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Research Article

# Another perspective on Kannan contraction

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**Abstract**

Inspired by the well-known result stating that if any iterate of a mapping is a Banach contraction on a complete metric space, then the mapping itself possesses a unique fixed point, we investigate that claim for a Kannan contraction but by retaining the left-hand side of the inequality as per the mapping itself. With an additional assumption of $k$-continuity, the existence and uniqueness of a fixed point is obtained for a new class of contractions, $m$-Kannan contraction, on a complete metric space. Several examples are given in order to substantiate many theoretical claims such as discontinuity at the unique limit point of the iterative sequence or the ones testifying that this class is wider than the class of Kannan mappings.

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## 1. Introduction

Banach fixed point theorem [1] has many valuable corollaries and generalizations. Starting from the well-known corollary of Banach fixed point theorem that if an iterate of the mapping is a Banach contraction, then a mapping itself possesses a unique fixed point, we are investigating this claim for Kannan contractive condition. It is obvious that if $n$th iterate of a mapping is a Kannan contraction, then, by the same approach as for Banach contraction, the mapping obtaines a unique fixed point if the underlying metricc space is complete. Hence, we present a modified approach in which the right-hand side of the inequality presents original Kannan contractive condition imposed on the mapping.
The importance of Kannan fixed point theorem [7] can be seen through two significant aspects-the lack of continuity request and through the carachterization of completeness of an underlying metric space. The first aspect was noticed by Kannan in 1968. since [7] contains an example of a mapping that is a Kannan contraction, but not continuous. Conell [5] gave an example of a Banach contraction having a fixed point on a metric space which is not complete, hence refuted the claim that a metric space $(X, d)$ is complete if and only if any Banach contraction on $X$ possesses a fixed point. However, this claim was proven valid for the class of Kannan contractions in 1975. by Subrahmanyam [10]. The original idea of Kannan was thoroughly studied, extended and modified in numerous ways and it is still an on-going research as can be seen in [2, 3, 6, 9, 11] among many others. We

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intend to study existence and uniqueness of a new class of contractive mappings that will be named $m$-Kannan mappings fulfilling the condition

$$d(T^m x, T^m y) \leq q\left(d(x,Tx) + d(y,Ty)\right)$$

on $X$ for some $q \in \left[0, \frac{1}{2}\right)$ and $m \in \mathbb{N}$. Evidently, for $m = 1$ this mapping is a Kannan contraction so this problem will not be in the scope of this article. We will use different proof techniques in the cases $m = 2$ and $m > 2$, but in order to obtain existence of the fixed point some continuity assumptions are imposed. This is the main difference comparing to Kannan contraction, but it also opens an interesting problem, does there always exists some $k \in \mathbb{N}$ such that a Kannan contraction is a $k$-continuous mapping. The continuity assumption is sufficient, but not necessary as can be seen through examples. Important property of this class of mappings for any natural $m$ is that the iterative sequence converges and that the limit is uniquely determined for arbitrary initial point in a complete metric space. Additionally, there are examples of mappings satisfying above mentioned contractive condition and not possessing a fixed point in a complete metric space due to being discontinuous.
The value of this approach may be seen through the fact that if we are talking about non-expansive mappings with a convergent iterative sequence, then $d(T^n x, x)$ is decreasing in some surrounding of the fixed point, so we can potentially obtain a wider class of mappings. This is substantiated with the example of a mapping not being a Kannan contraction, not having a second iterate as a Kannan contraction, but being 2-Kannan contraction instead.
For the convenience of a reader, we collect basic definitions and results regarding this topic. For more detailed informtations regarding terminology see [4].

**Theorem 1.1.** *[1] Let $(X,d)$ be a complete metric space and $T : X \mapsto X$ a mapping such that there exists some $q \in [0,1)$ with the inequality*

$$d(Tx,Ty) \leq q d(x,y) \tag{1.1}$$

*fulfilled for all $x, y$ in $X$. The mapping $T$ has a unique fixed point in $X$ and, for arbitrary $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point of $T$.*

The answer to important question of the existence of a fixed point of a mapping if its iterate is a Banach contraction is given in the sequel:

**Lemma 1.2.** *[8] Let $(X,d)$ be a complete metric space and $T : X \mapsto X$ a mapping. If there exists $n \in \mathbb{N}$ such that $n$-th iterate of a mapping $T$, $T^n$, is a Banach contraction, then the mapping $T$ has a unique fixed point in $X$.*

