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Research Article

A Novel Subclass of Harmonic Functions: Coefficient Bounds, Distortion Bounds, and Closure Properties

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ABSTRACT

In this paper, we introduce a new subclass of harmonic functions that significantly improves our understanding of these functions in geometric function theory. We provide a comprehensive analysis of this subclass by deriving several important properties, including coefficient bounds and decay bounds, which are necessary to evaluate the behavior and limitations of functions in this class. Additionally, we establish sufficient coefficient conditions for harmonic functions to belong to this class. Moreover, we rigorously show that this subclass is closed under both convex combinations and convolutions, meaning that any convex combination or convolution of functions in this class will also belong to the class. These results provide valuable insights into the stability and applicability of the subclass and provide a solid framework for further theoretical explorations and practical applications in complex analysis.

1. Introduction

In the study of harmonic functions, any function $\mathfrak{f}$ within the class SH^0 can be expressed as $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where

$$\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} u_s z^s, \mathfrak{v}(z) = \sum_{s=2}^{\infty} v_s z^s. \quad (1)$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C}: |z| < 1\}$. If the condition $|\mathfrak{v}'(z)| < |\mathfrak{u}'(z)|$ holds in $\mathbb{E}$, then $\mathfrak{f}$ is locally univalent and sense-preserving in $\mathbb{E}$. It is important to note that, when $\mathfrak{v}(z)$ is identically zero, the class SH^0 reduces to the class S.

Let C and K denote the subclasses of S mapping $\mathbb{E}$ onto close-to-convex and convex domains, respectively. Similarly, CH^0 and KH^0 are subclasses of SH^0, mapping $\mathbb{E}$ onto these respective domains [1-3].

Consider an analytic function $\mathfrak{u}$, where Salagean [4] defined the differential operator D^n of $\mathfrak{u}$ as follows:

$$D^0\mathfrak{u}(z) = \mathfrak{u}(z), \quad (2)$$

$$D^1\mathfrak{u}(z) = D\mathfrak{u}(z) = z\mathfrak{u}'(z), \quad (3)$$

$$D^n\mathfrak{u}(z) = D\left(D^{n-1}\mathfrak{u}(z)\right) \quad (4)$$

where $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$. For $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, Jahangiri et al. [5] defined the modified Salagean operator of $\mathfrak{f}$ as

$$D^n\mathfrak{f}(z) = D^n\mathfrak{u}(z) + (-1)^n\overline{D^n\mathfrak{v}(z)} \quad (5)$$

where

$$D^n\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} s^n u_s z^s, \qquad D^n\mathfrak{v}(z) = \sum_{s=2}^{\infty} s^n v_s z^s. \quad (6)$$

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Denote by $ACH(n)$ the class of functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ and satisfy

$$Re\left\{\left(D^n\mathfrak{u}(z)\right)'\right\} > \left|\left(D^n\mathfrak{v}(z)\right)'\right|. \qquad (7)$$

It can be shown that the PH^0 class, as investigated by Ponnussamy et al. [6], is obtained for $n = 0$, and that the WH^0class, as investigated by Nagpal and Ravichandran [7], is obtained for $n = 1$. Furthermore, by selecting specific parameter values, the following well-known function classes can also be derived as special cases:

$ACH(0) \equiv PH(q \to 1^-, 0)$	[8]
$ACH(0) \equiv AH^0(1,0,0)$	[9]
$ACH(1) \equiv AH^0(1,1,0)$	[9]
$ACH(0) \equiv GH(0,1,0)$	[10]
$ACH(0) \equiv RH(0,0)$	[11]
$ACH(1) \equiv RH(1,0)$	[11]
$ACH(1) \equiv RH(1,1,0)$	[12]
$ACH(1) \equiv WH^0(3,1)$	[13]

For more information on function classes defined by high-order differential inequalities, refer to [14-17].

The class $AC(n)$ consists of functions $\mathfrak{u} \in S$ that satisfy the inequality:

$$Re\left\{\left(D^n\mathfrak{u}(z)\right)'\right\} > 0. \qquad (8)$$

2. Geometric Properties of the Class $ACH(n)$

The first result provides a relationship between the function spaces $AC(n)$ and $ACH(n)$.

Theorem 2.1. The mapping $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in ACH(n)$ if and only if $\mathfrak{U}_\epsilon = \mathfrak{u} + \epsilon\mathfrak{v} \in AC(n)$ for each ϵ ($|\epsilon| = 1$).

