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# On Lcem Matrices Over Unique Factorization Domains

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**Abstract**: In this paper, we extend the concept of lcem matrices beyond the classical domain of natural integers into the domain of unique factorization domains. We investigate the structure of these matrix types when applied to both arbitrary sets and gced-closed sets. Furthermore, we find the determinant, the trace and the inverse of such matrices. To simplify these ideas, we employ domains such as the Gaussian integers domain and the domain of polynomials defined over finite fields.

**Keywords:** Lcem matrices, Exponential divisor, Unique factorization domain

## Introduction

An integer $d = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, $p_i$ is prime, is an exponential divisor of $m = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ if $a_i \mid b_i$ for every $1 \le i \le k$ and is denoted by $d \mid_e m$. By convention, $1 \mid_e 1$ and 1 is not an exponential divisor for every $m > 1$. If $n$ and $m$ have the same prime factors, then they have a common exponential divisor. Let gced $(m,n)$ (resp. lcem$(m,n)$) be the greatest common exponential divisor (resp. the least common exponential multiple) of two integers $m$ and $n$, aslo denoted by $(m,n)_e$ (resp. $[m,n]_e$). By convention, $(1,1)_e = [1,1]_e = 1$ and $(1,m)_e$ and $[1,m]_e$ do not exist for every $m > 1$. If $m = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ and $n = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$, then

$$(m,n)_e = p_1^{gcd(b_1,c_1)} p_2^{gcd(b_2,c_2)} \dots p_k^{gcd(b_k,c_k)}$$

and

$$[m,n]_e = p_1^{[b_1,c_1]} p_2^{[b_2,c_2]} \dots p_k^{[b_k,c_k]}$$

with $[b_i, c_i]$ is the least common multiple of $b_i$ and $c_i$. Two integers $m = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$ and $n = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$ are exponentially coprime if $\gcd(b_i, c_i) = 1$ for every $1 \le i \le k$.

If $T = \{x_1, x_2, \dots, x_n\}$ is a well-ordered set of $n$ distinct positive integers with $x_1 < x_2 < \dots < x_n$, then the gcd (resp. lcm) matrix on $T$ is an $n \times n$ matrix defined as $(T) = \gcd(x_i, x_j)$ (resp. $[T]_{m\times m} = [x_i, x_j]$. The $n \times n$ power gcd (resp. power lcm) matrix on $T$ is $(T^r) = \gcd(x_i, x_j)^r$ (resp. $[T^r] = [x_i, x_j]^r$), where $r$ is any real number. A set $T = \{x_1, x_2, \dots, x_n\}$ is said to be a factor closed (resp. a gcd-closed) set if $x$ is an element of $T$ for any divisor $x$ of $x_i$ (resp. if $\gcd(x_i, x_j)$ for all $x_i$ and $x_j$) in $T$.