Kannan [7] presented a class of contractive mappings differing from the Banach contractions and also containg some discontinuous mappings. The main result of [7] is the following:

**Theorem 1.3.** *[7] Let $(X,d)$ be a complete metric space and $T : X \mapsto X$ a mapping such that*

$$d(Tx,Ty) \leq q\left(d(x,Tx) + d(y,Ty)\right) \tag{1.2}$$

*holds for some $q \in \left[0, \frac{1}{2}\right)$ and for all $x, y$ in $X$. A mapping $T$ has a unique fixed point $x^* \in X$ and the sequence $(T^n x)$ converges to the fixed point with the estimation $d(T^n x, x^*) \leq K\left(\frac{q}{1-q}\right)^{n-1} d(x,Tx)$ for any $n \in \mathbb{N}$ and any $x \in X$.*

In the sequel we will use the notion of $k$-continuouity.

**Definition 1.4.** If $(X,d)$ is a metric space and $T : X \mapsto X$ a mapping, then $T$ is $k$-continuous mapping for some $k \in \mathbb{N}$ if $T^k$ is a continuous mapping.

Recall that on a metric space, continuity of a mapping is equivalent to sequential continuity. Also, continuous mapping is $k$-continuous for any $k \in \mathbb{N}$, but reverse obviously do not hold for $k \geq 2$.

**Example 1.5.** If $X = \mathbb{R}$ is equipped with Euclidean metric $d$ and $T : X \mapsto X$ defined by

$$Tx = \begin{cases} 0, & x \in \mathbb{R} \setminus \{1\} \\ 2, & x = 1 \end{cases},$$

then $T$ is discontinuous on $X$, but it is 2-continuous mapping.

## 2. Main results

Based on difference between proof techniques, we will separate discussion on the iterates of a mapping for $n = 2$ and $n > 2$.

**Theorem 2.1.** *If $(X, d)$ is a complete metric space and a mapping $T$ is a $k$-continuous mapping such that for any $x, y \in X$*

$$d(T^2x, T^2y) \leq q\,(d(x, Tx) + d(y, Ty)) \tag{2.1}$$

*holds for some $q \in \left[0, \frac{1}{2}\right)$, then $T$ has a unique fixed point in $X$ and for arbitrary initial point $x \in X$ the iterative sequence $(T^n x)$ converges to the fixed point of the mapping $T$.*

***Proof.*** Observe a mapping $T$ on a complete metric space $(X, d)$ satisfying the contractive condition (2.1) for some $q \in \left[0, \frac{1}{2}\right)$. Let $x_0 \in X$ be arbitrary and define $x_n = T^n x_0$ for any $n \in \mathbb{N}$.
If $q = 0$, the mapping $T^2$ is a constant mapping and it has a unique fixed point in $X$ denoted with $x^*$. If $Tx^* = y$, then $T^2y = T(T^2x^*) = Tx^* = y$ which asserts that $y = x^*$. Moreover, for any $x \in X$, $T^n x = x^*$ for $n \geq 2$, so the iterative sequence $(T^n x)$ converges to the fixed point $x^*$.
If $q \neq 0$, denote with $d_n$ a distance $d(x_n, x_{n+1})$. In order to estimate $d(x_n, x_m)$ observe that due to the inequality

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i$$

it is sufficient to show that the series $\sum_i d_i$ is convergent in order to claim that the sequence $(x_n)$ is a Cauchy sequence.
Since

$$d_n \leq q(d_{n-1} + d_{n-2})$$

we can observe that the upper bound for $d_n$ is $q(d_{n-1} + d_{n-2})$ and since all arguments are non-negative, the sum $\sum_{i=0}^{\infty} d_i$ is maximal if the sequence $(d_n)$ satisfies the recurrence relation

$$a_n = q(a_{n-1} + a_{n-2})$$

for any $n \geq 2$ and initial conditions determined by $d_0$ and $d_1$. After solving this linear difference equation, it follows that

$$a_n = A\left(\frac{q + \sqrt{q^2 + 4q}}{2}\right)^n + B\left(\frac{q - \sqrt{q^2 + 4q}}{2}\right)^n$$

where

$$A = -\frac{-2d_1 + qd_0 - \sqrt{q(4+q)}d_0}{2\sqrt{q(4+q)}}$$

$$B = d_0 + \frac{-2d_1 + qd_0 - \sqrt{q(4+q)}d_0}{2\sqrt{q(4+q)}}.$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} d_n &\le \sum_{n=0}^{\infty} a_n \\
&= \sum_{n=0}^{\infty} \left( A \left( \frac{q+\sqrt{q^2+4q}}{2} \right)^n + B \left( \frac{q-\sqrt{q^2+4q}}{2} \right)^n \right) \\
&= -A \frac{2}{-2+q+\sqrt{q(4+q)}} + B \frac{2}{2-q+\sqrt{q(4+q)}} \\
&= \frac{d_0(1-q)+d_1}{1-2q} \\
&< \infty
\end{aligned}$$

