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Compact operators on the Motzkin sequence space $c_0(\mathcal{M})$

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Motzkin numbers, Sequence spaces, Hausdorff measure of non-compactness, Compact operators

Abstract — The concept of non-compactness measure is extremely beneficial for functional analysis in theories, such as fixed point and operator equations. Apart from these, the Hausdorff measure of non-compactness also has some applications in the theory of sequence spaces which is an interesting topic of functional analysis. One of these applications is to obtain necessary and sufficient conditions for the matrix operators between Banach coordinate (BK) spaces to be compact. In line with these explanations, in this study, the necessary and sufficient conditions for a matrix operator to be compact from the Motzkin sequence space $c_0(\mathcal{M})$ to the sequence space $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$ are presented by using Hausdorff measure of non-compactness.

Subject Classification (2020): 11B83, 46A45

1. Introduction

The linear space containing all sequences of real or complex numbers is symbolized by ω. Each linear subspace Γ of ω is referred to as a sequence space. Some prominent instances of sequence spaces are c (convergent sequences' space), c_0 (null sequences' space), ℓ_∞ (bounded sequences' space) and ℓ_p (p-summable sequences' space). The aforementioned spaces are Banach spaces due to the norms $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{s\in\mathbb{N}} |u_s|$ and $\|u\|_{\ell_p} = \left(\sum_s |u_s|^p\right)^{1/p}$ for $1 \leq p < \infty$, where the notion $\sum_s$ purports the summation $\sum_{s=0}^{\infty}$ and $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Moreover, the acronym cs denotes the spaces of all convergent series. A Banach space wherein each coordinate functional f_s, defined by $f_s(u) = u_s$, exhibits continuity and is named a Banach coordinate (BK) space. Given spaces $\Gamma, \Psi \subseteq \omega$, the set $M(\Gamma * \Psi)$ is defined as follows:

$$M(\Gamma * \Psi) = \left\{\mu = (\mu_r) \in \omega : \mu u = (\mu_r u_r) \in \Psi, \text{ for all } u \in \Gamma\right\}$$

In that case, the beta dual of the set Γ is given by $\Gamma^\beta = M(\Gamma * cs)$. For an infinite matrix $D = (d_{rs})$ possessing entries from the real or complex field, D_r denotes the r^{th} row. The D-transform of $u = (u_s) \in \omega$, denoted by $(Du)_r$, is described as $\sum_{s=0}^{\infty} d_{rs}u_s$ such that the series converges for all $r \in \mathbb{N}$.

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Consider the sequence spaces Γ and Ψ. A matrix D is called a matrix mapping from Γ to Ψ if, for all $u \in \Gamma$, the image Du belongs to Ψ. The class of all such matrices that effectuate a mapping from Γ to Ψ is denoted by $(\Gamma : \Psi)$. Additionally, the notation Γ_D is employed to represent the set of all sequences for which the D-transform is contained in Γ, as expressed by:

$$\Gamma_D = \{u \in \omega : Du \in \Gamma\} \tag{1.1}$$

It is acknowledged that the matrix domain Γ_D constitutes a sequence space. Furthermore, if Γ is identified as a BK-space and D is triangular, then Γ_D also forms a BK-space, with norm defined by $\|u\|_{\Gamma_D} = \|Du\|_{\Gamma}$, as elucidated in the literature. In light of this principle, a plethora of intriguing BK-spaces have been the subject of scholarly investigation recently.

Obtaining new normed sequence spaces by using the special matrix and addressing some topics in these spaces, such as completeness, inclusion relations, Schauder basis, duals, matrix transformations, compact operators, and core theorems, have been seen as an important field of study since past years and many valuable researches have been carried out in this subject. When the researchers want to reach basic and advanced concepts on the subjects mentioned above, the sources to be consulted can be referred to as [1–10], and monographs [11–13].