Proof. Suppose $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in ACH(n)$ and $\mathfrak{U}_\epsilon = \mathfrak{u} + \epsilon\mathfrak{v}$ for each ϵ ($|\epsilon| = 1$),

$$\begin{aligned} Re\left\{\left(D^n\mathfrak{U}_\epsilon(z)\right)'\right\} &= Re\left\{\left(D^n\mathfrak{u}(z)\right)' + \epsilon\left(D^n\mathfrak{v}(z)\right)'\right\} \\ &> Re\left\{\left(D^n\mathfrak{u}(z)\right)'\right\} - \left|\left(D^n\mathfrak{v}(z)\right)'\right| \\ &> 0. \end{aligned}$$

Thus $\mathfrak{U}_\epsilon \in AC(n)$. Conversely, let $\mathfrak{U}_\epsilon = \mathfrak{u} + \epsilon\mathfrak{v} \in AC(n)$. We have

$$Re\left\{\left(D^n\mathfrak{u}(z)\right)'\right\} = Re\left\{-\epsilon\left(D^n\mathfrak{v}(z)\right)'\right\}.$$

With appropriate choice of ϵ ($|\epsilon| = 1$); it follows that

$$Re\left\{\left(D^n\mathfrak{u}(z)\right)'\right\} > \left|\left(D^n\mathfrak{v}(z)\right)'\right|.$$

So, $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in ACH(n)$.

The next results provide a coefficient bound for functions in the $ACH(n)$class.

Theorem 2.2. Let $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in ACH(n)$ then for $s \geq 2$,

$$|v_s| \leq \frac{1}{s^{n+1}}. \qquad (9)$$

Equality is achieved by the function $f(z) = z + \frac{1}{s^{n+1}}\overline{z}^s$.

Proof. Suppose that $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in ACH(n)$. If the series expansion of the function $\mathfrak{v}(re^{i\theta})$ is used for $0 \leq r < 1$ and $\theta \in \mathbb{R}$, the following inequality is obtained:

$$\begin{aligned} &r^{s-1}s^{n+1}|v_s| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|\left(D^n\mathfrak{v}(re^{i\theta})\right)'\right| d\theta \\ &< \frac{1}{2\pi}\int_0^{2\pi} Re\left\{\left(D^n\mathfrak{u}(re^{i\theta})\right)'\right\} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} Re\left\{1 + \sum_{s=2}^{\infty} s^{n+1}u_s r^{s-1}e^{i(s-1)\theta}\right\} d\theta \\ &= 1. \end{aligned}$$

Allowing $r \to 1^-$, we prove the result (9).

Theorem 2.3. Let $\mathfrak{f} = \mathfrak{u} + \bar{\mathfrak{v}} \in ACH(n)$ then for $s \geq 2$,

$$|u_s| + |v_s| \leq \frac{2}{s^{n+1}}.$$

Equality is achieved by the function $\mathfrak{f}(z) = z + \frac{2}{s^{n+1}} z^s$.

Proof. Suppose that $\mathfrak{f} = \mathfrak{u} + \bar{\mathfrak{v}} \in ACH(n)$, then from Theorem 2.2 $\mathfrak{U}_\epsilon = \mathfrak{u} + \epsilon\mathfrak{v} \in AC(n)$ for each ϵ ($|\epsilon| = 1$). Thus for each ϵ ($|\epsilon| = 1$), we have

$$Re\left\{\left(D^n\left(\mathfrak{u}(z) + \epsilon\mathfrak{v}(z)\right)\right)'\right\} > 0.$$

Therefore, $\left(D^n\left(\mathfrak{u}(z) + \epsilon\mathfrak{v}(z)\right)\right)' = P(z)$ can be achieved by an analytic function P having a positive real component in $\mathbb{E}$ and of the form $P(z) = 1 + \sum_{s=1}^{\infty} p_s z^s$. Then, we derive

$$m^{n+1}(u_s + \epsilon v_s) = p_{s-1} \quad for\ s \geq 2. \qquad (10)$$

Since $Re\{P(z)\} > 0$, we have $|p_s| \leq 2$ for $s \geq 1$. Hence, by equation (10), we get

$$s^{n+1}|u_s + \epsilon v_s| \leq 2 \quad for\ s \geq 2.$$

Since ϵ ($|\epsilon| = 1$) is arbitrary, it follows that the proof is concluded.

Now, we give a sufficient condition for a function to be in the class $ACH(n)$.