Smith (Smith, 1875) showed that if $T = \{1,2,\dots,n\}$, then $\det(T) = \phi(1)\phi(2)\dots\phi(n)$ and $\det[T] = \phi(1)\pi(1)\phi(2)\pi(2)\dots\phi(n)\pi(n)$, where $\phi$ is Euler's totient function and $\pi$ is a multiplicative function such that $\pi(p^k) = -p$, $p$ is a prime number. Moreover Smith showed that his results are true for factor-closed sets. Beslin and Ligh (Beslin, 1989b), factorized the gcd matrices and showed that it is non-singular. In Beslin and Ligh (1989a,1992) factorized the gcd matrices if $T$ is a gcd-closed set over the domain of integers and computed their determinants. Bourque and Ligh (1992) extended Smith's result on lcm matrices by showing that the determinant of the lcm matrix defined on a gcd-closed set $T = \{x_1, x_2, \dots, x_m\}$ is the product $\prod_{k=1}^{m} x_k^2 \beta_k$ where $\beta_k = \sum_{\substack{d \mid x_k \\ d \nmid x_t,\, x_t < x_k}} g(d)$, with the arithmetical function $g$ defined by $g(n) = \frac{1}{n}\sum_{d \mid n} \mu(d)$ and the function $\mu$ is the well-known Mobius function. Borque and Ligh (1995) conjectured that the lcm matrix on a gcd-closed set is invertible. In (Hong, 1999) Hong did systematic research on the conjecture of Bourque-Ligh and he showed that the Bourque-Ligh conjecture is true only for $n \le 7$. Also, Hong proved that this conjecture is true for a certain class of lcm-closed sets, see (Hong, 2005). Hong showed that if $n \le 3$, then for any lcm-closed set $T = \{x_1, x_2, \dots, x_n\}$, the gcd matrix on $T$ divides the lcm matrix on $T$ in the ring $M_n(\mathbf{Z})$ of the $n \times n$ matrices over the integers. For $n \ge 4$, there exists a lcm-closed set $T$ such that the gcd matrix on $T$ does not divide the lcm matrix on $T$ in the ring $M_n(\mathbf{Z})$, see Hong (2002). Beslin and El-Kassar (1989) expanded the notion of gcd matrices and Smith's determinant to Unique Factorization Domains (UFDs). Furthermore, there have been analogous adaptations of gcd and lcm matrices to Principal Ideal Domains (PIDs) and Euclidean domains (Eds). For further details, readers can consult Awad et al. (2020, 2023), as well as El-Kassar et al. (2009, 2010).

A set $T = \{x_1, x_2, \dots, x_n\}$ is called an exponential factor closed (resp. a gced-closed) set if the exponential divisor of every element of $T$ belongs to $T$ (resp. if $(x_i, x_j)_e \in T$ for every $x_i, x_j \in T$). If $T$ is an exponential factor closed set of distinct positive integers that are arranged in increasing order, then the $n \times n$ matrix $(T)_e$ (resp. $[T]_e$)= $t_{ij}$ having $t_{ij} = (x_i, x_j)_e$ (resp. $[x_i, x_j]_e$), as its $ij^{th}$ entry is called the gced (resp. lcem) matrix defined on $T$.

It is well known that $(\mathbf{Z}^+ \backslash \{1\}, |_e)$ is a poset under the exponential divisibility relation but not a lattice, since the gced does not always exist. More details are given in the next section. Korkee and Haukkanen (2009) embedded this poset in a lattice and studied the lcem matrices as an analogue of the lcm matrices. Raza and Waheed, (2015a, 2015b, 2012) gave structure theorems and calculated the determinant of gced and lcem matrices defined on an ordered set $T$. Zeid et al. (2022) extended the gced matrices from the domain of natural integers to the unique factorization domain and gave the structure of these types of matrices defined on both arbitrary sets and gced-closed sets.

In this paper, we study the lcem matrices as an analog of the gced matrices. Examples are given in the domains $\mathbf{Z}[i]$ and $\mathbf{Z}_p[x]$, where $p$ is a prime integer, to describe what has been done.

Throughout this paper, the following notations will be used

- $D$ is a unique factorization domain (UFD)
- $p_i$ is a prime element in $D$.
- $a_i$, $b_i$ and $c_i$ are positive integers.
- $z \sim w$ means $z$ and $w$ are two associates.
- $T = x_1, x_2, \dots, x_n$ is a finite ordered set (in increasing order) of nonzero, non-unit, and non-associate elements in $D$.

