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Global Behavior of a Nonlinear System of Difference Equations

Fatma Hilal Gümüş [1,†] **and Raafat Abo-Zeid** [2,‡,*]

[1] *Department of Business Administration, Faculty of Economics and Administrative Science, Zonguldak Bülent Ecevit University, Türkiye*
[2] *Department of Basic Science, The Higher Institute for Engineering & Technology, Al-Obour, Cairo, Egypt*
[†] *gumus@beun.edu.tr*, [‡] *abuzead73@yahoo.com*
[*] *Corresponding Author*

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Abstract

In this paper, we study the admissible solutions of the nonlinear system of difference equations

$$x_{n+1}=\frac{y_n}{x_n}, \quad y_{n+1}=\frac{y_n}{\breve{a}x_n+\breve{b}y_n}, \quad n=0,1,\ldots,$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial values x_0, y_0 are nonzero real numbers. In case $\breve{b}<0$ and $\breve{a}^2<-4\breve{b}$, we show that there are eventually periodic solutions when either $tan^{-1}\frac{\sqrt{-4\breve{b}-\breve{a}^2}}{\breve{a}} \in]\frac{\pi}{2},\pi[$ (with $\breve{a}<0$) is a rational multiple of π or $tan^{-1}\frac{\sqrt{-4\breve{b}-\breve{a}^2}}{\breve{a}} \in]0,\frac{\pi}{2}[$ (with $\breve{a}>0$) as well.

1. Introduction

Difference equations and systems of difference equations occur in the applications of mathematics in growth and decay models, physics, economics, biology, circuit analysis, dynamical systems and other fields. It can be appeared as an approximation to solutions of differential equations. To study the behavior of the solutions to systems of difference equations, we may be able derive its solutions otherwise, we can investigate its long-term behaviors via the stability of its equilibrium points.

In [1], Kudlak et al. studied the existence of unbounded solutions of the system of difference equations

$$x_{n+1}=\frac{x_n}{y_n},\ y_{n+1}=x_n+\gamma_n y_n,\ n=0,1,\ldots,$$

where $0<\gamma_n<1$ and the initial values are positive real numbers.

Camouzis et al. [2], studied the global behavior of the system of difference equations

$$x_{n+1}=\frac{\alpha_1+\gamma_1 y_n}{x_n},\ y_{n+1}=\frac{\beta_2 x_n+\gamma_2 y_n}{B_2 x_n+C_2 y_n},\ n=0,1,\ldots, \tag{1.1}$$

with nonnegative parameters and positive initial conditions. They studied the boundedness character of the system (1.1) in its special cases.

In [3], Camouzis et al. studied the solutions of the system

$$x_{n+1}=\frac{y_n}{x_n},\ y_{n+1}=\frac{\gamma_2 y_n}{A_2+B_2 x_n+y_n},\ n=0,1,\ldots,$$

with nonnegative parameters and positive initial conditions.

Cinar [4], studied the positive solutions of the system of difference equations

$$x_{n+1}=\frac{1}{y_n},\ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}},\ n=0,1,\ldots,$$

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where the initial values x_0, y_0, x_{-1} and y_{-1} are positive real numbers.

Clark and Kulenovic [5], studied the global stability properties and asymptotic behavior of solutions of the system of difference equation

$$x_{n+1} = \frac{x_n}{a+cy_n},\ y_{n+1} = \frac{y_n}{b+dx_n},\ n = 0,1,\ldots,$$

where a,b,c,d are positive real numbers and the initial values x_0, y_0 are nonnegative real numbers. For more on difference equations, see [6]-[27] and the references therein. For more on systems of difference equations that are solved in closed form, see [28]-[33] and the references therein.

In this paper, we study the admissible solutions of the nonlinear system of difference equations

$$x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{y_n}{\breve{a}x_n + \breve{b}y_n}, \quad n = 0,1,\ldots, \tag{1.2}$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial values x_0, y_0 are nonzero real numbers.

Consider the k^{th}-order difference equation

$$x_{n+1} = h(x_n, x_{n-1}, \ldots, x_{n-k+1}),\ n = 0,1,\ldots. \tag{1.3}$$

where the initial values $x_0, x_{-1}, \ldots,$ and x_{-k+1} are real numbers. The set

$$H = \{(x_0, x_{-1}, \ldots, x_{-k+1}) \in \mathbb{R}^k : x_n \text{ is undefined for some } n \in \mathbb{N}\},$$

is called the Forbidden set to Equation (1.3). The complement of the Forbidden set is called the Good set. Any solution $\{x_n\}_{n=-k+1}^{\infty}$ to Equation (1.3) with initial values belongs to the Good set is well-defined or admissible solution to Equation (1.3).