2. Preliminaries

Constructing new sequence spaces as domains of special infinite matrices, as the application of summability theory to sequence spaces, has emerged as an important research area in recent years. With this in mind, Başarır and Kara [14] first obtained an infinite matrix using the Fibonacci number sequence and then constructed new sequence spaces with the help of this matrix and comprehensively examined their various properties. Inspired by the mentioned work, various researchers later obtained sequence spaces with similar ideas using number sequences Lucas, Padovan, Pell, Leanardo, Catalan, Bell, Schröder, Motzkin, and Mersenne. The Schröder matrix $\widetilde{S} = (\widetilde{S}_{rs})$ [15] is defined by

$$\widetilde{S}_{rs} = \begin{cases} \dfrac{S_s S_{r-s}}{S_{r+1} - S_r}, & 0 \leq s \leq r \\ 0, & s > r \end{cases}$$

Recently, the domains $c_0(\widetilde{S}), c(\widetilde{S}), \ell_p(\widetilde{S})$, and $\ell_\infty(\widetilde{S})$ of $\widetilde{S}$ in c_0, c, ℓ_p, and ℓ_∞, respectively, are studied by Dağlı [15, 16]. Quite recently, the construction of sequence spaces using Catalan and Motzkin numbers has been investigated by Karakaş and Dağlı [17]. They studied $\ell_p(\widetilde{C})$ and $\ell_\infty(\widetilde{C})$ where $\widetilde{C} = (\widetilde{c}_{rs})$ is described by

$$\widetilde{c}_{rs} = \begin{cases} \dbinom{r}{s} \dfrac{M_s}{C_{r+1}}, & 0 \leq s \leq r \\ 0, & s > r \end{cases}$$

For relevant literature, see [18–28].

2.1. Motzkin Numbers and Associated Sequence Spaces

The first basic information about the Motzkin number sequence, one of the most interesting number sequences, is obtained from Motzkin's study [29]. The r^{th} Motzkin number represents the number of different situations in which n distinct points on a circle can be joined by non-intersecting chords in mathematics. To point out a detail here, the chords do not need to touch all points on the circle in each case. The first few terms of the Motzkin number sequence $(M_r)_{r\in\mathbb{N}}$, which has an important place in combinatorics, number theory, and geometry, are given as follows:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, \cdots$$

The recurrence relations of M_r are given the following way:

$$M_r = M_{r-1} + \sum_{s=0}^{r-2} M_s M_{r-s-2} = \frac{2r+1}{r+1} M_{r-1} + \frac{3r-3}{r+2} M_{r-2}$$

Another relation provided by the Motzkin numbers is given below:

$$M_{r+2} - M_{r+1} = \sum_{s=0}^{r} M_s M_{r-s}, \text{ for } r \geq 0$$

The generating function $m(u) = \sum_{r=0}^{\infty} M_r u^r$ of the Motzkin numbers satisfies

$$u^2 + [m(u)]^2 + (u-1)m(u) + 1 = 0$$

and is described by

$$m(u) = \frac{1 - u - \sqrt{1 - 2u - 3u^2}}{2u^2}$$

The expression on Motzkin numbers with the help of integral function is as follows:

$$M_r = \frac{2}{\pi} \int_0^{\pi} \sin^2 u \, (2cosu + 1)^r \, du$$

They have the asymptotic behavior

$$M_r \sim \frac{1}{2\sqrt{\pi}} \left(\frac{3}{r}\right)^{\frac{3}{2}} 3^r, \quad r \to \infty$$

Furthermore, it is known that

$$\lim_{r\to\infty} \frac{M_r}{M_{r-1}} = 3$$

In addition to the basic information stated above, readers can benefit from the studies Aigner [30], Barrucci et al. [31], and Donaghey and Shapiro [32] for more comprehensive content about Motzkin numbers and related subjects.