## Over Unique Factorization Domains

As in the integer case, a non-zero element $d = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ in $D$ is an exponential divisor of $m = p_1^{c_1} p_2^{c_2} \dots p_k^{c_k}$ if $a_i \mid c_i$ for every $1 \le i \le$, denoted by $d \mid_e m$. Note that a unit $u$ in $D$ is not an exponential divisor for any nonzero, non-unit element $m$ in $D$ and by convention $u \mid_e v$ for any unit $v$ in $D$. Two elements $a$ and $b$ in $D$ are associates if $a = ub$, where $u$ is a unit in $D$. Two elements in $D$ have a common exponential divisor if and only if they have the same prime factors. By convention, $(u, v)_e$ and $[u, v]_e$ does not exist for any non-zero, non unit element $a$ in $D$. A subset $T = \{x_1, x_2, \dots, x_n\}$ of $D$ is a gced-closed set if $(x_i, x_j)_e$ is also an element of $T$ for all

$x_i$, $x_j$ in $T$, $1 \leq i,j \leq n$. For example, the subset $T = \{1+3i, -2+4i, -1+7i, -12-16i\}$ of $\mathbf{Z}[i]$ is a gced-closed set, while the set $R = \{-2+4i, -1+7i, -12-16i\}$ is not.

### Exponential Convolution

Consider the two functions $f$ and $g$ defined on $D$. Define the exponential convolution of $f$ and $g$ of a non-zero element $m = \prod_{i=1}^{k} p_i^{c_i}$ in $D$ as:

$$(f \odot g)(m) = \sum_{a_1 b_1 = c_1} \cdots \sum_{a_k b_k = c_k} f\left(p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}\right) g\left(p_1^{b_1} p_2^{b_2} \ldots p_k^{b_k}\right).$$

Using the Möbius inversion exponential formula,

$$g(m) = \sum_{d|_e m} f(d) \mu^{(e)}\left(\frac{m}{d}\right)$$

if $f(m) = \sum_{d|_e m} g(d)$, and $\mu^{(e)}(u) = 1$ and $\mu^{(e)}(m) = \mu(c_1)\mu(c_2)\ldots\mu(c_k)$.

### Ordering

In our study, we consider the two particular domains, the domain of Gaussian integers $\mathbf{Z}[i]$ and the domain of polynomials over finite fields $\mathbf{Z}_p[x]$. It is well known that these two domains are not ordered. We use a well-defined linear ordering defined on these domains so that any two elements are comparable.

### Ordering in $\mathbf{Z}[i]$

Let $T = \{z_1, z_2, \ldots, z_n\}$ be a subset of $\mathbf{Z}[i]$. Define an ordering on $T$ as follows: If $q(z_i) < q(z_j)$, then $z_i < z_j$. If $q(z_i) = q(z_j)$, where $z_i \sim a+ib$, and $z_j \sim c+id$, such that $a\ b,\ c,\ d \geq 0$, then $z_i < z_j$ if $b < d$. The valuation function $q$ is defined as $q(a+ib) = a^2 + b^2$. In this case, the relation $<$ is a well-defined linear ordering on $T$.

### Ordering in $\mathbf{Z}_p[x]$

Let $T = \{f_1, f_2, \ldots, f_n\}$ be a subset of $\mathbf{Z}_p[x]$, where $p$ is a prime integer. Define an ordering on $T$ as follows: if $\deg(f_i) < \deg(f_j)$, then $f_i < f_j$ and if $\deg(f_i) = \deg(f_j)$ with $f_i = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and $f_j = x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$ with $0 \leq a_j, b_j \leq p-1$, then $f_i(x) < f_j(x)$ if $a_{j_0} < b_{j_0}$, where $j_0$ is the smallest index $j$ such that $a_j \neq b_j$. Again, the relation $<$ is a well-defined linear ordering on $T$.
Note that an non-zero element $a$ in $D$ is positive if $0 < a$, where 0 is the zero element in $D$ and $<$ is the ordering defined on $D$.