since $2q-1 \neq 0$.
Consequently, $(x_n)$ is a Cauchy sequence in a complete metric space, so there exists some $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.
Note that for an arbitrary initial point $y_0 \in X$ the sequence of successive approximations $(T^n y_0)$ is convergent as previously discussed but its limit is also $x^*$ because of the estimation

$$d(T^{n+1}x_0, T^{n+1}y_0) \le d(T^{n-1}x_0, T^n x_0) + d(T^{n-1}y_0, T^n y_0)$$

that yields to the conclusion that $x^* = \lim_{n\to\infty} T^n y_0$.
Accordingly,

$$\lim_{n\to\infty} T^n x^* = x^*. \tag{2.2}$$

If $T$ is a $k$-continuous mapping, then $\lim_{n\to\infty} T^k(x_n) = T^k x^*$ but on the other side

$$\lim_{n\to\infty} T^k(x_n) = \lim_{n\to\infty} T^{n+k} x_0 = x^*,$$

and $T^k x^* = x^*$. If $k=1$, then $T$ has a fixed point $x^*$ in $X$ as well as $T^2$. If $k=2$ the mapping $T^2$ has a fixed point $x^*$ and for this case as well for $k \ge 3$ we make further estimations. By taking into the account $T^{nk}x^* = x^*$ for any $n \in \mathbb{N}$, it follows

$$\begin{aligned}
d(x^*, Tx^*) &= d(T^{nk}x^*, T^{nk+1}x^*) \\
&\le d(T^{nk-2}x^*, T^{nk-1}x^*) + d(T^{nk-1}x^*, T^{nk}x^*)
\end{aligned}$$

and as $n \to \infty$, we conclude that $Tx^* = x^*$.

Further, assume that $Ty = y$, then

$$d(x^*, y) \le q\,(d(x^*, Tx^*) + d(y, Ty)) = 0.$$

Consequently, the mapping $T$ has a unique fixed point $x^*$ in $X$ and, as already was proven, the iterative sequence $(T^n x)$ converges to $x^*$ for any initial point $x \in X$.

□

**Remark 2.2.** The mapping $T$ fulfilling the condition (2.1) has a unique fixed point if and only if $T^2$ has a unique fixed point. Evidently, the set of fixed points of a mapping $T$ is a subset of the set of fixed point of a mapping $T^2$ due to

$$T^2 x^* = T(Tx^*) = Tx^* = x^*.$$

For the converse inclusion observe that if $z \in X$ is such that $T^2 z = z$, then

$$T^n z = \begin{cases} z, & \text{if } n \text{ is even} \\ Tz, & \text{if } n \text{ is odd} \end{cases}$$

for any $n \in \mathbb{N}$. But, $(T^n z)$ is convergent sequence according to initial considerations, so it must be $Tz = z$.

**Example 2.3.** Very well-known example testifying the independence of Banach contraction from Kannan contraction is a mapping $T : [0,1] \mapsto [0,1]$ defined by $Tx = \frac{x}{2}$ for any $x \in [0,1]$ while the domain is equipped with Euclidean metric and thus complete metric space.
If $x = 0$ and $y = 1$, then

$$d(Tx, Ty) = \frac{1}{2} = d(x, Tx) + d(y, Ty),$$

so $T$ is not a Kannan contraction on $X$.
Apart from that, for any $x, y \in [0,1]$,

$$\begin{aligned} d(T^2x, T^2y) &= \left| \frac{x}{4} - \frac{y}{4} \right| \\ &= \frac{1}{3} \left| \frac{3x}{4} - \frac{3y}{4} \right| \\ &\leq \frac{1}{3} \left( \frac{3x}{4} + \frac{3y}{4} \right) \\ &= \frac{1}{3} \left( d(x, Tx) + d(y, Ty) \right) \end{aligned}$$

implying that $T$ is a 2-Kannan contraction on $[0,1]$ for $q = \frac{1}{3}$.

Meanwhile, any Kannan contraction is a 2-Kannan contraction.