Theorem 2.4. Let $\mathfrak{f} = \mathfrak{u} + \bar{\mathfrak{v}} \in SH^0$ with

$$\sum_{s=2}^{\infty} s^{n+1}(|u_s| + |v_s|) \leq 1 \qquad (11)$$

then $\mathfrak{f} \in ACH(n)$. Equality holds for the function $\mathfrak{f}(z) = z + \frac{1}{s^{n+1}} z^s$.

Proof. Suppose that $\mathfrak{f} = \mathfrak{u} + \bar{\mathfrak{v}} \in SH^0$. Then using (11),

$$Re\left\{\left(D^n\mathfrak{u}(z)\right)'\right\} = Re\left\{1 + \sum_{s=2}^{\infty} s^{n+1} u_s z^{s-1}\right\}$$

$$> 1 - \sum_{s=2}^{\infty} s^{n+1}|u_s| \geq \sum_{s=2}^{\infty} s^{n+1}|v_s|$$

$$> \left|\sum_{s=2}^{\infty} s^{n+1} v_s z^{s-1}\right| = \left|\left(D^n\mathfrak{v}(z)\right)'\right|.$$

Hence, $\mathfrak{f} \in ACH(n)$.

This theorem highlights the distortion bounds for functions in the $ACH(n)$ class, showing how the function's derivatives can change or stretch values.

Theorem 2.5. Let $\mathfrak{f} \in ACH(n)$. Then

$$|z| + 2\sum_{s=2}^{\infty} \frac{(-1)^{s-1}|z|^s}{s} \leq |D^n\mathfrak{f}(z)|$$

and

$$|D^n\mathfrak{f}(z)| \leq |z| + 2\sum_{s=2}^{\infty} \frac{|z|^s}{s}.$$

Equality is satisfied for the function $\mathfrak{f}(z) = z + \sum_{s=2}^{\infty} \frac{2}{s^n} z^s$.

Proof. Let $\mathfrak{f} \in ACH(n)$. Then using Theorem 2.2, $\mathfrak{U}_\epsilon = \mathfrak{u} + \epsilon\mathfrak{v} \in AC(n)$ for each ϵ ($|\epsilon| = 1$). Moreover, there is an analytic function $\omega(z)$ such that

$$\left(D^n\mathfrak{U}_\epsilon(z)\right)' = \frac{1 + \omega(z)}{1 - \omega(z)} \qquad (12)$$

with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{E}$.

Hence, we get

$$D^n\mathfrak{U}_\epsilon(z) = \int_0^z \frac{1 + \omega(t)}{1 - \omega(t)} dt = \int_0^{|z|} \frac{1 + \omega(re^{i\theta})}{1 - \omega(re^{i\theta})} e^{i\theta} dr.$$

Moreover using Schwarz Lemma, we have

$$|D^n\mathfrak{U}_\epsilon(z)| = \left|\int_0^{|z|} \frac{1 + \omega(re^{i\theta})}{1 - \omega(re^{i\theta})} e^{i\theta} dr\right|$$

$$\leq \int_0^{|z|} \frac{1 + r}{1 - r} dr$$

and

$$|D^n \mathfrak{U}_\epsilon(z)| = \left| \int_0^{|z|} \frac{1+\omega(re^{i\theta})}{1-\omega(re^{i\theta})} e^{i\theta} dr \right|$$
$$\geq \int_0^{|z|} Re\left\{\frac{1+\omega(re^{i\theta})}{1-\omega(re^{i\theta})}\right\} dr$$
$$\geq \int_0^{|z|} \frac{1-r}{1+r} dr.$$

Since

$$|D^n \mathfrak{U}_\epsilon(z)| = |D^n \mathfrak{u}(z) + \epsilon D^n \mathfrak{v}(z)|$$
$$\leq 1 + 2\sum_{s=1}^{\infty} |z|^s$$

and

$$|D^n \mathfrak{U}_\epsilon(z)| = |D^n \mathfrak{u}(z) + \epsilon D^n \mathfrak{v}(z)|$$
$$\geq 1 + 2\sum_{s=1}^{\infty} (-1)^s |z|^s,$$

in particular, we get

$$|D^n \mathfrak{u}(z)| + |D^n \mathfrak{v}(z)| \leq 1 + 2\sum_{s=1}^{\infty} |z|^s$$

and

$$|D^n \mathfrak{u}(z)| - |D^n \mathfrak{v}(z)| \geq 1 + 2\sum_{s=1}^{\infty} (-1)^s |z|^s.$$