## LCEM Matrices Over Unique Factorization Domains

Let $T = \{x_1, x_2, \ldots, x_n\}$ be a subset of $D$. Again, the lcem matrix $[T]_{(e)}$ defined on $T$ is the $n \times n$ matrix whose $ij^{th}$ entry is $(x_{ij})_{(e)} = [x_i, x_j]_e$, the least common exponential multiple of $x_i$ and $x_j$ in D. Let $R = \{y_1, y_2, \ldots, y_m\}$ be the minimal gced-closed set containing $T$ (gced closure of $T$), such that $y_1 < y_2 < \cdots < y_m$. Define the function $f(w)$ as follows:

$$f(w) = \sum_{a_1 b_1 = c_1} \cdots \sum_{a_r b_r = c_r} \frac{1}{p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}} \mu^{(e)}\left(p_1^{b_1} p_2^{b_2} \ldots p_r^{b_r}\right),$$

where $w = p_1^{c_1} p_2^{c_2} \ldots p_r^{c_r} \in D$.

**Theorem 3.1.** *Let $T = \{x_1, x_2, \dots, x_n\}$ be a gced-closed set in $D$. Then,* $\sum_{x_k|_e(x_i,x_j)_e}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right) = \sum_{d|_e(x_i,x_j)_e} g(d)$.

*Proof.* Let $d \mid_e (x_i, x_j)_e$ and let $S = \{x_{k_1}, x_{k_2}, \dots, x_{k_r}\}$ be an ordered subset of $T$ such that $x_{k_m} \mid_e (x_i, x_j)_e$ and $d \mid_e x_{k_m}$ for every $1 \le m \le r$. Then $d \mid_e (x_{k_1}, x_{k_2}, \dots, x_{k_r})_e$ which is an element in $T$ as $T$ is a gced-closed set. Since $T$ is an ordered set, then $(x_{k_1}, x_{k_2}, \dots, x_{k_r})_e = x_{k_1}$. But $d \mid x_{k_1}$and $d \nmid_e x_r$ whenever $x_r < x_{k_1}$ as $x_{k_1}$ is the minimal element in $S$. So, each divisor of $(x_i, x_j)_e$ is found once in the sum. Hence,

$$\sum_{x_k|_e(x_i,x_j)_e}\left(\sum_{\substack{d|_e x_k\\ d\nmid_e x_r\\ x_r<x_k}} g(d)\right) = \sum_{d|_e(x_i,x_j)_e} g(d).$$

**Theorem 3.2.** $[T]_{(e)} = C\Phi C^t$*, where the $n \times m$ matrix $C = (c_{ij})$ is defined as:* $c_{ij} = \begin{cases} x_i, & \text{if } y_j \mid_e x_i \\ 0, & \text{else} \end{cases}$*, and $\Phi$ is an $m \times m$ diagonal matrix define as:* $\Phi = diag\left(\sum_{d|_e y_1} f(d), \sum_{\substack{d|_e y_2\\ d\nmid_e y_1}} f(d), \dots, \sum_{\substack{d|_e y_m\\ d\nmid_e y_r\\ y_r<y_m}} f(d)\right)$.

*Proof.* The $ij^{th}$ entry of $C\Phi C^t$ is

$$\begin{aligned}(C\Phi C^t)_{ij} &= \sum_{k=1}^{m} c_{ik}\left(\sum_{\substack{d|_e y_k\\ d\nmid_e y_r\\ y_r<y_k}} f(d)\right) c_{jk} = \sum_{\substack{y_k|_e x_i\\ y_k|_e x_j}} x_i\, x_j \left(\sum_{\substack{d|_e y_k\\ d\nmid_e y_r\\ y_r<y_k}} f(d)\right)\\ &= x_i x_j \sum_{y_k|_e(x_i,x_j)_e}\left(\sum_{\substack{d|_e y_k\\ d\nmid_e y_r\\ y_r<y_k}} f(d)\right) = x_i x_j \sum_{d|_e(x_i,x_j)_e} f(d).\end{aligned}$$