**Remark 2.4.** Assume that a mapping $T$ is a Kannan contraction on a metric space $(X, d)$ for some contractive constant $q \in \left[0, \frac{1}{2}\right)$. Let $x, y \in X$ be arbitrary and recall that

$$d(Tz, T^2z) \leq \frac{q}{1-q} d(z, Tz)$$

for any $z \in X$. Then,

$$\begin{aligned} d(T^2x, T^2y) &\leq q d(Tx, T^2x) + q d(Ty, T^2y) \\ &\leq \frac{q^2}{1-q} \left( d(x, Tx) + d(y, Ty) \right). \end{aligned}$$

In addition, $\frac{q^2}{1-q} \leq q < \frac{1}{2}$ for $q \in \left[0, \frac{1}{2}\right)$.

**Theorem 2.5.** *If $(X, d)$ is a complete metric space and a mapping $T$ is such that for any $x, y \in X$ the inequality (2.1) holds for some $q \in \left[0, \frac{1}{2}\right)$, then for arbitrary initial point $x \in X$ the iterative sequence $(T^n x)$ converges in $(X, d)$ to the same point $x^* \in X$ and $Fix(T) \subseteq \{x^*\}$.*

***Proof.*** The first part of the claim of the theorem is based on the first part of the proof of Theorem 2.1 where, as already mentioned, continuity presumption was not utilised in any manner. The claim regarding the set of fixed points of the mapping $T$ is then obvious since the iterative sequence initiated by a fixed point is constant. □

**Corollary 2.6.** *The 2-Kannan contraction on a complete metric space has at most one fixed point.*

**Example 2.7.** Let $X = [0,1]$ and a mapping $T : X \mapsto X$ defined by

$$Tx = \begin{cases} \frac{x}{4}, & x \in (0,1] \\ \frac{1}{4}, & x = 0 \end{cases}.$$

The mapping $T$ is not a Kannan contraction as can be seen through observing $x = 0$ and $y_n = \frac{1}{2^n}$ for $n \in \mathbb{N} \setminus \{1\}$. Indeed,

$$d(Tx, Ty_n) = \frac{1}{4} - \frac{1}{2^{n+2}} = \frac{2^n - 1}{2^{n+2}}$$

and

$$d(x, Tx) + d(y_n, Ty_n) = \frac{1}{4} + \frac{3}{2^{n+2}},$$

so it is impossible to find a constant $q \in \left[0, \frac{1}{2}\right)$ such that

$$\frac{1}{4} - \frac{1}{2^{n+2}} \leq q \left( \frac{1}{4} + \frac{3}{2^{n+2}} \right)$$

as $n \to \infty$ since $q \geq \frac{2^n - 1}{2^n + 3}$ for $n \geq 2$. Already for $n = 3$ we have

$$\begin{aligned} d(Tx, Ty_3) &= \frac{7}{32} \\ &\geq \frac{1}{2} \left( d(x, Tx) + d(y_3, Ty_3) \right) \\ &= \frac{11}{64}. \end{aligned}$$

Hence, we will investigate the properties of its second iterate $T^2$ determined by

$$T^2 x = \begin{cases} \frac{x}{16}, & x \in (0,1] \\ \frac{1}{16}, & x = 0 \end{cases}.$$

In order to estimate the distance $d(T^2x, T^2y)$, we discuss on several different options and assume that $q = \frac{1}{4}$.
(1) If $x, y \in (0,1]$, then

$$\begin{aligned} d(T^2x, T^2y) &= \frac{|x - y|}{16} \\ &= \frac{1}{4} \frac{|x - y|}{4} \\ &\leq \frac{1}{4} \frac{x + y}{4} \\ &\leq \frac{1}{4} \left( d(x, Tx) + d(y, Ty) \right). \end{aligned}$$

(2) If $x = 0$ and $y \in (0,1]$, or equivalently vice-versa, then

$$\begin{aligned} d(T^2x, T^2y) &= \frac{1 - y}{16} \\ &= \frac{1}{4} \left( \frac{1}{4} - \frac{y}{4} \right) \\ &\leq \frac{1}{4} d(x, Tx) \\ &\leq \frac{1}{4} \left( d(x, Tx) + d(y, Ty) \right). \end{aligned}$$

Consequently, for any $x, y \in X$ the inequality (2.1) holds, but neither $T$ nor $T^2$ possess a fixed point in a complete metric space $(X, d)$.

It is also sufficient to assume that the mapping $T$ is continuous (or 2-continuous) at the limit of iterative sequences and thus following conclusion holds.