The remainder of this subsection will provide information about the study conducted by Erdem et al. [27]. It is given the Motzkin matrix $\mathcal{M} = (\mathfrak{m}_{rs})_{r,s\in\mathbb{N}}$ constructed with the help of Motzkin numbers as follows:

$$\mathfrak{m}_{rs} := \begin{cases} \dfrac{M_s M_{r-s}}{M_{r+2} - M_{r+1}}, & 0 \leq s \leq r \\ 0, & s > r \end{cases} \tag{2.1}$$

Furthermore, the Motzkin matrix $\mathcal{M}$ is conservative, that is $\mathcal{M} \in (c : c)$ and it is given the inverse $\mathcal{M}^{-1} = (\mathfrak{m}_{rs}^{-1})$ of the Motzkin matrix $\mathcal{M}$ as

$$\mathfrak{m}_{rs}^{-1} := \begin{cases} (-1)^{r-s} \dfrac{M_{s+2} - M_{s+1}}{M_r} \pi_{r-s}, & 0 \leq s \leq r \\ 0, & s > r \end{cases} \tag{2.2}$$

where $\pi_0 = 0$ and

$$\pi_r = \begin{vmatrix} M_1 & M_0 & 0 & 0 & \cdots & 0 \\ M_2 & M_1 & M_0 & 0 & \cdots & 0 \\ M_3 & M_2 & M_1 & M_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ M_r & M_{r-1} & M_{r-2} & M_{r-3} & \cdots & M_1 \end{vmatrix}$$

for all $r \in \mathbb{N}\backslash\{0\}$. From its definition, it is clear that $\mathcal{M}$ is a triangle. Furthermore, $\mathcal{M}$-transform of $u = (u_s) \in \omega$ is expressed with

$$\nu_r := (\mathcal{M}u)_r = \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s, r \in \mathbb{N} \tag{2.3}$$

The Motzkin sequence spaces $c(\mathcal{M})$ and $c_0(\mathcal{M})$, which are BK-spaces constructed as the domain of the Motzkin matrix, are given by

$$c(\mathcal{M}) = \left\{ u = (u_s) \in \omega : \lim_{r\to\infty} \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s \text{ exists} \right\}$$

and

$$c_0(\mathcal{M}) = \left\{ u = (u_s) \in \omega : \lim_{r\to\infty} \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s = 0 \right\}$$

and in [27], the authors studied some algebraic and topological properties of newly described spaces.

Consider the sets $\aleph_1$ and $\aleph_2$ as

$$\aleph_1 = \left\{ t = (t_s) \in \omega : \sum_{r=s}^{\infty} (-1)^{r-s} \frac{M_{s+2} - M_{s+1}}{M_r} \pi_{r-s} t_r \text{ exists for each } s \in \mathbb{N} \right\}$$

$$\aleph_2 = \left\{ t = (t_s) \in \omega : \sup_{r\in\mathbb{N}} \sum_{s=0}^{r} \left| \sum_{i=s}^{r} (-1)^{i-s} \frac{M_{s+2} - M_{s+1}}{M_i} \pi_{i-s} t_i \right| < \infty \right\}$$

Then, the β-dual of $c_0(\mathcal{M})$ is determined as $\{c_0(\mathcal{M})\}^{\beta} = \aleph_1 \cap \aleph_2$.

In this article, the necessary and sufficient conditions are presented for a matrix operator to be compact from the Motzkin sequence space $c_0(\mathcal{M})$ to the sequence space $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$ by using Hausdorff measure of non-compactness.

3. Compact Operators on the Motzkin Sequence Space $c_0(\mathcal{M})$

The current section intends to determine the compactness conditions of an operator from $c_0(\mathcal{M})$ to Ψ by using the Hausdorff measure of non-compactness, where $\Psi \in \{c_0, c, \ell_\infty, \ell_1\}$. We may start by reminding the basic concepts and results in this section.