By Möbius inversion exponential formula we have,

$$\sum_{d|_e m} f(d) = \frac{1}{m}.$$

Hence,

$$(C\Phi C^t)_{ij} = \frac{x_i x_j}{(x_i, x_j)_e} = [x_i, x_j]_e.$$

**Theorem 3.3.** The determinant of the matrix $[T]_{(e)}$ is given by

$$det([T]_{(e)}) = \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} \left(det\left(C_{(k_1,k_2,\ldots,k_n)}\right)\right)^2 \prod_{i=1}^{n} \left( \sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r} < y_{k_i}}} f(d) \right),$$

*where $C_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of $C$ consisting of $k_1^{th}, k_2^{th}, \ldots, k_n^{th}$ columns of $C$.*

*Proof.* Let $D_e$ be an extension field of $D(x)$, the field of fractions of $D$, in which $\sqrt{\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r} < y_{k_i}}} f(d)}$ exists.

$[T]_{(e)} = C\Phi C^t = EE^t$, where $E = C\Phi^{\frac{1}{2}}$. Apply the Cauchy-Binet formula to get

$$\begin{aligned} det([T]_{(e)}) &= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} \left(det\left(E_{(k_1,k_2,\ldots,k_n)}\right)\right)\left(det\left(E^t_{(k_1,k_2,\ldots,k_n)}\right)\right) \\ &= \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} \left(detE_{(k_1,k_2,\ldots,k_n)}\right)^2, \end{aligned}$$

where $E_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of $E$ consisting of $k_1^{th}, k_2^{th}, \ldots, k_n^{th}$ columns of $E$. Moreover,

$$detE_{(k_1,k_2,\ldots,k_n)} = detC_{(k_1,k_2,\ldots,k_n)} \sqrt{\prod_{i=1}^{n} \left( \sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r} < y_{k_i}}} f(d) \right)}.$$

Hence,

$$det([T]_{(e)}) = \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} \left(detC_{(k_1,k_2,\ldots,k_n)}\right)^2 \prod_{i=1}^{n} \left( \sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r} < y_{k_i}}} f(d) \right).$$

**Remark 1**. *If $<$ is the ordering defined on $D$, then $\sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r} < y_{k_i}}} f(d) > 0$.*

**Example 3.1**. *Let $T = \{-2+4i, -1+7i, -12-16i\}$ which is not gcd-closed set in $\mathbf{Z}[i]$. Its gcd-closure is $R = \{1+3i, -2+4i, -1+7i, -12-16i\}$. The lcm matrix $[T]_{(e)}$ defined on $T$ is:*

$$[T]_{(e)} = \begin{bmatrix} -2+4i & -8+6i & -12-16i \\ -8+6i & -1+7i & -12-16i \\ -12-16i & -12-16i & -12-16i \end{bmatrix}.$$

*The matrix $C$ is*

$$C=\begin{bmatrix} -2+4i & -2+4i & 0 & 0 \\ -1+7i & 0 & -1+7i & 0 \\ -12-16i & -12-16i & -12-16i & -12-16i \end{bmatrix},$$

*And*

$$\Phi=\begin{bmatrix} \frac{1}{1+3i} & 0 & 0 & 0 \\ 0 & \frac{-1}{5}+\frac{i}{10} & 0 & 0 \\ 0 & 0 & \frac{-3}{25}+\frac{4i}{25} & 0 \\ 0 & 0 & 0 & \frac{19}{100}+\frac{2i}{25} \end{bmatrix}.$$

*Then,*

$$C\Phi C^t=\begin{bmatrix} -2+4i & -8+6i & -12-16i \\ -8+6i & -1+7i & -12-16i \\ -12-16i & -12-16i & -12-16i \end{bmatrix}=[T]_{(e)}.$$