2. Case $\breve{a}\breve{b} = 0$

In this section, we shall investigate the case $\breve{a}\breve{b} = 0$.

Assume that $\breve{a} = 0$. Then the solution of system (1.2) is

$$\begin{cases} x_{2n} = \dfrac{x_0}{\breve{b}y_0} & ,n = 1,2,\ldots, \\[2ex] x_{2n+1} = \dfrac{y_0}{x_0} & ,n = 1,2,\ldots, \\[2ex] y_n = \dfrac{1}{\breve{b}} & ,n = 1,2\ldots. \end{cases} \tag{2.1}$$

It is clear that, every admissible solution of system (1.2) is eventually 2-periodic.

In fact, for any admissible solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1.2), we have

$$(x_{2n+1}, y_{2n+1}) = (x_{2n-1}, y_{2n-1}) = \left(\frac{y_0}{x_0}, \frac{1}{\breve{b}}\right),\ n = 1,2,\ldots,$$

and

$$(x_{2n+2}, y_{2n+2}) = (x_{2n}, y_{2n}) = \left(\frac{x_0}{\breve{b}y_0}, \frac{1}{\breve{b}}\right),\ n = 1,2,\ldots.$$

Now assume that $\breve{b} = 0$. Then the solution of system (1.2) is

$$\begin{cases} x_n = \dfrac{1}{\breve{a}} & ,n = 2,3,\ldots, \\[2ex] y_n = \dfrac{1}{\breve{a}^2} & ,n = 2,3,\ldots. \end{cases} \tag{2.2}$$

In this case, every admissible solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1.2) converges to $\left(\dfrac{1}{\breve{a}}, \dfrac{1}{\breve{a}^2}\right)$.

3. Case $\check{a}\check{b} > 0$

In this section, we shall derive the admissible solutions of system (1.2) and investigate the global stability of its equilibrium points when $\check{a}\check{b} > 0$.

3.1. Case $\check{a} > 0$ and $\check{b} > 0$

Assume that $\check{a}$ and $\check{b}$ are positive real numbers. For system (1.2), we can write

$$u_{n+1} = \check{a} + \frac{\check{b}}{u_n}, \ n = 0, 1, ..., \tag{3.1}$$

where

$$u_n = \frac{x_n}{y_n}, \text{ with } u_0 = \frac{x_0}{y_0}.$$

Solving Equation (3.1) and substituting in system (1.2), we can write the admissible solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{\check{b}y_0\theta_{n-2} + x_0\theta_{n-1}}{\check{b}y_0\theta_{n-1} + x_0\theta_n} & ,n = 1, 2, ..., \\ \\ y_n = \dfrac{\check{b}y_0\theta_{n-2} + x_0\theta_{n-1}}{\check{b}y_0\theta_n + x_0\theta_{n+1}} & ,n = 1, 2..., \end{cases} \tag{3.2}$$

where $\theta_j = \dfrac{t_1^j - t_2^j}{\sqrt{\check{a}^2 + 4\check{b}}}$, $t_1 = \dfrac{\check{a} + \sqrt{\check{a}^2 + 4\check{b}}}{2}$ and $t_2 = \dfrac{\check{a} - \sqrt{\check{a}^2 + 4\check{b}}}{2}$, $j = -1, 0, ...$

The forbidden set for system (1.2) can be written as

$$F_1 = \bigcup_{j=1}^{2}\{(v_1, v_2) \in \mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1, v_2) \in \mathbb{R}^2 : v_1 = -\frac{\theta_n}{\theta_{n+1}}\check{b}v_2\}.$$

The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = \frac{\bar{y}}{\check{a}\bar{x} + \check{b}\bar{y}}.$$

Then we have two equilibrium points $E_1(\bar{x}_1, \bar{y}_1)$ and $E_2(\bar{x}_2, \bar{y}_2)$, where $\bar{x}_1$ and $\bar{x}_2$ are the solutions of the equation

$$\check{b}x^2 + \check{a}x - 1 = 0.$$

Consider the associated system of system (1.2)

$$G_1(x, y) = (y/x, y/(\check{a}x + \check{b}y)). \tag{3.3}$$

The Jacobian matrix corresponding to system (3.3) at an equilibrium point of system (1.2) is

$$J_{G_1}(\bar{x}, \bar{y}) = \begin{pmatrix} -1 & \frac{1}{\bar{x}} \\ -\check{a}\bar{y} & \check{a}\bar{x} \end{pmatrix}.$$

For more results on the stability of difference equations, see [24].