Let $\mathcal{U}_\Gamma$ represents the unit sphere of normed space Γ. The acronym $\mathfrak{B}(\Gamma : \Psi)$ denotes every bounded (continuous) linear operators' family from the Banach space Γ to Banach space Ψ. In that case, $\mathfrak{B}(\Gamma : \Psi)$ is Banach with $\|\mathcal{L}\| = \sup_{u\in\mathcal{U}_\Gamma} \|\mathcal{L}u\|$. We express the notation

$$\|u\|_\Gamma^\diamond = \sup_{x\in\mathcal{U}_\Gamma} \left| \sum_s u_s x_s \right|$$

for $u = (u_s) \in \omega$, with the assumption of the series is convergent for BK-space $\Gamma \supset \Omega$ and $u \in \Gamma^\beta$, where Ω represents all finite sequences' space. Furthermore, a linear operator $\mathcal{L} : \Gamma \to \Psi$ is called as compact operator if the domain of $\mathcal{L}$ is all of Γ and the sequence $(\mathcal{L}(u))$ possesses a convergent subsequence in Ψ for the spaces Γ and Ψ and every bounded sequence $u = (u_s) \in \Gamma$.

Consider the metric space Γ. Then, the Hausdorff measure of non-compactness of a bounded set $A \subseteq \Gamma$ is symbolized by $\chi(A)$ and stated in the following way:

$$\chi(A) = \inf \left\{ \epsilon > 0 : A \subseteq \cup_{j=0}^{n} A(u_j, m_j), u_j \in \Gamma, m_j < \epsilon, n \in \mathbb{N} \right\}$$

where $A(u_j, m_j)$ represents the open ball centered at u_j with radius m_j for all $j \in \{0, 1, 2, \ldots, n\}$. More descriptive information about the Hausdorff measure of non-compactness can be found in [33].

The Hausdorff measure of non-compactness for $\mathcal{L}$, symbolized as $\|\mathcal{L}\|_\chi$, is characterized as $\|\mathcal{L}\|_\chi = \chi(\mathcal{L}(\mathcal{U}_\Gamma))$. There exists a crucial relationship between the concepts of the Hausdorff measure of non-compactness and compact operators, specifically, "A linear operator $\mathcal{L}$ is compact iff $\|\mathcal{L}\|_\chi = 0$".

For further investigation of sequence spaces and the application of the Hausdorff measure of non-compactness in characterizing compact operators between BK-spaces, readers are encouraged to consult the literature [33–38].

Lemma 3.1. [33] $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$ and $\|u\|_\Gamma^\diamond = \|u\|_{\ell_1}$ for $u \in \ell_1$ and $\Gamma \in \{\ell_\infty, c, c_0\}$.

Lemma 3.2. [36] Consider the BK-spaces Γ and Ψ. Then, $(\Gamma : \Psi) \subseteq \mathfrak{B}(\Gamma : \Psi)$. More clearly, for any $D \in (\Gamma : \Psi)$, there corresponds a linear operator $\mathcal{L}_D \in \mathfrak{B}(\Gamma : \Psi)$ such that $\mathcal{L}_D(u) = Du$, for all $u \in \Gamma$.

Lemma 3.3. [33] Let $\Omega \subseteq \Gamma$ be any BK-space and $D \in (\Gamma : \Psi)$. Then,

$$\|\mathcal{L}_D\| = \|D\|_{(\Gamma:\Psi)} = \sup_{r\in\mathbb{N}} \|D_r\|_\Gamma^\diamond < \infty$$

Theorem 3.4. [33] For $u = (u_m) \in c_0$, consider that $A \subseteq c_0$ is bounded and the operator $\lambda_m : c_0 \to c_0$ is described with $\lambda_m(u) = (u_0, u_1, u_2, u_3, ..., u_m, 0, 0, ...)$. Then, for the identity operator I on c_0, we have

$$\chi(A) = \lim_{m\to\infty} \left(\sup_{u\in A} \|(I - \lambda_m)(u)\| \right)$$

We can give the following results for $x = (x_s)$,$y = (y_s) \in \omega$ connected to each other by the relation

$$y_s = \sum_{j=s}^{\infty} (-1)^{j-s} \frac{M_{s+2} - M_{s+1}}{M_j} \pi_{j-s} x_j$$

for all $s \in \mathbb{N}$.