$$\begin{aligned}\det[T]_{(e)} &= \begin{vmatrix} -2+4i & -2+4i & 0 \\ -1+7i & 0 & -1+7i \\ -12-16i & -12-16i & -12-16i \end{vmatrix}^2 \\ &\times \sum_{d|_e y_1} f(d) \sum_{\substack{d|_e y_2 \\ d\nmid_e y_1}} f(d) \sum_{\substack{d|_e y_3 \\ d\nmid_e y_r \\ y_r<y_3}} f(d) \\ &+ \begin{vmatrix} -2+4i & -2+4i & 0 \\ -1+7i & 0 & 0 \\ -12-16i & -12-16i & -12-16i \end{vmatrix}^2 \sum_{d|_e y_1} f(d) \sum_{\substack{d|_e y_2 \\ d\nmid_e y_1}} f(d) \sum_{\substack{d|_e y_4 \\ d\nmid_e y_r \\ y_r<y_4}} f(d) \\ &+ \begin{vmatrix} -2+4i & 0 & 0 \\ -1+7i & -1+7i & 0 \\ -12-16i & -12-16i & -12-16i \end{vmatrix}^2 \sum_{d|_e y_1} f(d) \sum_{\substack{d|_e y_3 \\ d\nmid_e y_r \\ y_r<y_3}} f(d) \sum_{\substack{d|_e y_4 \\ d\nmid_e y_r \\ y_r<y_4}} f(d) \\ &+ \begin{vmatrix} -2+4i & 0 & 0 \\ 0 & -1+7i & 0 \\ -12-16i & -12-16i & -12-16i \end{vmatrix}^2 \sum_{\substack{d|_e y_2 \\ d\nmid_e y_1}} f(d) \sum_{\substack{d|_e y_3 \\ d\nmid_e y_r \\ y_r<y_3}} f(d) \sum_{\substack{d|_e y_4 \\ d\nmid_e y_r \\ y_r<y_4}} f(d) \\ &= 2968-5176i.\end{aligned}$$

**Corollary 3.4**. *Let $T=\{x_1,x_2,\dots,x_n\}$ be a gced-closed subset of $D$. Then,*

$$det[T]_{(e)}=\prod_{k=1}^{n} x_k^2 \left( \sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} f(d) \right).$$

***Proof***. Since $T$ is is gced-closed set, the matrix $C$ is lower triangular with diagonal $(x_1,x_2,\dots,x_n)_n$. As a result,

$$det[T]_{(e)} = \prod_{k=1}^{n} x_k^2 \left( \sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} f\,(d) \right).$$

**Corollary 3.5.** *Let $T = \{x_1, x_2, \dots, x_n\}$ be a subset of $D$, then $tr\big([T]_{(e)}\big) = \sum_{i=1}^{n} x_i$.*

**Theorem 3.6.** *Let $T = \{x_1, x_2, \dots, x_n\}$ be a subset of $D$. Then,*

$$det\big([T]_{(e)}\big) = \prod_{k=1}^{n} x_k^2 \left( \sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} f\,(d) \right)$$

*if and only if $T$ is gced-closed.*

***Proof***. The necessary condition follows from Corollary 3.4. Now, assume that $T$ is not a gced-closed set and the equality holds. Theorem 3.3 gives

$$det\big([T]_{(e)}\big) = \sum_{1\le k_1<k_2<\dots<k_n\le m} \big(detC_{(k_1,k_2,\dots,k_n)}\big)^2 \prod_{i=1}^{n} \left( \sum_{\substack{d|_e y_{k_i} \\ d\nmid_e y_{k_r} \\ y_{k_r}<y_{k_i}}} f\,(d) \right).$$

This sum runs over the all combinations of the $k_i^{th}$ columns of the matrix $C$, where $1 \le i \le n$. In each combination we get a new term in this sum, as $y_{k_i}$ related to the chosen column $k_i$. Since $T$ is a subset of $R$, then

$$det\big([T]_{(e)}\big) = \prod_{k=1}^{n} \left( \sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} f\,(d) \right) + s,$$

where $s > 0$. Consequently,

$$det\big([T]_{(e)}\big) > \prod_{k=1}^{n} \left( \sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} f\,(d) \right)$$

which contradicts the necessary condition that equality holds.