Theorem 3.1. *The following statements are true:*

1. *The equilibrium point $E_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) is locally asymptotically stable.*
2. *The equilibrium point $E_2(\bar{x}_2, \bar{y}_2)$ of system (1.2) is unstable (saddle point).*

Proof. The eigenvalues of the Jacobian matrix $J_{G_1}(\bar{x}, \bar{y})$ are $\lambda_1 = 0$ and $\lambda_2 = -\check{b}\bar{y}$. Then $|\lambda_2| = \check{b}\bar{y} = 1 - \check{a}\bar{x}$.

1. For the equilibrium point $E_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) we have that

$$0 < \bar{x}_1 = -\frac{\check{a}}{2\check{b}} + \frac{\sqrt{\check{a}^2 + 4\check{b}}}{2\check{b}} < \frac{1}{\check{a}}.$$

   This implies that

$$0 < \lambda_2 = 1 - \check{a}\bar{x}_1 < 1,$$

   and the result follows.

2. For the equilibrium point $E_2(\bar{x}_2,\bar{y}_2)$ of system (1.2) there is nothing to say, since $\bar{x}_2 = -\frac{\check{a}}{2\check{b}} - \frac{\sqrt{\check{a}^2+4\check{b}}}{2\check{b}} < 0$.

□

Theorem 3.2. *The equilibrium point $E_1(\bar{x}_1,\bar{y}_1)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). Then using the solution form (3.2) we get

$$\begin{aligned} x_n &= \frac{\check{b}y_0\theta_{n-2}+x_0\theta_{n-1}}{\check{b}y_0\theta_{n-1}+x_0\theta_n} \\ &= \frac{\theta_{n-2}}{\theta_{n-1}}\frac{\check{b}y_0+x_0\frac{\theta_{n-1}}{\theta_{n-2}}}{\check{b}y_0+x_0\frac{\theta_n}{\theta_{n-1}}} \qquad \to \bar{x}_1 \text{ as } n\to\infty, \end{aligned}$$

where $\frac{\theta_n}{\theta_{n-1}} \to t_1$ as $n\to\infty$. Similarly,

$$\begin{aligned} y_n &= \frac{\check{b}y_0\theta_{n-2}+x_0\theta_{n-1}}{\check{b}y_0\theta_n+x_0\theta_{n+1}} \\ &= \frac{\theta_{n-2}}{\theta_n}\frac{\check{b}y_0+x_0\frac{\theta_{n-1}}{\theta_{n-2}}}{\check{b}y_0+x_0\frac{\theta_{n+1}}{\theta_n}} \qquad \to \bar{y}_1 \text{ as } n\to\infty. \end{aligned}$$

Then the equilibrium point $E_1(\bar{x}_1,\bar{y}_1)$ of system (1.2) is a global attractor of all admissible solutions of system (1.2). In view of Theorem (3.1), we conclude that the equilibrium point $E_1(\bar{x}_1,\bar{y}_1)$ of system (1.2) is globally asymptotically stable. □

3.2. Case $\check{a}<0$ and $\check{b}<0$

Assume that $\check{a}$ and $\check{b}$ are negative real numbers. We can write $\check{a}=-a$ and $\check{b}=-b$ for some positive reals a and b.
For system (1.2), we can write

$$u_{n+1} = -a - \frac{b}{u_n},\ n=0,1,..., \tag{3.4}$$

where

$$u_n = \frac{x_n}{y_n},\ \text{with } u_0 = \frac{x_0}{y_0}.$$

We shall consider three cases:
Case $a^2>4b$
Solving Equation (3.4) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{by_0\psi_{n-2}-x_0\psi_{n-1}}{by_0\psi_{n-1}-x_0\psi_n} & ,n=1,2,..., \\[2ex] y_n = \dfrac{by_0\psi_{n-2}-x_0\psi_{n-1}}{by_0\psi_n-x_0\psi_{n+1}} & ,n=1,2..., \end{cases} \tag{3.5}$$

where $\psi_j = \frac{t_+^j - t_-^j}{\sqrt{a^2-4b}}$, $t_+ = \frac{-a+\sqrt{a^2-4b}}{2}$ and $t_- = \frac{-a-\sqrt{a^2-4b}}{2}$, $j=-1,0,....$
The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = -\frac{\bar{y}}{a\bar{x}+b\bar{y}}.$$