Lemma 3.5. Let us consider the sequence $x = (x_s) \in (c_0(\mathcal{M}))^\beta$. In that case, $y = (y_s) \in \ell_1$ and

$$\sum_s x_s u_s = \sum_s y_s \nu_s \tag{3.1}$$

for all $u = (u_s) \in c_0(\mathcal{M})$.

Lemma 3.6. For all $x = (x_s) \in (c_0(\mathcal{M}))^\beta$, the following statement is held.

$$\|x\|_{c_0(\mathcal{M})}^\diamond = \sum_s |y_s| < \infty$$

Proof. It is achieved from the Lemma 3.5 that $y = (y_s) \in \ell_1$ and (3.1) holds for $x = (x_s) \in (c_0(\mathcal{M}))^\beta$. Since, $\|u\|_{c_0(\mathcal{M})} = \|\nu\|_{c_0}$, we reach that "$u \in \mathcal{U}_{c_0(\mathcal{M})}$ if and only if $\nu \in \mathcal{U}_{c_0}$". Thus, we can write the equality

$$\|x\|_{c_0(\mathcal{M})}^\diamond = \sup_{u\in\mathcal{U}_{c_0(\mathcal{M})}} \left| \sum_s x_s u_s \right| = \sup_{\nu\in\mathcal{U}_{c_0}} \left| \sum_s y_s \nu_s \right| = \|y\|_{c_0}^\diamond$$

By the aid of the Lemma 3.1, we obtain that

$$\|x\|_{c_0(\mathcal{M})}^\diamond = \|y\|_{c_0}^\diamond = \|y\|_{\ell_1} = \sum_s |y_s| < \infty$$

□

Consider the matrices $H = (h_{rs})$ and $D = (d_{rs})$ as

$$h_{rs} = \sum_{j=s}^{\infty} (-1)^{j-s} \frac{M_{s+2} - M_{s+1}}{M_j} \pi_{j-s} d_{rj}$$

for all $r, s \in \mathbb{N}$ whenever the series is convergent.

Lemma 3.7. For $\Psi \subseteq \omega$ and infinite matrix $D = (d_{rs})$, if $D \in (c_0(\mathcal{M}) : \Psi)$, in that case $H \in (c_0 : \Psi)$ and $Du = H\nu$ for all $u \in c_0(\mathcal{M})$.

Proof. It is obvious by Lemma 3.5. □

Lemma 3.8. If $D \in (c_0(\mathcal{M}) : \Psi)$, then it is achieved that

$$\|\mathcal{L}_D\| = \|D\|_{(c_0(\mathcal{M}):\Psi)} = \sup_{r\in\mathbb{N}} \left(\sum_s |h_{rs}| \right) < \infty$$

where $\Psi \in \{c_0, c, \ell_\infty\}$.

Lemma 3.9. [34] Consider the BK-space $\Gamma \supset \Omega$. Each of the following results are well known:

i. Let $D \in (\Gamma : c_0)$. Then, $\|\mathcal{L}_D\|_\chi = \limsup_r \|D_r\|_\Gamma^\diamond$ and $\mathcal{L}_D$ is compact iff $\lim_r \|D_r\|_\Gamma^\diamond = 0$.

ii. Let Γ possesses AK property or $\Gamma = \ell_\infty$ and $D \in (\Gamma : c)$. In that case,

$$\frac{1}{2} \limsup_r \|D_r - \sigma\|_\Gamma^\diamond \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \|D_r - \sigma\|_\Gamma^\diamond$$

and $\mathcal{L}_D$ is compact iff $\lim_r \|D_r - \sigma\|_\Gamma^\diamond = 0$ for $\sigma = (\sigma_s)$ and $\sigma_s = \lim_r d_{rs}$.