Assume Γ is the radial segment extending from 0 to z, we get

$$|D^n \mathfrak{f}(z)| \leq \int_\Gamma (|D^n \mathfrak{u}(\zeta)| + |D^n \mathfrak{v}(\zeta)|)|d\zeta|$$
$$\leq \int_0^{|z|} \left(1 + 2\sum_{s=1}^{\infty} |z|^s\right) dt$$
$$= |z| + 2\sum_{s=1}^{\infty} \frac{|z|^{s+1}}{s+1}$$
$$= |z| + 2\sum_{s=2}^{\infty} \frac{|z|^s}{s}$$

and

$$|D^n \mathfrak{f}(z)| \geq \int_\Gamma (|D^n \mathfrak{u}(\zeta)| - |D^n \mathfrak{v}(\zeta)|)|d\zeta|$$
$$\geq \int_0^{|z|} \left(1 + 2\sum_{s=1}^{\infty} (-1)^s |z|^s\right) dt$$
$$= |z| + 2\sum_{s=2}^{\infty} \frac{(-1)^{s-1}|z|^s}{s}.$$

Next theorem shows that the $ACH(n)$ class is closed under convex combinations, meaning that any convex combination of functions in $ACH(n)$ will also belong to $ACH(n)$.

Theorem 2.6. The class $ACH(n)$ is closed under convex combinations.

Proof. Suppose $\mathfrak{f}_k = \mathfrak{u}_k + \overline{\mathfrak{v}_k} \in ACH(n)$ and $\sum_{k=1}^{\infty} c_k = 1$ $(0 \leq c_k \leq 1)$. The convex combination of functions f_k may be written as

$$\mathfrak{f}(z) = \sum_{k=1}^{\infty} c_k \mathfrak{f}_k(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)}$$

where

$$\mathfrak{u}(z) = z + \sum_{k=1}^{\infty} c_k \mathfrak{u}_k(z)$$

and

$$\mathfrak{v}(z) = \sum_{k=1}^{\infty} c_k \mathfrak{v}_k(z).$$

Both $\mathfrak{u}$ and $\mathfrak{v}$ are analytic functions within the open unit disk $\mathbb{E}$, satisfying the conditions $D^n\mathfrak{u}(0) = D^n\mathfrak{v}(0) = (D^n\mathfrak{u})'(0) - 1 = (D^n\mathfrak{v})'(0) = 0$ and

$$Re\left\{\left(D^n \mathfrak{u}(z)\right)'\right\} = Re\left\{\sum_{k=1}^{\infty} c_k \left(\left(D^n \mathfrak{u}_k(z)\right)'\right)\right\}$$
$$> \sum_{k=1}^{\infty} c_k \left|\left(D^n \mathfrak{v}_k(z)\right)'\right| \geq \left|\left(D^n \mathfrak{v}(z)\right)'\right|$$

showing that $\mathfrak{f} \in ACH(n)$.

If a sequence $\{u_s\}_{s=0}^{\infty}$ of non-negative real numbers satisfies the following criteria, it is termed a "convex null sequence": as $s \to \infty$, a_s approaches 0, and the inequality

$$u_0 - u_1 \geq u_1 - u_2 \geq \cdots \geq u_{s-1} - u_s \geq \cdots \geq 0$$

holds.

We shall require the following Lemma 2.7, Lemma 2.8 and Lemma 2.9 to prove results of convolution.

Lemma 2.7. [18] When $\{a_s\}_{s=0}^{\infty}$ is a convex null sequence, then the function

$$Q(z) = \frac{a_0}{2} + \sum_{s=1}^{\infty} a_s z^s$$

is analytic, and the real part of $Q(z)$ is positive within the open unit disk $\mathbb{E}$.

Lemma 2.8. [19] Suppose the function Φ is analytic within the domain $\mathbb{E}$, satisfying $\Phi(0) = 1$ and $Re\,[\Phi(z)] > 1/2$ throughout $\mathbb{E}$. For any analytic function $\mathfrak{U}$ defined in $\mathbb{E}$, the function $\Phi * \mathfrak{U}$ maps to values within the convex hull of the image of $\mathbb{E}$ under $\mathfrak{U}$.

Lemma 2.9. Let $\mathfrak{U} \in AC(n)$, then $Re\left\{\frac{\mathfrak{U}(z)}{z}\right\} > \frac{1}{2}$.

