.99	-56.84
	18	NA	-44.53	-43.47	NA	-51.09	-49.96	NA	-57.62	-56.47
	19	NA	-44.74	-43.71	NA	-51.16	-50.11	NA	-57.85	-56.72
	20	Inf	-45.12	-44.07	NA	-51.31	-50.20	NA	-57.61	-56.49
	21	-43.1	-44.9	-43.8	NA	-51.11	-50.03	NA	-57.65	-56.58
	22	-43.1	-44.7	-43.6	NA	-51.19	-50.12	NA	-57.85	-56.76
	23	-43.4	-44.8	-43.8	-49.31	-51.26	-50.20	NA	-57.67	-56.60
	24	-43.4	-44.8	-43.8	-49.87	-51.52	-50.47	NA	-57.67	-56.60
	26	-43.5	-44.8	-43.8	-49.76	-51.20	-50.17	NA	-57.98	-56.98
	27	-43.8	-45.0	-44.0	-49.69	-51.09	-50.09	-55.77	-57.51	-56.44
	28	-43.5	-44.7	-43.7	-49.84	-51.18	-50.16	-56.12	-57.68	-56.68
	29	-43.2	-44.4	-43.4	-49.49	-50.76	-49.76	-56.48	-57.92	-56.92
	30	-43.5	-44.7	-43.7	-50.13	-51.34	-50.36	-56.29	-57.70	-56.67
	31	-43.5	-44.7	-43.7	-50.26	-51.40	-50.41	-56.63	-57.93	-56.94
	32	-43.5	-44.6	-43.6	-49.85	-51.01	-50.04	-56.63	-57.90	-56.91
	33	-43.6	-44.7	-43.7	-49.97	-51.19	-50.20	-56.18	-57.39	-56.40
	34	-44.0	-45.0	-44.1	-49.94	-51.09	-50.13	-56.27	-57.43	-56.48
	35	-43.7	-44.7	-43.7	-49.70	-50.88	-49.87	-56.19	-57.35	-56.38
	36	-43.7	-44.8	-43.8	-50.16	-51.25	-50.32	-56.72	-57.89	-56.92
	37	-43.7	-44.6	-43.6	-50.14	-51.19	-50.24	-56.36	-57.49	-56.54
	38	-43.9	-44.7	-43.8	-50.04	-51.10	-50.14	-56.34	-57.44	-56.52
	39	-43.9	-44.7	-43.7	-49.91	-50.93	-49.96	-56.31	-57.37	-56.45
	40	0	0	0	-50.16	-51.09	-50.17	-56.54	-57.55	-56.63
	41	0	0	0	-50.49	-51.39	-50.44	-56.46	-57.46	-56.55
	42	0	0	0	-50.08	-51.05	-50.05	-56.62	-57.66	-56.69
	43	0	0	0	-50.29	-51.14	-50.21	-56.67	-57.64	-56.69
	44	0	0	0	-50.29	-51.11	-50.12	-56.71	-57.65	-56.71
	45	0	0	0	0	0	0	-56.63	-57.58	-56.64
	46	0	0	0	0	0	0	-56.57	-57.42	-56.55
	47	0	0	0	0	0	0	-56.84	-57.72	-56.77
	48	0	0	0	0	0	0	-56.78	-57.57	-56.63
	49	0	0	0	0	0	0	-56.85	-57.64	-56.65