iii. Let $D \in (\Gamma : \ell_\infty)$. In that case, $0 \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \|D_r\|_\Gamma^\diamond$ and $\mathcal{L}_D$ is compact if $\lim_r \|D_r\|_\Gamma^\diamond = 0$.

iv. Let $D \in (\Gamma : \ell_1)$. In that case,

$$\lim_j \left(\sup_{E\in\mathcal{F}_j} \left\| \sum_{r\in E} D_r \right\|_\Gamma^\diamond \right) \leq \|\mathcal{L}_D\|_\chi \leq 4. \lim_j \left(\sup_{E\in\mathcal{F}_j} \left\| \sum_{r\in E} D_r \right\|_\Gamma^\diamond \right)$$

and $\mathcal{L}_D$ is compact iff $\lim_j \left(\sup_{E\in\mathcal{F}_j} \|\sum_{r\in E} D_r\|_\Gamma^\diamond \right) = 0$, where $\mathcal{F}$ symbolizes all finite subsets' family of $\mathbb{N}$ and $\mathcal{F}_j$ is the subfamily of $\mathcal{F}$ occuring of subsets of $\mathbb{N}$ with elements that are greater than j.

Theorem 3.10. *i.* For $D \in (c_0(\mathcal{M}) : \ell_\infty)$,

$$0 \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \sum_s |h_{rs}|$$

holds.

ii. For $D \in (c_0(\mathcal{M}) : c)$,

$$\frac{1}{2} \limsup_r \sum_s |h_{rs} - h_s| \leq \|\mathcal{L}_D\|_\chi \leq \limsup_r \sum_s |h_{rs} - h_s|$$

holds where $h_s = \lim_{r\to\infty} h_{rs}$ for each $s \in \mathbb{N}$.

iii. For $D \in (c_0(\mathcal{M}) : c_0)$,

$$\|\mathcal{L}_D\|_\chi = \limsup_r \sum_s |h_{rs}|$$

holds.

iv. For $D \in (c_0(\mathcal{M}) : \ell_1)$,

$$\lim_j \|D\|_{(c_0(\mathcal{M}):\ell_1)}^{(j)} \leq \|\mathcal{L}_D\|_\chi \leq 4. \lim_j \|D\|_{(c_0(\mathcal{M}):\ell_1)}^{(j)}$$

holds where $\|D\|_{(c_0(\mathcal{M}):\ell_1)}^{(j)} = \sup_{E\in\mathcal{F}_j} \left(\sum_s |\sum_{r\in E} h_{rs}|\right)$ for all $j \in \mathbb{N}$.

Proof. *i.* Let $D \in (c_0(\mathcal{M}) : \ell_\infty)$. From the convergence of $\sum_{s=0}^\infty d_{rs} u_s$ for all $r \in \mathbb{N}$, it is observed that $D_r \in (c_0(\mathcal{M}))^\beta$. From Lemma 3.6, we reach that

$$\|D_r\|^\diamond_{c_0(\mathcal{M})} = \|H_r\|^\diamond_{c_0} = \|H_r\|_{\ell_1} = \left(\sum_s |h_{rs}|\right)$$

From Lemma 3.9-*iii*, it is obtain

$$0 \le \|\mathcal{L}_D\|_\chi \le \limsup_r \sum_s |h_{rs}|$$

ii. If $D \in (c_0(\mathcal{M}) : c)$, in that case $H \in (c_0 : c)$ from Lemma 3.7. Thus, from Lemma 3.9-*ii*, we obtain that

$$\frac{1}{2} \limsup_r \|H_r - h\|^\diamond_{c_0} \le \|\mathcal{L}_D\|_\chi \le \limsup_r \|H_r - h\|^\diamond_{c_0}$$

where $h = (h_s)$ and $h_s = \lim_{r\to\infty} h_{rs}$ for all $s \in \mathbb{N}$. Hence, from Lemma 3.1, $\|H_r - h\|^\diamond_{c_0} = \|H_r - h\|_{\ell_1} = \sum_s |h_{rs} - h_s|$, for all $r \in \mathbb{N}$.