Proof. Consider $\mathfrak{U}$ belonging to the class $AC(n)$, defined as $\mathfrak{U}(z) = z + \sum_{s=2}^{\infty} U_s z^s$. Then, the inequality

$$Re\left\{1 + \sum_{s=2}^{\infty} s^{n+1} U_s z^{s-1}\right\} > 0 \quad (z \in \mathbb{E})$$

can be equivalently expressed as $Re\{P(z)\} > \frac{1}{2}$ within the open unit disk $\mathbb{E}$, where

$$P(z) = 1 + \frac{1}{2}\sum_{s=2}^{\infty} s^{n+1} U_s z^{s-1}.$$

Consider a sequence $\{u_s\}_{s=0}^{\infty}$ defined by

$$u_0 = 1 \; ve \; u_{s-1} = \frac{2}{s^{n+1}} \quad \text{for } s \geq 2.$$

It is evident that the sequence $\{u_s\}_{s=0}^{\infty}$ forms a convex null sequence. By applying Lemma 2.7, we conclude that

$$Q(z) = \frac{1}{2} + \sum_{s=2}^{\infty} \frac{2}{s^{n+1}} z^{s-1}$$

is an analytic function and $Re\{Q(z)\} > 0$ within $\mathbb{E}$. Expressing

$$\frac{\mathfrak{U}(z)}{z} = P(z) * \left(1 + \sum_{s=2}^{\infty} \frac{2}{s^{n+1}} z^{s-1}\right),$$

and using Lemma 2.8, we arrive at the conclusion that $Re\left\{\frac{\mathfrak{U}(z)}{z}\right\} > \frac{1}{2}$ for $z \in \mathbb{E}$.

Theorem 2.10. Let $\mathfrak{U}_k \in AC(n)$ for $k = 1,2$. Then $\mathfrak{U}_1 * \mathfrak{U}_2 \in AC(n)$.

Proof. Suppose $\mathfrak{U}_1(z) = z + \sum_{s=2}^{\infty} U_s z^s$ and $\mathfrak{U}_2(z) = z + \sum_{s=2}^{\infty} V_s z^s$. Then the convolution of $\mathfrak{U}_1(z)$ and $\mathfrak{U}_2(z)$ is defined by

$$\mathfrak{U}(z) = (\mathfrak{U}_1 * \mathfrak{U}_2)(z) = z + \sum_{s=2}^{\infty} U_s V_s z^s.$$

Then, we have

$$\left(D^n \mathfrak{U}(z)\right)' = \left(D^n \mathfrak{U}_1(z)\right)' * \frac{\mathfrak{U}_2(z)}{z}. \tag{13}$$

Since $\mathfrak{U}_1 \in AC(n)$, we get $Re\left\{\left(D^n \mathfrak{U}_1(z)\right)'\right\} > 0$. Moreover using Lemma 2.9, $Re\left\{\frac{\mathfrak{U}_2(z)}{z}\right\} > \frac{1}{2}$ in $\mathbb{E}$. Now applying Lemma 2.8 to (13) yields $Re\left\{\left(D^n \mathfrak{U}_1(z)\right)'\right\} > 0$ in $\mathbb{E}$. Thus, $\mathfrak{U} = \mathfrak{U}_1 * \mathfrak{U}_2 \in AC(n)$.

The next theorem shows that the $ACH(n)$ class is closed under the convolution operation, meaning that the convolution of functions in $ACH(n)$ will also belong to $ACH(n)$.

Theorem 2.11. Let $\mathfrak{f}_k \in ACH(n)$ for $k = 1,2$. Then $\mathfrak{f}_1 * \mathfrak{f}_2 \in ACH(n)$.

Proof. Suppose $\mathfrak{f}_k = \mathfrak{o}_k + \mathfrak{v}_k \in ACH(n)$ with $k = 1,2$. The convolution $\mathfrak{f}_1 * \mathfrak{f}_2 = \mathfrak{u}_1 * \mathfrak{u}_2 + \overline{\mathfrak{v}_1 * \mathfrak{v}_2}$ is defined as the convolution of the individual components of f_1 and f_2. To prove that $\mathfrak{f}_1 * \mathfrak{f}_2 \in ACH(n)$ we need to prove that $\mathfrak{U}_\epsilon = \mathfrak{u}_1 * \mathfrak{u}_2 + \epsilon(\mathfrak{v}_1 * \mathfrak{v}_2) \in AC(n)$ for each ϵ ($|\epsilon| = 1$). By Teorem 2.10, the class $AC(n)$ is closed under convolutions for each ϵ ($|\epsilon| = 1$), $\mathfrak{u}_i + \epsilon\mathfrak{v}_i \in AC(n)$ for $i = 1,2$. we can assert that $AC(n)$ includes both $\mathfrak{U}_1$ and $\mathfrak{U}_2$, where