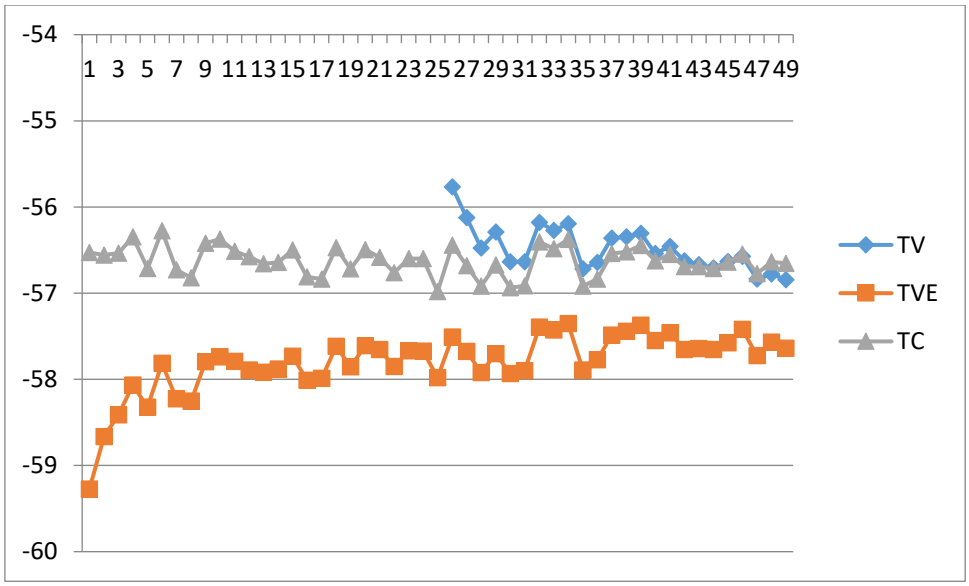

Figure 1. Critical values with 0.05 significance level for tests for goodness to Exponentiated Exponential distribution versus bandwidth for n = 100

Observing the graphs of critical values for n = 100 in Figure 1, it becomes evident that as the bandwidth w increases, the critical values for the TVE test remain relatively constant. On the other hand, the critical values for the TV test tend to stabilize as they decrease, and the critical values for the TC test show an increasing trend in stability.

3.2. Exponentiated Exponential under Complete Sample

This section analyzes the best and worst powers concerning the choice of bandwidth w under various distributions and parameter scenarios, assuming that the true distribution is the Exponentiated Exponential distribution.

Table 5. If the true distribution is $Weibull(5,3)$, the best bandwidth w and corresponding powers (Complete Sample)

n	K-S	TV			TVE			TC		
	Power	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power
10	0	4	-0.53472	0.0057	1	-2.80384	0.0064	1	-0.87421	0.0063
20	0.0025	6	-6.5173	0.0015	6	-8.0926	0.0015	6	-6.98912	0.0016
30	0.01	8	-12.1032	4.00E-04	4	-14.0371	6.00E-04	4	-12.7877	6.00E-04
40	0.0193	14	-18.4543	2.00E-04	4	-20.2122	2.00E-04	4	-18.8493	2.00E-04
50	0.0338	20	-24.7359	1.00E-04	3	-26.3433	1.00E-04	10	-24.9431	1.00E-04
60	0.0563	1	NA	0	1	-33.5308	0	1	-30.9908	0
70	0.0777	1	NA	0	1	-39.903	0	1	-37.2755	0
80	0.1141	1	NA	0	18	-44.9686	1.00E-04	18	-43.9219	1.00E-04
90	0.1491	1	NA	0	1	-52.6165	0	1	-49.8152	0
100	0.1678	1	NA	0	1	-59.2637	0	1	-56.5043	0

Table 6. If the true distribution is $Weibull(5,3)$, worst bandwidth w and corresponding powers (Complete Sample)

n	TV			TVE			TC		
	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power
10	2	NA	0	4	-2.16067	0.0047	4	-0.87718	0.0048
20	5	Inf	0	9	-7.74306	0.0012	1	-6.56822	0.0011
30	7	NA	0	14	-13.4926	4.00E-04	14	-12.4562	4.00E-04
40	19	-18.7924	0	19	-19.7692	0	19	-18.7544	0
50	24	-25.0013	0	24	-25.9611	0	24	-24.9456	0
60	29	-31.2907	0	29	-32.1708	0	29	-31.2009	0
70	34	-37.6988	0	34	-38.5237	0	34	-37.5905	0
80	39	-43.9757	0	39	-44.7696	0	39	-43.8098	0
90	44	-50.1323	0	44	-50.9229	0	44	-49.9419	0
100	49	-56.5206	0	49	-57.2733	0	49	-56.3515	0

It was understood that TV, TVE, and TC tests could not be performed against the worst w, and they did not have power; this situation is somewhat important for selecting w for TV, TVE, and TC tests. When Tables 5 and 6 are examined for testing the suitability of $Weibull(5,3)$ the population with A distribution to the Exponentiated Exponential distribution, the power of TV, TVE, and TC tests is almost equal to the best bandwidth w, and the power of K-S test is higher than TV, TVE, and TC tests.