iii. Consider that $D \in (c_0(\mathcal{M}) : c_0)$. From the relation $\|D_r\|^\diamond_{c_0(\mathcal{M})} = \|H_r\|^\diamond_{c_0} = \|H_r\|_{\ell_1} = (\sum_s |h_{rs}|)$ for each $r \in \mathbb{N}$ and from Lemma 3.9-*i*, we see $\|\mathcal{L}_D\|_\chi = \limsup_r \sum_s |h_{rs}|$.

iv. Let $D \in (c_0(\mathcal{M}) : \ell_1)$. By Lemma 3.7, we reach that $H \in (c_0 : \ell_1)$. It follows from Lemma 3.9 that

$$\lim_j \left(\sup_{E\in\mathcal{F}_j} \left\|\sum_{r\in E} H_r\right\|^\diamond_{c_0}\right) \le \|\mathcal{L}_D\|_\chi \le 4.\lim_j \left(\sup_{E\in\mathcal{F}_j} \left\|\sum_{r\in E} H_r\right\|^\diamond_{c_0}\right)$$

Furthermore, Lemma 3.1 implies that

$$\left\|\sum_{r\in E} H_r\right\|^\diamond_{c_0} = \left\|\sum_{r\in E} H_r\right\|_{\ell_1} = \left(\sum_s \left|\sum_{r\in E} h_{rs}\right|\right)$$

□

Thus, using the theorem given above, we can give the following result.

Corollary 3.11. *i.* For $D \in (c_0(\mathcal{M}) : \ell_\infty)$, $\mathcal{L}_D$ is compact if

$$\lim_r \sum_s |h_{rs}| = 0$$

ii. For $D \in (c_0(\mathcal{M}) : c)$, $\mathcal{L}_D$ is compact iff

$$\lim_r \sum_s |h_{rs} - h_s| = 0$$

iii. For $D \in (c_0(\mathcal{M}) : c_0)$, $\mathcal{L}_D$ is compact iff

$$\lim_r \sum_s |h_{rs}| = 0$$

iv. For $D \in (c_0(\mathcal{M}) : \ell_1)$, $\mathcal{L}_D$ is compact iff

$$\lim_j \|D\|^{(j)}_{(c_0(\mathcal{M}):\ell_1)} = 0$$

where $\|D\|^{(j)}_{(c_0(\mathcal{M}):\ell_1)} = \sup_{E\in\mathcal{F}_j} (\sum_s |\sum_{r\in E} h_{rs}|)$, for all $j \in \mathbb{N}$.

4. Conclusion

Obtaining new normed sequence spaces using special matrices and addressing some intriguing topics such as completeness, inclusion relations, Schauder basis, α-, β- and γ-duals, matrix transformations, compact operators, core theorems and geometric properties in these spaces has been considered an important field of study in recent years and many valuable researches have been done on this subject. Furthermore, the idea of using special number sequences to obtain sequence spaces has begun to be used by authors. In this context, as an application of matrix summability methods to Banach spaces theory, in this study, it is presented the necessary and sufficient conditions for a matrix operator to be compact from the Motzkin sequence space $c_0(\mathcal{M})$ constructed by the aid of Motzkin number sequence to the sequence space $\mu \in \{\ell_\infty, c, c_0, \ell_1\}$ by using Hausdorff measure of non-compactness. It is noted here that the characterization of compact operators on sequence spaces by using Hausdorff's measure of non-compactness will constitute the focus of our future research endeavors. In future work, researchers can investigate the compactness of operators on different sequence spaces, taking into account those that have not been studied before.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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