$\mathfrak{U}_1 = (\mathfrak{u}_1 - \mathfrak{v}_1) * (\mathfrak{u}_2 - \epsilon\mathfrak{v}_2)$ ve $\mathfrak{U}_2 = (\mathfrak{u}_1 + \mathfrak{v}_1) * (\mathfrak{u}_2 + \epsilon\mathfrak{v}_2)$.

combinations, we can form the function

$$\mathfrak{U}_\epsilon = \frac{1}{2}(\mathfrak{U}_1 + \mathfrak{U}_2) = \mathfrak{u}_1 * \mathfrak{u}_2 + \epsilon(\mathfrak{v}_1 * \mathfrak{v}_2)$$

belongs to $AC(n)$. Hence $ACH(n)$ is closed under convolution.

In the following example, we illustrate the application of the theoretical results.

Example 2.12. Let the functions $\mathfrak{f}(z) = \mathfrak{u}_1(z) + \overline{\mathfrak{v}_1}(z) = z + 0.075z^2 + 0.075\overline{z}^3$ and $\mathfrak{g}(z) = \mathfrak{u}_2(z) + \overline{\mathfrak{v}_2}(z) = z + 0.05z^2 - 0.05\overline{z}^3$ be given. Since $\mathfrak{u}_1(z) = z + 0.075z^2$, $\mathfrak{v}_1(z) = 0.075\overline{z}^3$, $\mathfrak{u}_2(z) = z + 0.075z^2$ and $\mathfrak{v}_2(z) = 0.075\overline{z}^3$ a straightforward computation yields $\left|\frac{\mathfrak{v}_1'(z)}{\mathfrak{u}_1'(z)}\right| < 1$ and $\left|\frac{\mathfrak{v}_2'(z)}{\mathfrak{u}_2'(z)}\right| < 1$. Consequently, the functions $\mathfrak{f}$ and $\mathfrak{g}$ belong to the SH^0 class. Moreover, since $4|0.075| + 9|0.075| = 0.975 < 1$ and $4|0.05| + 9|0.05| = 0.065 < 1$, by Theorem 2.4 the functions $\mathfrak{f}$ belongs to the $ACH(1)$ class. On the other hand, the function $\mathfrak{h}(z) = \mathfrak{f}(z) * \mathfrak{g}(z) = \mathfrak{u}_3(z) + \overline{\mathfrak{v}_3}(z) = z + 0.00375z^2 - 0.00375\overline{z}^3$, is similarly easily obtained to belong to the class $ACH(1)$.

The images of the unit disk $\mathbb{E}$ under the mappings $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are shown in Figure 1, Figure 2, and Figure 3, respectively.

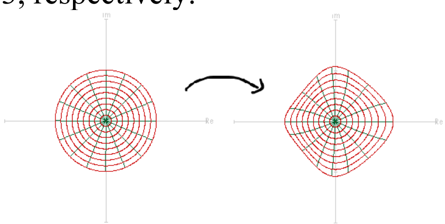

Figure 1. Image of the unit disk under the function $\mathfrak{f}$.

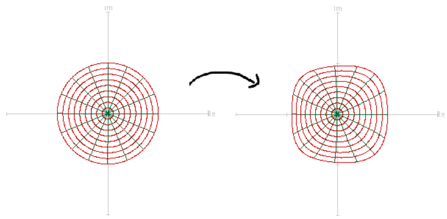

Figure 2. Image of the unit disk under the function $\mathfrak{g}$.

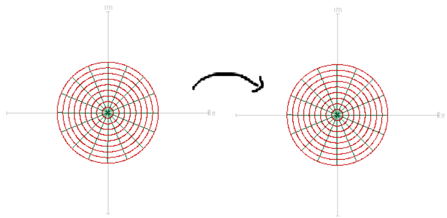

Figure 3. Image of the unit disk under the function $\mathfrak{h}$.

3. Conclusion

In this paper, we introduce a new subclass of harmonic functions. We also thoroughly analyze its properties. We derived precise coefficient bounds and distortion bounds that characterize the behavior of functions within this subclass. Additionally, we established sufficient conditions for coefficients that ensure the functions meet the desired criteria. Our results also demonstrate that this subclass is closed under both convex combinations and convolutions, highlighting its robustness and applicability. These findings contribute to the broader understanding of harmonic functions and provide a foundation for future research in geometric function theory.

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