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Research Article

Introduction to Soft Metric Preserving Functions

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Abstract: In this study, we aim to present the notion of soft metric preserving functions (SMPFs) which allows us to transform a soft metric into another one. We study some properties of SMPFs and investigate some characterizations to decide whether a soft function is soft metric preserving or not. Then, we show that the soft topology induced by soft metric was not preserved under SMPFs, present the stronger concept for these functions and also research the relationships of this concept with continuity.

Keywords: Soft function, soft metric, completeness, metric preserving function.

1. Introduction

In mathematical analysis and topology, metric spaces are fundamental structures that provide a rigorous way to measure distances between elements. A metric space is defined by a set paired with a distance function (metric) that satisfies the following conditions: non-negativity, identity of indiscernibles, symmetry, and triangle inequality. In the study of metric spaces, understanding how functions interact with the underlying metric structure is essential for many areas of mathematics, including topology, analysis, and geometry. When we use of a metric space endowed with a given metric d, it is often useful to exchange d for a different metric which is more suitable for our purposes. This possibility is crucial for applications where the integrity of the distance relationships must be maintained, such as in data analysis, machine learning, and various forms of geometric transformations. The importance of this concept can be interpreted that a particular property is fulfilled or not by a subset or a mapping when verifying such a property is a difficult task in the metric space. Actually, the first studies of this type functions was shown by Wilson [34]. The notion of metric preserving function (MPF) was given in [7] as follows: $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called a MPF if $d_f : U \times U \to \mathbb{R}^+$ by $d_f(u_1, u_2) = f(d(u_1, u_2))$ for all $u_1, u_2 \in U$ is a metric on U whenever (U, d) is a metric space. A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be amenable if $f(u) = 0 \Leftrightarrow u = 0$. Also, f is called subadditive if $f(u_1) + f(u_2) \geq f(u_1 + u_2)$ for all $u_1, u_2 \in \mathbb{R}^+$. If the topology

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generated by transformed metric coincides with the topology generated by original metric to be transformed, then f is called a strong metric preserving function (S-MPF). The metric preserving functions have applications in various mathematical and practical contexts such as geometry and topology, computer graphics and image processing, data analysis and machine learning, physics and engineering. Some quality research related to this concept in the settings of different views can be found in [4–6, 12, 13, 15, 18, 21, 23–27, 30].

In the literature, different set theories have been presented after Zadeh [35] introduced the fuzzy set theory since there are situations that the traditional classical methods do not capable of solving complex problems, especially including uncertain data, in many fields such as engineering, economics, environmental sciences, computer sciences, medical sciences, and etc. One of these theories is the soft set (SS) theory given by Molodtsov [22] and defined as a parameterized family of sets where the parameter takes value over an arbitrary set. This theory has been applied in different areas successfully since this notion was initiated [3, 8, 9, 19, 28, 31–33]. Further, Majumdar and Samanta [20] gave the the idea of soft mappings and studied images and inverse images of crisp sets and SSs under soft mappings. In 2012, Das and Samanta [10] defined the notion of the soft element (SE), and in particular, the soft real number which is interpreted as the extension of fuzzy number in the sense of Dubois and Prade [14].

Classical metric spaces may not be suitable for dealing with problems involving uncertainties, vagueness, or imprecision, which are common in real-world applications. To overcome these limitations, the concept of a soft metric space (SMS) has been introduced by the authors of [11] as an extension of classical metric spaces. SMSs integrate the ideas of SSs and fuzzy sets to handle uncertainty and imprecision more effectively. SSs provide a flexible mathematical framework for modeling situations where traditional methods struggle, especially in cases involving incomplete or partially known data. The primary distinction between classical metric spaces and SMSs lies in the nature of the metric itself. In classical metric spaces, the distance between any two points is a single, exact number. In contrast, in SMS we represent this distance as a ”soft value,” which is effectively a set of possible values or a range that encapsulates the uncertainty or fuzziness inherent in the measurement. The authors of [11] also study the topological properties of these spaces and gave Banach’s fixed point theorem and Cantor’s intersection theorem. Some different types of fixed point theorems in the soft setting for SMSs can be found in [1, 2, 16, 17]. Recently, Taşköprü and Altıntaş [29] described the soft functions by means of SEs as particular soft mappings in the sense of [20].

In this paper, we study on the soft mappings given by Taşköprü and Altıntaş [29]. We introduce the notion of SMPFs that can be taken as soft generalization of the metric preserving functions. We study some features of SMPFs and investigate some characterizations that allow us practical usefulness in the applications. Then, we show that the soft topology generated by soft metric was not preserved under SMPFs, present the stronger concept for these functions and also investigate the relationships of this concept with continuity.