**Corollary 2.8.** *If 2-Kannan contraction on a complete metric space does not have a fixed point, then the unique limit point of all iterative sequences $(T^n x)$ for any $x \in X$ is the point of discontinuity of the mapping $T$.*

**Example 2.9.** Analyzing the Example 2.7 we observe that $x = 0$ is a point of discontinuity of the mapping $T$ and at the same time the limit of any iterative sequence since for $x \in (0, 1]$ we have $T^n x = \frac{x}{4^n}$ for any $n \in \mathbb{N}$ while for $x = 0$ we get $T^n x = \frac{1}{4^n}$, $n \in \mathbb{N}$. Either-way, $\lim\limits_{n\to\infty} T^n x = 0$.

Basically what can be seen from the proof of Theorem 2.1 that it is sufficient for $T^2$ to be continuous at the limit of iterative sequence $(T^n x)$ in order to have the fixed point. In general, it is not a necessary assumption.

**Example 2.10.** Let $X = [0, 1] \cup \{2, 3\}$ be equipped with Euclidean metric and $T : X \mapsto X$ a mapping defined by

$$Tx = \begin{cases} 2, & x = \frac{1}{2n},\ n \in \mathbb{N} \\ 3, & x = \frac{1}{2n-1},\ n \in \mathbb{N} \\ 0, & x \in [0, 1] \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\} \\ \frac{2}{3}, & x \in \{2, 3\} \end{cases}.$$

Obviously,

$$T^2 x = \begin{cases} \frac{2}{3}, & x = \frac{1}{n},\ n \in \mathbb{N} \\ 0, & x \in ([0, 1] \cup \{2, 3\}) \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\} \end{cases}.$$

It is relevant to observe $d(T^2 x, T^2 y)$ only if $x = \frac{1}{n}$ for some $n \in \mathbb{N}$ and $y \in ([0, 1] \cup \{2, 3\}) \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$ (or vice-versa due to the symmetry) since this distance is equal to zero in all other cases. Let $q = \frac{4}{9} \in \left[0, \frac{1}{2}\right)$.
If $x = \frac{1}{2n}$ for some $n \in \mathbb{N}$, then $d(x, Tx) \geq \frac{3}{2}$ and if $x = \frac{1}{2n-1}$ for some $n \in \mathbb{N}$, then $d(x, Tx) \geq 2$. Anyway,

$$\begin{aligned} d(T^2 x, T^2 y) &= \frac{2}{3} \\ &\leq q d(x, Tx) \\ &\leq q\left(d(x, Tx) + d(y, Ty)\right), \end{aligned}$$

implying that $T$ is a 2-Kannan contraction on a complete metric space. The mapping $T$ has a unique fixed point $x = 0$, but neither $T$ nor $T^2$ is continuous at $x = 0$ as we may observe the sequence $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$ that converges to 0. The sequence $(Tx_n)$ is divergent, while $(T^2 x_n)$ is a constant sequence and converges to $\frac{2}{3} \neq T^2 0$.

**Remark 2.11.** If we use the conclusion of the proof of Theorem 2.1, which is obtained without any continuity assumption, that any iterative sequence $(T^n x)$ converges to the same limit point $x^*$, then we may observe a restriction of the mapping $T$ on the orbit of $x^*$.
Let $O(x^*) = \{T^n x^* \mid n \in \mathbb{N}_0\}$ and $T_0 : O(x^*) \mapsto O(x^*)$ is a restriction of the mapping $T$, i.e., $T_0 x = Tx$ for any $x \in O(x^*)$.
The mapping $T$ is well-defined and the orbit $O(x^*)$ is closed subset of the complete metric space, hence complete itself with a restriction of metric $d$. It is easy to deduce that $T$ has a fixed point $x^* \in X$ if and only if $T_0$ is a continuous mapping.

Indeed, if $T_0$ is continuous mapping, then

$$\begin{aligned} Tx^* &= T\left(\lim_{n\to\infty} T^n x^*\right) \\ &= \lim_{n\to\infty} T^{n+1}x^* \\ &= x^*. \end{aligned}$$

Converse, if $x^*$ is a fixed point of the mapping $T$, then $O(x^*) = \{x^*\}$ and $T_0$ is continuous on $O(x^*)$.

## 3. m-Kannan mapppings

Previously presented concept of 2-Kannan mappings has shown some interesting properties and can be further extended for arbitrary natural number $m$ through the notion of $m$-Kannan mappings.