Then we have two equilibrium points $L_+(\bar{x}_+,\bar{y}_+)$ and $L_-(\bar{x}_-,\bar{y}_-)$, where $\bar{x}_+$ and $\bar{x}_-$ are the solutions of the equation

$$bx^2+ax+1=0.$$

Theorem 3.3. *The following statements are true:*

1. *The equilibrium point $L_+(\bar{x}_+,\bar{y}_+)$ of system (1.2) is locally asymptotically stable.*

2. *The equilibrium point $L_-(\bar{x}_-, \bar{y_-})$ of system (1.2) is unstable (saddle point).*

Proof. Consider the associated system of system (1.2)

$$G_2(x,y) = (y/x, -y/(ax+by)). \tag{3.6}$$

The Jacobian matrix corresponding to system (3.6) at an equilibrium point of system (1.2) is

$$J_{G_2}(\bar{x},\bar{y}) = \begin{pmatrix} -1 & \frac{1}{\bar{x}} \\ -a\bar{y} & -a\bar{x} \end{pmatrix}. \tag{3.7}$$

The eigenvalues of the Jacobian matrix $J_{G_2}(\bar{x},\bar{y})$ are $\lambda_1 = 0$ and $\lambda_2 = -1 - a\bar{x}$.

1. For the equilibrium point $L_+(\bar{x}_+, \bar{y_+})$ of system (1.2) we have that

$$-\frac{2}{a} < \bar{x}_+ = -\frac{a}{2b} + \frac{\sqrt{a^2-4b}}{2b} < -\frac{1}{a}.$$

   This implies that

$$0 < \lambda_2 = -1 - a\bar{x}_+ < 1,$$

   and the result follows.
2. For the equilibrium point $L_-(\bar{x}_-, \bar{y}_-)$ of system (1.2), we have

$$\bar{x}_- = -\frac{a}{2b} - \frac{\sqrt{a^2-4b}}{2b} < -\frac{2}{a}.$$

   Then

$$\lambda_2 = -1 - a\bar{x}_- > 1.$$

   Therefore, the equilibrium point $L_-(\bar{x}_-, \bar{y}_-)$ of system (1.2) is unstable (saddle point).

□

Theorem 3.4. *The equilibrium point $L_+(\bar{x}_+, \bar{y_+})$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). For the global attractivity of the equilibrium point $L_+(\bar{x}_+, \bar{y_+})$, it is sufficient to see that

$$\frac{\psi_n}{\psi_{n-1}} \to t_- \text{ as } n \to \infty.$$

In view of Theorem (3.3), we conclude that the equilibrium point $L_+(\bar{x}_+, \bar{y_+})$ of system (1.2) is globally asymptotically stable. □

Case $a^2 = 4b$

Suppose that $a^2 = 4b$. Solving Equation (3.4) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = -\dfrac{2}{a}\dfrac{ay_0(n-2)+2x_0(n-1)}{ay_0(n-1)+2x_0 n} & ,n = 1,2,\ldots, \\ y_n = \left(-\dfrac{2}{a}\right)^2 \dfrac{ay_0(n-2)+2x_0(n-1)}{ay_0 n+2x_0(n+1)} & ,n = 1,2.\ldots \end{cases} \tag{3.8}$$

Theorem 3.5. *The unique equilibrium point $L\left(-\frac{2}{a}, \frac{4}{a^2}\right)$ of system (1.2) is nonhyperbolic point.*

Proof. There is nothing to say except that, the eigenvalues of the Jacobian matrix (4.8) are

$$\lambda_1 = 0 \text{ and } \lambda_2 = -1 - a\bar{x} = -1 - a\left(-\frac{2}{a}\right) = 1.$$

□

From the solution form (3.8), we conclude that, every admissible solution for system (1.2) converges to the unique equilibrium point $L\left(-\frac{2}{a},\frac{4}{a^2}\right)$.