2. Preliminaries

In this part, we recall some necessary notions such as SS, SE, soft function, soft metric and soft topology that will be used in the other sections. Suppose that U is an universal set, A is a non-empty set of parameters and $\mathfrak{B}(\mathbb{R})$ is the collection of all non-empty bounded subsets of the set $\mathbb{R}$.

Definition 2.1 *[10, 19, 22] A pair (F, A) is said to be SS over U if F is a mapping of A into the set of all subsets of U (i.e., $F : A \to P(U)$). We denote SS (F, A) by F shortly.*

The complement of SS F is denoted by F^c where $F^c : A \to P(U)$ is a mapping given by $F^c(\gamma) = U \setminus F(\gamma)$ for all $\gamma \in A$. SS F over U is said to be

(1) A null SS and denoted by Φ if $F(\gamma) = \varnothing$ for all $\gamma \in A$,

(2) An absolute SS and denoted by $\widetilde{U}$ if $F(\gamma) = U$ for all $\gamma \in A$.

We will denote by $S(\widetilde{U})$ the collection of all SSs F for which $F(\gamma) \neq \varnothing$ for any $\gamma \in A$.

Also, all mappings $\epsilon : A \to U$ are said to be SEs of U. SE ϵ is said to belong to SS F and denoted by $\epsilon \tilde{\in} F$ if $\epsilon(\gamma) \in F(\gamma)$ for all $\gamma \in A$. Here, we note that any family of SEs of SS can generate a soft subset of this SS. We will denote SS constructed from a collection $\mathscr{B}$ of SEs by $SS(\mathscr{B})$. Also, we will denote the collection of SEs of SS F by $SE(F)$.

Specially, a soft real set is a mapping $F : A \to \mathfrak{B}(\mathbb{R})$. If F is a single-valued function on A taking values in $\mathbb{R}$, then F is called SE of $\mathbb{R}$ or a soft real number. If F is a single-valued function on A taking values in $\mathbb{R}^+$, then F is said to be a non-negative soft real number. The set of all non-negative soft real numbers is denoted by $\mathbb{R}(A)^*$. Also, the notations $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are used to denote soft real numbers whereas $\bar{u}$, $\bar{v}$, $\bar{w}$ are used to denote a special type of soft real numbers such as $\bar{u}(\gamma) = u$ for all $\gamma \in A$ which is called a constant soft real number. For instance, $\bar{0}$ is the soft real number where $\bar{0}(\gamma) = 0$ for all $\gamma \in A$. The collection of all non-negative constant soft real numbers is denoted by $\overline{\mathbb{R}(A)^*}$.

Definition 2.2 *[10] The soft orderings are defined for soft real numbers $\widetilde{u_1}$, $\widetilde{u_2}$ as follows:*

(1) $\widetilde{u_1} \tilde{\leq} \widetilde{u_2}$ if $\widetilde{u_1}(\gamma) \leq \widetilde{u_2}(\gamma)$, for all $\gamma \in A$,

(2) $\widetilde{u_1} \tilde{\geq} \widetilde{u_2}$ if $\widetilde{u_1}(\gamma) \geq \widetilde{u_2}(\gamma)$, for all $\gamma \in A$,

(3) $\widetilde{u_1} \tilde{<} \widetilde{u_2}$ if $\widetilde{u_1}(\gamma) < \widetilde{u_2}(\gamma)$, for all $\gamma \in A$,

(4) $\widetilde{u_1}\widetilde{>}\widetilde{u_2}$ *if* $\widetilde{u_1}(\gamma) > \widetilde{u_2}(\gamma)$, *for all* $\gamma \in A$.

Definition 2.3 *[11] Let* $F, G \in S(\widetilde{U})$.

(1) The complement of F *is denoted by* $F^{\complement}$ *and defined by* $F^{\complement} = SS(\mathscr{B})$ *where* $\mathscr{B} = \{\widetilde{u_1}\widetilde{\in}\widetilde{U} : \widetilde{u_1}\widetilde{\notin}F^c\}$.

(2) F *is said to be a soft subset of* G *and denoted by* $F \Subset G$ *if every SE of* F *is also SE of* G.

(3) The union of F *and* G *is denoted by* $F \Cup G$ *and defined by* $F \Cup G = SS(\mathscr{B})$ *where* $\mathscr{B} = \{\widetilde{u}\widetilde{\in}\widetilde{U} : \widetilde{u}\widetilde{\in}F$ *or* $\widetilde{u}\widetilde{\in}G\}$, *i.e.,* $F \Cup G = SS(SE(F) \cup SE(G))$.