Table 7. If the true distribution is $EG(0.4,2)$, worst bandwidth w and corresponding powers (Complete Sample)

n	K-S	TV			TVE			TC		
	Power	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power
10	4.00E-04	3	-0.33979	0.839	4	-2.26843	0.8384	1	-0.83	0.8561
20	0.001	7	-6.54705	0.987	7	-8.00128	0.9852	1	-6.75347	0.9888
30	0.0034	9	-12.3847	0.999	5	-14.0542	0.9989	3	-12.7581	0.9995
40	0.0042	11	-18.1826	1	7	-20.0335	1	2	-18.6572	1
50	0.0036	13	-24.0919	1	1	-26.932	1	1	-24.5333	1
60	0.006	16	-30.6575	1	1	-33.5186	1	1	-30.9845	1
70	0.0059	18	-36.6944	1	1	-39.8808	1	1	-37.3195	1
80	0.0096	21	-43.1355	1	1	-46.1896	1	1	-43.5357	1
90	0.0115	23	-49.3148	1	1	-52.494	1	1	-49.7867	1
100	0.0114	26	-55.7665	1	1	-59.2729	1	1	-56.5291	1

Table 8. If the true distribution is $EG(0.4,2)$, the best bandwidth w and corresponding powers (Complete Sample)

n	TV			TVE			TC		
	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power	w	Critical Values (%95)	Power
10	2	NA	0	1	-2.84598	0.8183	4	-0.9794	0.8428
20	5	Inf	0	1	-8.64008	0.9788	9	-6.64341	0.985
30	7	NA	0	1	-14.6716	0.9984	14	-12.5628	0.9991
40	10	Inf	0	4	-20.3324	0.9995	19	-18.9264	0.9997
50	12	NA	0	24	-25.8575	1	24	-24.8451	1
60	15	Inf	0	29	-32.1584	1	29	-31.2042	1
70	17	NA	0	34	-38.7469	1	34	-37.7665	1
80	20	Inf	0	39	-44.4957	1	39	-43.535	1
90	22	NA	0	44	-51.0945	1	44	-50.1238	1
100	25	Inf	0	49	-57.6001	1	49	-56.6904	1

Upon examining Tables 7 and 8 for assessing the suitability of a population with an actual distribution to the Exponentiated Exponential Distribution, the following observations can be made:

The powers of the TV, TVE, and TC tests when using the best bandwidth are nearly equal, and they all surpass the power of the K-S (Kolmogorov-Smirnov) test. Among these, the TC test exhibits slightly superior performance.

It is worth noting that the TV test cannot be executed effectively when employing the worst bandwidth w, emphasizing the significance of carefully selecting w for the TV test.

The power of both the TVE and TC tests experiences a decline as the sample size is reduced to n = 10, 20, 30, and 40, particularly when the worst bandwidth w is chosen. Thus, the choice of bandwidth w for these tests becomes somewhat crucial in such scenarios.

3.3. Exponentiated Exponential under Censored Sample

The Exponentiated Exponential test critical values, derived from simulation, are utilized to investigate the best and worst powers for different distributions and parameter settings, contingent on the bandwidth *w* selection. We used the Noughabi [21] censoring scheme. This is assuming that the real distribution follows an exponential distribution. Tables 9 and 10 present the findings from these analyses.

Table 9. Best bandwidth w and corresponding power obtained for different alternative distributions (Progressive Type Censored Sample)