Case $a^2<4b$

Suppose that $a^2<4b$. Solving Equation (3.4) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n=\frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0\sin(n-2)\alpha-x_0\sin(n-1)\alpha}{\sqrt{b}y_0\sin(n-1)\alpha-x_0\sin n\alpha} & ,n=1,2,..., \\ y_n=\frac{1}{b}\frac{\sqrt{b}y_0\sin(n-2)\alpha-x_0\sin(n-1)\alpha}{\sqrt{b}y_0\sin n\alpha-x_0\sin(n+1)\alpha} & ,n=1,2..., \end{cases} \tag{3.9}$$

where $\alpha=tan^{-1}\frac{-\sqrt{4b-a^2}}{a}\in]\frac{\pi}{2},\pi[$.

Theorem 3.6. *Assume that $a^2<4b$. If $\alpha=\frac{l}{k}\pi$ is a rational multiple of π (l and k are relatively positive prime integers) such that $\frac{k}{2}<l<k$. Then every admissible solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2) is eventually k-periodic.*

Proof. Assume that $\alpha=\frac{l}{k}\pi$ is a rational multiple of π (l and k are relatively positive prime integers) such that $\frac{k}{2}<l<k$ and let $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2). Then for $n\geq 1$, we have

$$\begin{aligned} x_{n+k} &= \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0\sin(n+k-2)\alpha-x_0\sin(n+k-1)\alpha}{\sqrt{b}y_0\sin(n+k-1)\alpha-x_0\sin n+k\alpha} \\ &= \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0(-1)^l\sin(n+k-2)\alpha-x_0(-1)^l\sin(n+k-1)\alpha}{\sqrt{b}y_0(-1)^l\sin(n+k-1)\alpha-x_0(-1)^l\sin n+k\alpha} \\ &= \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0\sin(n-2)\alpha-x_0\sin(n-1)\alpha}{\sqrt{b}y_0\sin(n-1)\alpha-x_0\sin n\alpha} \\ &= x_n. \end{aligned}$$

Similarly, we can see that $y_{n+k}=y_n$ for all $n\geq 1$.

Therefore, the admissible solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ of system (1.2) is eventually k-periodic (in fact except for the initial point (x_0,y_0)). □

The forbidden set for system (1.2) depends on the relation between a and b. For system (1.2) we have the following:

1. If $a^2>4b$, then the forbidden set of system (1.2) is

$$F_2=\bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2:v_j=0\}\cup\bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2:v_1=\frac{\theta_n}{\theta_{n+1}}bv_2\}.$$

2. If $a^2=4b$, then the forbidden set of system (1.2) is

$$F_3=\bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2:v_j=0\}\cup\bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2:v_1=-\frac{n}{n+1}(\frac{a}{2})v_2\}.$$

3. If $a^2<4b$, then the forbidden set of system (1.2) is

$$F_4=\bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2:v_j=0\}\cup\bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2:v_1=\sqrt{b}\frac{\sin n\alpha}{\sin(n+1)\alpha}v_2\}.$$

4. Case $\check{a}\check{b}<0$

In this section, we shall derive the solution of system (1.2) and investigate the global stability of its equilibrium points when $\check{a}\check{b}<0$.

4.1. Case $\breve{a} < 0$ and $\breve{b} > 0$

Assume that $\breve{a} = -a < 0$ and $\breve{b} = b > 0$. Then we can write system (1.2) as

$$u_{n+1} = -a + \frac{b}{u_n},\ n = 0, 1, ..., \tag{4.1}$$

where

$$u_n = \frac{x_n}{y_n},\ \text{with}\ u_0 = \frac{x_0}{y_0}.$$

Solving Equation (4.1) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{by_0\acute{\theta}_{n-2} + x_0\acute{\theta}_{n-1}}{by_0\acute{\theta}_{n-1} + x_0\acute{\theta}_n} & ,n = 1, 2, ..., \\ \\ y_n = \dfrac{by_0\acute{\theta}_{n-2} + x_0\acute{\theta}_{n-1}}{by_0\acute{\theta}_n + x_0\acute{\theta}_{n+1}} & ,n = 1, 2..., \end{cases} \tag{4.2}$$

where $\acute{\theta}_j = \dfrac{\acute{t}_1^j - \acute{t}_2^j}{\sqrt{a^2+4b}}$, $\acute{t}_1 = \dfrac{-a+\sqrt{a^2+4b}}{2}$ and $\acute{t}_2 = \dfrac{-a-\sqrt{a^2+4b}}{2}$, $j = -1, 0, ...$