**Inverse of the LCEM Matrix**

Let $T = \{x_1, x_2, \dots, x_n\}$ be a gced-closed subset of $D$ and let the $n \times n$ matrix $C = \big(c_{ij}\big)$ be defined as

$$c_{ij} = \begin{cases} x_i, & \text{if } y_j|_e x_i \\ 0, & \text{else} \end{cases}$$

**Theorem 3.1.1.** *The inverse of $C$ is the $n \times n$ matrix $W = (w_{ij})$ with*

$$w_{ij} = \begin{cases} \dfrac{1}{x_j} \displaystyle\sum_{\substack{d|_e \frac{x_i}{x_j} \\ d\nmid_e \frac{x_r}{x_j},\, x_r < x_i}} \mu^{(e)}(d), & \text{if } x_j|_e x_i \\ 0, & \text{else} \end{cases}.$$

*Proof.* The $ij^{th}$ entry of $CW$ is given by

$$(cw)_{ij} = \sum_{k=1}^{n} c_{ik}\, w_{kj} = \sum_{\substack{x_k|_e x_i \\ x_j|_e x_k}} \frac{x_i}{x_j} \left( \sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r < x_k}} \mu^{(e)}(d) \right) = \frac{x_i}{x_j} \sum_{\frac{x_k}{x_j}|_e \frac{x_i}{x_j}} \left( \sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r < x_k}} \mu^{(e)}(d) \right).$$

A similar argument to that given in Theorem 3.3,

$$\sum_{\frac{x_k}{x_j}|_e \frac{x_i}{x_j}} \left( \sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r < x_k}} \mu^{(e)}(d) \right) = \sum_{d|_e \frac{x_i}{x_j}} \mu^{(e)}(d).$$

Therefore,

$$\frac{x_i}{x_j} \sum_{d|_e \frac{x_i}{x_j}} \mu^{(e)}(d) = \frac{x_i}{x_j} \mu^2 \left( \frac{x_i}{x_j} \right) = \begin{cases} 1, & \text{if } x_j = x_i \\ 0, & \text{else} \end{cases}.$$

**Theorem 3.1.2.** *The inverse of the lcem matrix $[T]_{(e)}$ is the matrix $[M]_{(e)} = (m_{ij})_{(e)}$ which is defined as:*

$$(m_{ij}) = \frac{1}{x_i x_j} \sum_{\substack{x_i|_e x_k \\ x_j|_e x_k}} \left( \sum_{\substack{d|_e x_k/x_i \\ d\nmid_e x_r/x_i \\ x_r < x_k}} \mu^{(e)}(d) \frac{1}{\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r < x_k}} g(d)} \sum_{\substack{d|_e x_k/x_j \\ d\nmid_e x_r/x_j \\ x_r < x_k}} \mu^{(e)}(d) \right).$$

*Proof.* $[M]_{(e)} = [T]_{(e)}^{-1} = (C\Phi C^t)^{-1} = W^t \Phi^{-1} W$, where $W = C^{-1}$ and

$$\Phi^{-1} = diag\left(\frac{1}{\sum_{d|_e x_1} f(d)}, \frac{1}{\sum_{\substack{d|_e x_2 \\ d\nmid_e x_1}} f(d)}, \ldots, \frac{1}{\sum_{\substack{d|_e x_n \\ d\nmid_e x_r \\ x_r<x_n}} f(d)}\right).$$

So,

$$m_{ij} = (W^t\Phi^{-1}W)_{ij} = \sum_{k=1}^{n} w_{ki} \frac{1}{\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} f(d)} w_{kj}$$

$$= \frac{1}{x_i x_j} \sum_{\substack{x_i|_e x_k \\ x_j|_e x_k}} \left( \sum_{\substack{d|_e \frac{x_k}{x_i} \\ d\nmid_e \frac{x_r}{x_i} \\ x_r<x_k}} \mu^{(e)}(d) \frac{1}{\sum_{\substack{d|_e x_k \\ d\nmid_e x_r \\ x_r<x_k}} g(d)} \sum_{\substack{d|_e \frac{x_k}{x_j} \\ d\nmid_e \frac{x_r}{x_j} \\ x_r<x_k}} \mu^{(e)}(d) \right).$$