**Definition 3.1.** If $(X, d)$ is a metric space and $T : X \mapsto X$ a mapping such that there exist some $q \in \left[0, \frac{1}{2}\right)$ and $m \in \mathbb{N}$ fulfilling the inequality

$$d(T^m x, T^m y) \le q\,(d(x, Tx) + d(y, Ty)) \tag{3.1}$$

for any $x, y \in X$, then the mapping $T$ is a $m$-Kannan mapping.

Note that the term of 1-Kannan contraction is equivalent to Kannan contraction. As was seen for the case $m = 2$ in Example 2.3, not any $m$-Kannan contraction is a Kannan contraction. Reverse statement do hold.

**Example 3.2.** If $(X, d)$ is a metric space and $T : X \mapsto X$ a Kannan contraction for some $q \in \left[0, \frac{1}{2}\right)$, then we will prove that for any $n \in \mathbb{N}$ and $x, y \in X$ we have

$$d(T^n x, T^{n+1}x) \le \left(\frac{q}{1-q}\right)^n (d(x, Tx) + d(y, Ty)) \tag{3.2}$$

and consequently

$$d(T^n x, T^n y) \le \frac{q^n}{(1-q)^{n-1}}\,(d(x, Tx) + d(y, Ty))\,. \tag{3.3}$$

For that purpose we will use the principle of mathematical induction twice.
Evidently, (3.2) holds for $n = 1$ since

$$d(Tx, T^2x) \le q\left(d(x, Tx) + d(Tx, T^2x)\right),$$

so suppose that (3.2) holds for some $n \in \mathbb{N}\setminus\{1\}$. Further,

$$\begin{aligned} d(T^{n+1}x, T^{n+2}x) &\le q\left(d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x)\right) \\ &\le \frac{q}{1-q} d(T^n x, T^{n+1}x) \\ &\le \left(\frac{q}{1-q}\right)^{n+1} (d(x, Tx) + d(y, Ty)) \end{aligned}$$

asserts that (3.2) holds for any $n \in \mathbb{N}$.
Kannan contractive condition is (3.3) for $n = 1$, so assume that (3.3) holds for some $n \in \mathbb{N}\setminus\{1\}$ and notice that

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &\le q\left(d(T^n x, T^{n+1}x) + d(T^n y, T^{n+1}y)\right) \\ &\le q\left(\frac{q}{1-q}\right)^n (d(x, Tx) + d(y, Ty)) \\ &\le \frac{q^{n+1}}{(1-q)^n}\,(d(x, Tx) + d(y, Ty)) \end{aligned}$$

further deduces that (3.3) holds for any natural $n$.
Therefore, if $T$ is a Kannan contraction on $X$, it is a $m$-Kannan contraction for any natural number $m$ as $\frac{q^m}{(1-q)^{m-1}} \in \left[0, \frac{1}{2}\right)$ for any $m \in \mathbb{N}$.

Different proof techniques will be applied in discussing existence and uniqueness of a fixed point of a class of $m$-Kannan mappings for arbitrary $m \in \mathbb{N}$.

**Theorem 3.3.** *If $(X, d)$ is a complete metric space and a mapping $T$ is a $k$-continuous mapping for some $k \in \mathbb{N}$ such that for some $q \in \left[0, \frac{1}{2}\right)$ and $m \in \mathbb{N} \setminus \{1, 2\}$ the inequality (3.1) holds for any $x, y \in X$ , then $T$ has a unique fixed point in $X$ and for arbitrary initial point $x \in X$ the iterative sequence $(T^n x)$ converges to the fixed point of the mapping $T$.*

***Proof.*** For arbitrary $x_0 \in X$ define a sequence $x_n = T^n x_0$ for $n \in \mathbb{N}$. The proof will be divided in three phases, to prove that $(x_n)$ is a Cauchy sequence, that it converges to the fixed point of the mapping $T$ while in the third part uniqueness of the fixed point will be addressed.
Denote with $a(x)$ the sum $\sum_{i=0}^{m-1} d(T^i x, T^{i+1} x)$ for any $x \in X$. We will prove that

$$d(T^{nm+l}x, T^{nm+l+1}x) \leq 2^{n-1} q^n a(x) \tag{3.4}$$

holds for all $n \in \mathbb{N}$ and $l \in \{0, \dots, m-1\}$.
To apply the principle of mathematical induction we need to confirm that this inequality holds for $n = 1$.
For $l \in \{0, \dots, m-2\}$, we have

$$\begin{aligned} d(T^{m+l}x, T^{m+l+1}x) &\leq q\left(d(T^l x, T^{l+1}x) + d(T^{l+1}x, T^{l+2}x)\right) \\ &\leq qa(x), \end{aligned}$$

while for $l = m-1$

$$\begin{aligned} d(T^{m+l}x, T^{m+l+1}x) &\leq q\left(d(T^l x, T^{l+1}x) + d(T^{l+1}x, T^{l+2}x)\right) \\ &\leq qd(T^{m-1}x, T^m x) + q^2\left(d(x, Tx) + d(Tx, T^2 x)\right) \\ &\leq qa(x). \end{aligned}$$