The forbidden set of system (1.2) can be written as

$$F_5 = \bigcup_{j=1}^{2}\{(v_1, v_2) \in \mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1, v_2) \in \mathbb{R}^2 : v_1 = -\frac{\acute{\theta}_n}{\acute{\theta}_{n+1}} bv_2\}.$$

The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}}\ \text{and}\ \bar{y} = \frac{\bar{y}}{-a\bar{x} + b\bar{y}}.$$

Then we have two equilibrium points $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ and $\acute{E}_2(\bar{x}_2, \bar{y}_2)$, where $\bar{x}_1$ and $\bar{x}_2$ are the solutions of the equation

$$bx^2 - ax - 1 = 0.$$

Theorem 4.1. *The following statements are true:*

1. *The equilibrium point $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) is unstable (saddle point).*
2. *The equilibrium point $\acute{E}_2(\bar{x}_2, \bar{y}_2)$ of system (1.2)is locally asymptotically stable.*

Proof. Consider the associated system of system (1.2)

$$G_3(x, y) = (y/x, y/(-ax + by)). \tag{4.3}$$

The Jacobian matrix corresponding to system (4.3) at an equilibrium point of system (1.2) is

$$J_{G_3}(\bar{x}, \bar{y}) = \begin{pmatrix} -1 & \frac{1}{\bar{x}} \\ a\bar{y} & -a\bar{x} \end{pmatrix}. \tag{4.4}$$

The eigenvalues of the Jacobian matrix $J_{G_3}(\bar{x}, \bar{y})$ are $\acute{\lambda}_1 = 0$ and $\acute{\lambda}_2 = -1 - a\bar{x}$.

1. For the equilibrium point $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ of system (1.2), we have

$$1 + a\bar{x}_1 = 1 + a(\frac{a}{2b} + \frac{\sqrt{a^2+4b}}{2b}) > 1.$$

Then

$$|\acute{\lambda}_2| = 1 + a\bar{x}_1 > 1.$$

Therefore, the equilibrium point $\acute{E}_1(\bar{x}_1, \bar{y}_1)$ of system (1.2) is unstable (saddle point).

2. For the equilibrium point $\acute{E}_2(\bar{\acute{x}}_2, \bar{\acute{y}}_2)$ of system (1.2) we have that

$$-\frac{2}{a} < \bar{\acute{x}}_2 = \frac{a}{2b} - \frac{\sqrt{a^2+4b}}{2b}.$$

This implies that

$$0 < |\acute{\lambda}_2| = |-1 - a\bar{\acute{x}}_2| < 1,$$

and the result follows.

□

Theorem 4.2. *The equilibrium point $\acute{E}_2(\bar{\acute{x}}_2, \bar{\acute{y}}_2)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). For the global attractivity of the equilibrium point $\acute{E}_2(\bar{\acute{x}}_2, \bar{\acute{y}}_2)$, it is sufficient to see that $\dfrac{\acute{\theta}_n}{\acute{\theta}_{n-1}} \to \acute{t}_2$ as $n \to \infty$.
In view of Theorem (4.1), we conclude that the equilibrium point $\acute{E}_2(\bar{\acute{x}}_2, \bar{\acute{y}}_2)$ of system (1.2) is globally asymptotically stable. □

4.2. Case $\check{a} > 0$ and $\check{b} < 0$

Assume that $\check{a} = a > 0$ and $\check{b} = -b < 0$. Then we can write system (1.2) as

$$u_{n+1} = a - \frac{b}{u_n}, \; n = 0, 1, ..., \tag{4.5}$$

where

$$u_n = \frac{x_n}{y_n}, \text{ with } u_0 = \frac{x_0}{y_0}.$$

We shall consider three cases:
Case $a^2 > 4b$
Solving Equation (4.5) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \dfrac{by_0\acute{\psi}_{n-2} - x_0\acute{\psi}_{n-1}}{by_0\acute{\psi}_{n-1} - x_0\acute{\psi}_n} & ,n = 1, 2, ..., \\ \\ y_n = \dfrac{by_0\acute{\psi}_{n-2} - x_0\acute{\psi}_{n-1}}{by_0\acute{\psi}_n - x_0\acute{\psi}_{n+1}} & ,n = 1, 2..., \end{cases} \tag{4.6}$$