Censoring Scheme No	Weibull(5.3)			EP(2.2)			EG(0.4.2)		
	w	Critical value (%95)	Power	w	Critical value (%95)	Power	w	Critical value (%95)	Power
1	1	-0.1517	0.1173	3	-0.2720	0.1197	3	-0.2720	0.2820
2	1	0.1370	0.0443	1	0.1370	0.0526	2	0.0703	**0.5159**
3	1	0.0892	0.0439	2	0.0221	0.0549	2	0.0221	0.5543
4	2	-0.2738	0.2276	5	-0.3449	0.1357	5	-0.3449	0.2479
5	1	0.2040	0.0577	2	0.1004	0.0612	2	0.1004	**0.6613**
6	4	-0.1666	0.2915	2	-0.1252	0.0787	5	-0.1584	0.1429
7	3	-0.2442	0.3088	7	-0.3110	0.1413	7	-0.3110	0.1987
8	1	0.2758	0.0905	3	0.0957	0.0850	2	0.1247	**0.6050**
9	7	-0.2472	0.3690	3	-0.1907	0.0998	3	-0.1907	0.0705
10	2	-0.1689	0.1990	4	-0.1947	0.1377	4	-0.1947	0.4651
11	1	0.0925	0.0460	2	0.0478	0.0529	2	0.0478	**0.7259**
12	1	0.0667	0.0490	1	0.0667	0.0555	2	0.0236	0.4914
13	3	-0.2909	**0.4518**	8	-0.3313	**0.1798**	8	-0.3313	0.3833
14	1	0.1828	0.0633	2	0.0884	0.0643	2	0.0884	**0.9376**
15	1	0.0850	0.1099	3	-0.0447	0.1005	3	-0.0447	0.1153
16	4	-0.2862	**0.5980**	13	-0.3383	**0.1832**	13	-0.3383	0.2810
17	1	0.2646	0.1171	7	0.0624	0.0991	3	0.0903	**0.9425**
18	1	0.1618	0.1611	5	-0.0351	0.1152	14	-0.0726	0.8783
19	3	-0.2291	**0.5032**	8	-0.2565	**0.1760**	8	-0.2565	0.5211
20	1	0.1216	0.0648	3	0.0475	0.0599	2	0.0617	**0.9717**
21	1	0.0791	0.0749	3	0.0053	0.0627	2	0.0186	0.6923
22	4	-0.3102	**0.7423**	16	-0.3577	**0.2134**	17	-0.3584	0.3614
23	1	0.2345	0.1196	6	0.0599	0.0864	3	0.0814	**0.9965**
24	4	-0.1055	0.3736	5	-0.1155	0.1557	5	-0.1155	0.1297
25	5	-0.2456	**0.8172**	21	-0.2939	**0.2190**	22	-0.2943	0.2748
26	1	0.2839	0.2222	8	0.0597	0.1315	4	0.0775	**0.9835**
27	2	0.0381	0.3167	8	-0.0449	0.1762	24	-0.0725	0.9340

Table 10. Worst bandwidth w and corresponding power obtained for different alternative distributions (Progressive Type Censored Sample)

Censoring Scheme No	Weibull(5.3)			EP(2.2)			EG(0.4.2)		
	w	Critical value (%95)	Power	w	Critical value (%95)	Power	w	Critical value (%95)	Power
1	3	-0.2720	0.0551	1	-0.1517	0.0593	1	-0.1517	0.1298
2	3	0.0343	0.0260	3	0.0343	0.0416	1	0.1370	**0.4659**
3	3	-0.0111	0.0194	3	-0.0111	0.0508	1	0.0892	0.4853
4	5	-0.3449	0.0832	1	-0.1355	0.0515	1	-0.1355	0.0734
5	4	0.0523	0.0137	5	0.0326	0.0527	1	0.2040	**0.5474**
6	1	0.0138	0.1423	5	-0.1584	0.0310	2	-0.1252	0.0697
7	7	-0.3110	0.2055	1	-0.0255	0.0520	1	-0.0255	0.0603
8	6	0.0474	0.0044	1	0.2758	0.0591	1	0.2758	**0.4321**
9	1	0.0216	0.2126	7	-0.2472	0.0321	7	-0.2472	0.0489
10	4	-0.1947	0.0391	1	-0.1092	0.0609	1	-0.1092	0.1846
11	3	0.0304	0.0309	4	0.0141	0.0467	1	0.0925	**0.6450**
12	4	-0.0085	0.0202	4	-0.0085	0.0450	1	0.0667	0.4309
13	9	-0.3290	**0.0120**	1	-0.1521	**0.0507**	1	-0.1521	0.0841
14	7	0.0309	0.0129	9	0.0179	0.0364	1	0.1828	**0.8564**
15	9	-0.0938	0.0091	1	0.0850	0.0554	9	-0.0938	0.0501
16	14	-0.3366	**0.0570**	1	-0.0700	**0.0499**	1	-0.0700	0.0605
17	11	0.0384	0.0014	1	0.2646	0.0557	1	0.2646	**0.8201**
18	14	-0.0726	0.0004	1	0.1618	0.0629	1	0.1618	0.6458
19	9	-0.2543	**0.0053**	1	-0.1362	**0.0479**	1	-0.1362	0.1178
20	6	0.0281	0.0174	9	0.0100	0.0286	1	0.1216	**0.9276**
21	8	-0.0274	0.0090	9	-0.0325	0.0416	1	0.0791	0.5432
22	19	-0.3561	**0.0043**	1	-0.1252	**0.0528**	1	-0.1252	0.0660
23	16	0.0267	0.0007	19	0.0229	0.0470	1	0.2345	**0.9719**
24	19	-0.1494	0.0063	19	-0.1494	0.0397	18	-0.1502	0.0550
25	24	-0.2933	**0.1099**	1	-0.0023	**0.0478**	1	-0.0023	0.0527
26	24	0.0282	0.0000	1	0.2839	0.0655	1	0.2839	**0.8965**
27	24	-0.0725	0.0000	1	0.1856	0.0710	1	0.1856	0.6935