Note that the assumption of $m > 2$ is important for the last conclusion.
In order to prove that (3.4) holds for any $n \in \mathbb{N}$ and $l \in \{0, \dots, m-1\}$, observe that

$$\begin{aligned} d(T^{(n+1)m+l}x, T^{(n+1)m+l+1}x) &\leq q\left(d(T^{nm+l}x, T^{nm+l+1}x) + d(T^{nm+l+1}x, T^{nm+l+2}x)\right) \\ &\leq q(2^{n-1}q^n a(x) + 2^{n-1}q^n a(x)) \\ &= 2^n q^{n+1} a(x) \end{aligned}$$

holds for any $l \in \{0, 1, \dots, m-2\}$.
If $l = m-1$, then

$$\begin{aligned} d(T^{(n+1)m+l}x, T^{(n+1)m+l+1}x) &\leq q\left(d(T^{(n+1)m-1}x, T^{(n+1)m}x) + d(T^{(n+1)m}x, T^{(n+1)m+1}x)\right) \\ &\leq q2^{n-1}q^n a(x) + q^2 d(T^{nm}x, T^{nm+1}x) \\ &+ q^2 d(T^{nm+1}x, T^{nm+2}x) \\ &\leq 2^{n-1}q^{n+1}a(x) + q^2(2^{n-1}q^n a(x) + 2^{n-1}q^n a(x)) \\ &= 2^{n-1}q^{n+1}(1+2q)a(x) \\ &< 2^n q^{n+1} a(x). \end{aligned}$$

Consequently, by the principle of mathematical induction, we conclude that (3.4) holds for any $n \in \mathbb{N}$ and $l \in \{0, \dots, m-1\}$.

Let $n_1, n_2 \in \mathbb{N}$ and $n_2 \geq n_1$, $k_i = [\frac{n_i}{m}]$ and $n_i = k_i m + l_i$ where $l_i \in \{0, 1, \dots m-1\}$ for $i = 1, 2$, then:

$$\begin{aligned} d(T^{n_1}x, T^{n_2}x) &\leq \sum_{i=n_1}^{n_2-1} d(T^i x, T^{i+1}x) \\ &\leq 2^{k_1-1}q^{k_1}(m - l_1)a(x) + \sum_{i=k_1+1}^{k_2-1} 2^{i-1}q^i m a(x) + 2^{k_2-1}q^{k_2}(l_2+1)a(x) \\ &\leq \sum_{i=k_1}^{k_2} 2^{i-1}q^i m a(x) \\ &\leq \sum_{i=k_1}^{+\infty} 2^{i-1}q^i m a(x). \end{aligned}$$

However, the series $\sum_{i=1}^{+\infty} 2^{i-1}q^i$ converges, so $\lim\limits_{n_1,n_2\to\infty} d(x_{n_1}, x_{n_2}) = 0$. As any Cauchy sequence in a complete metric space converges, notice $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Additionally, if $y \in X$ is arbitrary then we will in a same manner obtain that $(T^n y)$ is a convergent sequence and

$$d(x_{nm}, T^{nm}y) \leq qd(x_{(n-1)m}, x_{(n-1)m+1}) + qd(T^{(n-1)m}y, T^{(n-1)m+1}y),$$

leads to the conclusion that $(T^n y)$ converges to the same limit point $x^* \in X$ for any $y \in X$. Recall that the mapping $T$ is $k$-continuous, so

$$\lim_{n\to\infty} T^k(x_n) = T^k(\lim_{n\to\infty} x_n) = T^k x^*.$$

Taking into the account that $(T^k x_n)$ is a subsequence of a convergent sequence, it follows that $T^k x^* = x^*$.
Moreover,

$$\begin{aligned} d(Tx^*, x_{mn}) &= d(T^{kmn+1}x^*, x_{mn}) \\ &\leq q\left(d(T^{(kn-1)m+1}x^*, T^{(kn-1)m+2}x^*) + d(x_{m(n-1)}, x_{m(n-1)+1})\right) \end{aligned}$$