where $\acute{\psi}_j = \dfrac{\acute{t}_+^j - \acute{t}_-^j}{\sqrt{a^2-4b}}$, $\acute{t}_+ = \dfrac{a+\sqrt{a^2-4b}}{2}$ and $\acute{t}_- = \dfrac{a-\sqrt{a^2-4b}}{2}$, $j = -1, 0, ...$
The equilibrium points of system (1.2) satisfy the equations

$$\bar{x} = \frac{\bar{y}}{\bar{x}} \text{ and } \bar{y} = \frac{\bar{y}}{a\bar{x} - b\bar{y}}.$$

Then we have two equilibrium points $\acute{L}_+(\bar{\acute{x}}_+, \bar{\acute{y}}_+)$ and $\acute{L}_-(\bar{\acute{x}}_-, \bar{\acute{y}}_-)$, where $\bar{\acute{x}}_+$ and $\bar{\acute{x}}_-$ are the admissible solutions of the equation

$$bx^2 - ax + 1 = 0.$$

Theorem 4.3. *The following statements are true:*

1. *The equilibrium point $\acute{L}_+(\bar{\acute{x}}_+, \bar{\acute{y}}_+)$ of system (1.2) is unstable (saddle point).*
2. *The equilibrium point $\acute{L}_-(\bar{\acute{x}}_-, \bar{\acute{y}}_-)$ of system (1.2) is locally asymptotically stable.*

Proof. Consider the associated system of system (1.2)

$$G_4(x, y) = (y/x, y/(ax - by)). \tag{4.7}$$

The Jacobian matrix corresponding to system (4.7) at an equilibrium point of system (1.2) is

$$J_{G_4}(\bar{x}, \bar{y}) = \begin{pmatrix} -1 & \dfrac{1}{\bar{x}} \\ -a\bar{y} & a\bar{x} \end{pmatrix}. \tag{4.8}$$

The eigenvalues of the Jacobian matrix $J_{G_4}(\bar{x}, \bar{y})$ are $|\lambda_1| = 0$ and $|\lambda_2| = a\bar{x} - 1$.

1. For the equilibrium point $\acute{L}_+(\bar{x}_+, \bar{y}_+)$ of system (1.2) we have that
$$\bar{x}_+ = \frac{a}{2b} + \frac{\sqrt{a^2-4b}}{2b} > \frac{a}{2b} > \frac{2}{a}.$$
This implies that
$$|\lambda_2| = a\bar{x}_+ - 1 > 1,$$
and the result follows.
2. For the equilibrium point $\acute{L}_-(\bar{x}_-, \bar{y}_-)$ of system (1.2), we have $\frac{1}{a} < \bar{x}_- = \frac{a}{2b} - \frac{\sqrt{a^2-4b}}{2b} < \frac{2}{a}$. Then
$$|\lambda_2| = a\bar{x}_- - 1 < 1.$$
Therefore, the equilibrium point $\acute{L}_-(\bar{x}_-, \bar{y}_-)$ of system (1.2) is locally asymptotically stable.

□

Theorem 4.4. *The equilibrium point $\acute{L}_-(\bar{x}_-, \bar{y}_-)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be an admissible solution for system (1.2). For the global attractivity of the equilibrium point $\acute{L}_-(\bar{x}_-, \bar{y}_-)$, it is sufficient to see that $\frac{\psi_n}{\psi_{n-1}} \to \acute{t}_+$ as $n \to \infty$.
In view of Theorem (4.3), we conclude that the equilibrium point $\acute{L}_-(\bar{x}_-, \bar{y}_-)$ of system (1.2) is globally asymptotically stable. □

Case $a^2 = 4b$
Suppose that $a^2 = 4b$. Solving Equation (4.5) and substituting in system (1.2), we can write the admissible solution of system (1.2) as

$$\begin{cases} x_n = \frac{2}{a}\frac{ay_0(n-2) - 2x_0(n-1)}{ay_0(n-1) - 2x_0 n} & ,n = 1,2,..., \\ y_n = \left(\frac{2}{a}\right)^2 \frac{ay_0(n-2) - 2x_0(n-1)}{ay_0 n - 2x_0(n+1)} & ,n = 1,2.... \end{cases} \tag{4.9}$$