Examining Tables 9 and 10 will help determine whether a population with an $Weibull(5,3)$ $EP(2,2)$ or $EG(0.4,2)$ distribution is suitable for the exponential distribution. The censoring systems for the $Weibull(5,3)$ distribution show notable shifts in the authorities. When the initial observation is made, the level of censorship in the $R_m = 12$ and $R_i = 0, i \neq 12$-shaped censorship schemes is higher, outcomes that are comparable to the distribution $Weibull(5,3)$ are also obtained for distribution $EP(2,2)$, the powers for the $EP(2,2)$ dispersion are typically minimal, for the $EG(0.4,2)$ distribution, The censoring schemes indicate substantial shifts in the authorities. It has been established that the choice of bandwidth w is important since there is a difference between the distribution's best and worst powers. The powers are more significant in the $R_m = 20, R_i = 0$,i=1,2, ... ,m-1--shaped censoring schemes where all censorship is made after the last observation is collected.

3.4. Real Data Analysis in Bladder Cancer

This section demonstrates the proposed method with a real data set. Since this real dataset is publicly available, it does not require ethics committee approval. The actual data set is from Abbas et al. [22]. The proposed dataset representing recovery times (in months) of a random sample of 128 bladder cancer patients was used in Abbas et al. [22] (n=128, m=20).

The Real dataset shows the number of months that 128 patients with bladder cancer went into remission.

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46	1.76	2.02	2.02
2.07	2.09	2.23	2.26	2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31
3.36	3.36	3.48	3.52	3.57	3.64	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5
4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49	5.62	5.71	5.85	6.25
6.31	6.54	6.76	6.93	6.94	6.97	7.09	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66	7.87
7.93	8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66	10.75	11.25	11.64
11.79	11.98	12.02	12.03	12.07	12.63	13.11	13.29	13.8	14.24	14.76	14.77	14.83	15.96	16.62	17.12
17.14	17.36	18.1	19.13	20.28	21.73	22.69	23.63	25.74	25.82	32.15	34.26	36.66	43.01	46.12	79.05

Censored Data-1, m = 20

0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46
1.76	2.02	2.02	2.07	2.09	2.23	2.26						

We test exponentiation using the proposed procedure.

i. The value of the test statistics for w = 2 are calculated as follows: TA(w = 2) = −0.8017, normally 0.0981 are critical values corresponding to 0.05.

ii. The value of the test statistics for w = 3 are calculated as follows: TA(w = 3) = -2.3271, normally 0.0867 are critical values corresponding to 0.05.

iii. The value of the test statistics for w = 6 are calculated as follows: TA(w = 6) = −1.9900, normally 0.0607 are critical values corresponding to 0.05.

iv. The value of the test statistics for w = 9 are calculated as follows: TA(w = 9) = −0.4562, normally 0.0502 are critical values corresponding to 0.05.

Therefore, the hypothesis that Type-II censored people are from an exponentiated exponential distribution is accepted at the 0.05 significance level. Test statistics and corresponding values were calculated on censored datasets via R software.

4. Conclusion

In conclusion, utilizing the best and worst bandwidths, comparing critical values and power was performed under different conditions, based on Kullback-Leibler divergence, for both complete and censored sampling scenarios. It was shown that the test's power rose as the sample size grew in all comparisons. The differences between the best and worst bandwidths in most cases highlight the importance of the bandwidth selection (denoted as 'w') while addressing the goodness-of-fit testing problem.

Furthermore, it became evident that when bandwidth 'w' is generally selected for the TV test, the test cannot be effectively performed, highlighting the pivotal role of 'w' in this particular test. Among the tests with bandwidth comparisons, the TC test consistently emerged as the strongest, followed by the TV test, with the TVE test showing the lowest performance.

As a result of numerous comparisons, it can be concluded that the choice of bandwidth 'w' holds considerable importance, as substantial differences in power were observed between the best and worst bandwidths in nearly all cases.

To demonstrate the usefulness of our suggested research, we used a dataset of people who had been diagnosed with bladder cancer in the last phase of our investigation. Ultimately, models like this can help academics and medical practitioners with tasks like risk prediction, investigation of events, medical decision-making, and time-to-event analysis (survival analysis).

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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