**Example 3.1.1.** *Let $T = \{x(x+1), x(x+1)^2, x^2(x+1)^2\}$ which is gced-closed set in $\mathbf{Z}_2[x]$. The lcem matrix defined on $T$ is:*

$$[T]_{(e)} = \begin{bmatrix} x(x+1) & x(x+1)^2 & x^2(x+1)^2 \\ x(x+1)^2 & x(x+1)^2 & x^2(x+1)^2 \\ x^2(x+1)^2 & x^2(x+1)^2 & x^2(x+1)^2 \end{bmatrix}.$$

*Then,*

$$\begin{aligned} m_{11} &= \frac{1}{x^2(x+1)^2}\Big(\mu^{(e)}\big(x(x+1)\big)\frac{1}{f\big(x(x+1)\big)}\mu^{(e)}\big(x(x+1)\big) \\ &\quad + \mu^{(e)}(x(x+1)^2)\frac{1}{f(x(x+1)^2)}\mu^{(e)}(x(x+1)^2) \\ &\quad + \big[\mu^{(e)}\big(x^2(x+1)\big) + \mu^{(e)}(x^2(x+1)^2)\big]\frac{1}{f\big(x^2(x+1)\big) + f(x^2(x+1)^2)} \times \\ &\quad \big[\mu^{(e)}\big(x^2(x+1)\big) + \mu^{(e)}(x^2(x+1)^2)\big]\Big) \\ &= \frac{1}{x^2(x+1)}. \end{aligned}$$

$$\begin{aligned} m_{12} &= \frac{1}{x^2(x+1)^3}\Big(\mu^{(e)}(x(x+1)^2)\frac{1}{f(x(x+1)^2)}\mu^{(e)}\big(x(x+1)\big) \\ &\quad + \big[\mu^{(e)}\big(x^2(x+1)\big) + \mu^{(e)}(x^2(x+1)^2)\big]\frac{1}{f\big(x^2(x+1)\big) + f(x^2(x+1)^2)}\mu^{(e)}\big(x^2(x+1)\big)\Big) \\ &= -\frac{(x+1)^2}{x^2(x+1)^3} = \frac{1}{x^2(x+1)}. \end{aligned}$$

$$\begin{aligned} m_{13} &= \frac{1}{x^3(x+1)^3}\big[\mu^{(e)}\big(x^2(x+1)\big) + \mu^{(e)}(x^2(x+1)^2)\big]\frac{1}{f\big(x^2(x+1)\big) + f(x^2(x+1)^2)}\mu^{(e)}\big(x(x+1)\big) \\ &= 0 \end{aligned}$$

*Completing the computation, we get*

$$[M]_{(e)} = \begin{bmatrix} \frac{1}{x^2(x+1)} & \frac{1}{x^2(x+1)} & 0 \\ \frac{1}{x^2(x+1)} & \frac{x^2+x+1}{x^2(x+1)^3} & \frac{1}{x(x+1)^3} \\ 0 & \frac{1}{x(x+1)^3} & \frac{1}{x^2(x+1)^3} \end{bmatrix}.$$

## Conclusion

In conclusion, the lcem matrices defined on gced closed and non-gced closed sets over a unique factorization domain $D$ were considered. A complete characterization of their structure, determinant, trace, and inverse was given. Furthermore, the work done in the literature used the classical domain (domain of natural integers), which is an example of a UFD and therefore the previous research can be viewed a special case with the domain of integers representing our chosen UFD.

## Scientific Ethics Declaration

The authors declare that the scientific ethical and legal responsibility of this article published in EPSTEM journal belongs to the authors.


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