Theorem 4.5. *The unique equilibrium point $\acute{L}\left(\frac{2}{a}, \frac{4}{a^2}\right)$ of system (1.2) is nonhyperbolic point.*

Proof. There is nothing to say except that, the eigenvalues of the Jacobian matrix (4.8) are
$$\lambda_1 = 0 \text{ and } \lambda_2 = a\bar{x} - 1 = a(\frac{2}{a}) - 1 = 1.$$

□

From the admissible solution form (4.9), we conclude that, every admissible solution for system (1.2) converges to the unique equilibrium point $\acute{L}(\frac{2}{a}, \frac{4}{a^2})$.

Case $a^2 < 4b$
Suppose that $a^2 < 4b$. Solving Equation (4.5) and substituting in system (1.2), we can write the solution of system (1.2) as

$$\begin{cases} x_n = \frac{1}{\sqrt{b}}\frac{\sqrt{b}y_0 \sin(n-2)\beta - x_0\sin(n-1)\beta}{\sqrt{b}y_0\sin(n-1)\beta - x_0\sin n\beta} & ,n = 1,2,..., \\ y_n = \frac{1}{b}\frac{\sqrt{b}y_0\sin(n-2)\beta - x_0\sin(n-1)\beta}{\sqrt{b}y_0\sin n\beta - x_0\sin(n+1)\beta} & ,n = 1,2..., \end{cases} \tag{4.10}$$

where $\beta = tan^{-1}\frac{\sqrt{4b-a^2}}{a} \in]0, \frac{\pi}{2}[$.

Theorem 4.6. *Assume that $a^2 < 4b$. If $\beta = \frac{l}{k}\pi$ is a rational multiple of π (l and k are relatively positive prime integers) such that $0 < l < \frac{k}{2}$. Then every admissible solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (1.2) is eventually k-periodic.*

Proof. The proof is similar to that of Theorem (3.6) and is omitted. □

We end this subsection by introducing the forbidden set for system (1.2), which depends on the relation between a and b. For system (1.2) we have the following:

1. If $a^2 > 4b$, then the forbidden set of system (1.2) is

$$F_6 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \frac{\acute{\psi}_n}{\acute{\psi}_{n+1}} b v_2\}.$$

2. If $a^2 = 4b$, then the forbidden set of system (1.2) is

$$F_7 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \frac{n}{n+1}(\frac{a}{2})v_2\}.$$

3. If $a^2 < 4b$, then the forbidden set of system (1.2) is

$$F_8 = \bigcup_{j=1}^{2}\{(v_1,v_2)\in\mathbb{R}^2 : v_j = 0\} \cup \bigcup_{n=1}^{\infty}\{(v_1,v_2)\in\mathbb{R}^2 : v_1 = \sqrt{b}\frac{\sin n\beta}{\sin(n+1)\beta} v_2\}.$$

Conclusion

In this work, we derived and studied the admissible solutions of the nonlinear system of difference equations

$$x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{y_n}{\breve{a}x_n + \breve{b}y_n}, \quad n = 0,1,\ldots,$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial values x_0, y_0 are nonzero real numbers.

We discussed the linearized and global stability of the solutions for all nontrivial values of $\breve{a}$ and $\breve{b}$ as well as introduced the forbidden sets.

We showed under certain conditions that, there exist eventually periodic solutions when $\breve{a} < 0$ and $\breve{b} < 0$ as well as when $\breve{a} > 0$ and $\breve{b} < 0$.

We conjecture that the same results can be obtained for the system

$$x_{n+1} = \frac{y_{n-k}}{x_{n-k}}, \quad y_{n+1} = \frac{y_{n-k}}{\breve{a}x_{n-k} + \breve{b}y_{n-k}}, \quad n = 0,1,\ldots,$$

where $\breve{a}$, $\breve{b}$ are real numbers and the initial points (x_{-i}, y_{-i}), where $i = 0,1,\ldots,k$ are nonzero real numbers.

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ORCID
Fatma Hilal Gümüş https://orcid.org/0000-0002-6329-7142
Raafat Abo-Zeid https://orcid.org/0000-0002-1858-5583

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