for arbitrary $n \in \mathbb{N}$, thus

$$\lim_{n\to\infty} x_{mn} = Tx^* = x^*.$$

Accordingly, $T$ has a fixed point $x^* \in X$. For the uniqueness, let $Ty = y$. Then

$$\begin{aligned} d(x^*, y) =& d(T^m x^*, T^m y) \\ &\leq qd(x^*, Tx^*) + qd(y, Ty) \\ &= 0, \end{aligned}$$

and $x^*$ is the unique fixed point of the mapping $T$. □

**Theorem 3.4.** *If $(X, d)$ is a complete metric space and a mapping $T$ is such that for any $x, y \in X$ the inequality (3.1) for some $q \in \left[0, \frac{1}{2}\right)$ and natural $m > 2$, then for arbitrary initial point $x \in X$ the iterative sequence $(T^n x)$ converges in $(X, d)$ to the same point $x^* \in X$ and $Fix(T) \subseteq \{x^*\}$.*

***Proof.*** This conclusion is easily derived from the proof of Theorem 3.3 since in this part the presumption of continuity was not utilised. □

**Corollary 3.5.** *The $m$-Kannan contraction for $m \in \mathbb{N} \setminus \{1, 2\}$ on a complete metric space has at most one fixed point.*

**Example 3.6.** Let $X = [0,1]$ be equipped with Eucledean metric and a mapping $T : X \mapsto X$ defined by

$$Tx = \begin{cases} \frac{x}{2}, & x \in (0,1] \\ \frac{1}{2}, & x = 0 \end{cases}.$$

The mapping $T$ is not a Kannan contraction because of, par example, $x = 0$ and $y = \frac{1}{8}$ since $d(Tx, Ty) = \frac{7}{16}$ but

$$d(x, Tx) + d(y, Ty) = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$$

so we have $q \geq \frac{7}{9}$ which is impossible.
Furthermore, $T^2$ is also not a Kannan contraction as discussed in Example 2.7 as

$$T^2x = \begin{cases} \frac{x}{4}, & x \in (0,1] \\ \frac{1}{4}, & x = 0 \end{cases}.$$

However,

$$T^3x = \begin{cases} \frac{x}{8}, & x \in (0,1] \\ \frac{1}{8}, & x = 0 \end{cases},$$

satisfy (3.1). In order to prove that we will separately discuss case $x, y \in (0,1]$ and $x = 0, y \in (0,1]$ (equivalently $x \in (0,1], y = 0$).

(1) If $x, y \in (0,1]$, then

$$\begin{aligned} d(T^3x, T^3y) &= \frac{|x-y|}{8} \\ &= \frac{1}{4}\frac{|x-y|}{2} \\ &\leq \frac{1}{4}\frac{x+y}{2} \\ &\leq \frac{1}{4}\left(d(x,Tx) + d(y,Ty)\right). \end{aligned}$$

(2) If $x = 0$ and $y \in (0,1]$, and analogously $x \in (0,1], y = 0$, then

$$\begin{aligned} d(T^3x, T^3y) &= \frac{1-y}{8} \\ &= \frac{1}{4}\left(\frac{1}{2} - \frac{y}{2}\right) \\ &\leq \frac{1}{4}d(x,Tx) \\ &\leq \frac{1}{4}\left(d(x,Tx) + d(y,Ty)\right). \end{aligned}$$

Consequently, $T$ is a 3-Kannan contraction on a complete metric space $(X, d)$ but none of the iterates of $T$ possess a fixed point in $X$ which can be substantiated by the fact that the limit of iterative sequence is 0 which is a point of discontinuity of $T^n$ for any $n \in \mathbb{N}$.

## 4. Conclusion

Investigating the question of existence and uniqueness of mappings satisfying these types of contractive condition naturally raises from very well-known and widely applicable result regarding $n$-th iterate of a mapping being a Banach contraction. But when we discuss on Kannan contraction, discontinuity is an important property that is potentially preserved in the class of $m$-Kannan mappings for $m \in \mathbb{N}$ which can have at most one fixed point. In order to claim the existence of a fixed point we do need a presumption of $k$-continuity for some $k \in \mathbb{N}$ independent from the choice of $m$. Moreover, examples testify on the cases when lack of discontinuity at the limit point of iterative sequence, which is uniquely determined for this class of mappings, brings non-existence of a fixed point. Question that remains open is can the continuity assumption be replaced with some less restrictive request.

**There is no conflict of interest.**

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