(4) The intersection of F *and* G *is denoted by* $F \Cap G$ *and defined by* $F \Cap G = SS(\mathscr{B})$ *where* $\mathscr{B} = \{\widetilde{u}\widetilde{\in}\widetilde{U} : \widetilde{u}\widetilde{\in}F$ *and* $\widetilde{u}\widetilde{\in}G\}$, *i.e.,* $F \Cap G = SS(SE(F) \cap SE(G))$.

Definition 2.4 *[29] A soft mapping from* U *to* V *with parameter set* A *is denoted by the mapping* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$.

If $\{f_\gamma : \gamma \in A\}$ *is a collection of crisp mapping from* U *to* V, *then* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *is a soft mapping such that* $f(\widetilde{u})(\gamma) = f_\gamma(\widetilde{u}(\gamma))$ *for all* $\gamma \in A$. *Hence, every parameterized family of crisp mappings can be taken as a soft mapping. However, the converse of this statement is not satisfied in general as seen in [29].*

Theorem 2.5 *[29] If the soft mapping* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *satisfies the following condition (F), then* $f_\gamma : U \to V$ *defined by* $f_\gamma(\widetilde{u}(\gamma)) = f(\widetilde{u})(\gamma)$ *is a function.*

(F) $f(\widetilde{u})(\gamma) : \widetilde{u}(\gamma) = u$ *is a single-point set for all* $u \in U$ *and* $\gamma \in A$.

Definition 2.6 *[29] A soft mapping* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *is called a soft function if* f *satisfies the condition (F).*

A soft function f *is called injective if* $\widetilde{u_1} = \widetilde{u_2}$ *whenever* $f(\widetilde{u_1}) = f(\widetilde{u_2})$ *and surjective if* $f(SE(\widetilde{U})) = SE(\widetilde{V})$. *It is obvious that a soft function* $f : SE(\widetilde{U}) \to SE(\widetilde{V})$ *is injective (surjective) if and only if* $f_\gamma : U \to V$ *is injective (surjective) for all* $\gamma \in A$.

We also note that a soft function is a special soft mapping in the sense of [20].

Definition 2.7 *[11] A soft metric on SS* $\widetilde{U}$ *is a mapping* $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ *fulfilling the following axioms:*

(SM1) $\widetilde{u_1} = \widetilde{u_2}$ *iff* $d(\widetilde{u_1}, \widetilde{u_2}) = \bar{0}$,

(SM2) $d(\widetilde{u_1}, \widetilde{u_2}) = d(\widetilde{u_2}, \widetilde{u_1})$,

(SM3) $d(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\leq} d(\widetilde{u_1}, \widetilde{u_3}) + d(\widetilde{u_3}, \widetilde{u_2})$ *for all* $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3} \in SE(\widetilde{U})$.

SS $\widetilde{U}$ with a soft metric d on $\widetilde{U}$ is called SMS and denoted by the triplet $(\widetilde{U}, d, A)$ or $(\widetilde{U}, d)$ for short. If there exists a $\bar{k} \in \overline{\mathbb{R}(A)^*}$ such that $d(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\leq} \bar{k}$ for all $\widetilde{u_1}, \widetilde{u_2} \in \mathbb{R}(A)^*$, then $(\widetilde{U}, d)$ is called a bounded SMS.

Example 2.8 *[11] If $\{d_\gamma, \gamma \in A\}$ is a parameterized family of crisp metrics on a set U, then the mapping $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ defined by $d(\widetilde{u_1}, \widetilde{u_2})(\gamma) = d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))$, for all $\gamma \in A$ and $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$, is a soft metric on $\widetilde{U}$.*

Result 2.9 *[11] Let $\widetilde{U} = \overline{\mathbb{R}(A)^*}$ and define the mapping $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ by $d(\bar{u_1}, \bar{u_2}) = |\bar{u_1} - \bar{u_2}|$ for all $\bar{u_1}, \bar{u_2} \in \overline{\mathbb{R}(A)^*}$. Then, $(\widetilde{U}, d)$ is SMS.*

Proposition 2.10 *[11] If $(\widetilde{U}, d)$ is SMS, then the mapping $d_\gamma : U \times U \to \mathbb{R}^+$ defined by $d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma)) = d(\widetilde{u_1}, \widetilde{u_2})(\gamma)$, for all $\gamma \in A$, is a metric on U provided that d satisfies the following condition:*

(SM4) $\{d(\widetilde{u_1}, \widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$ is a singelton set for all $(a, b) \in U \times U$ and $\gamma \in A$.

Definition 2.11 *[11] Let $(\widetilde{U}, d)$ be SMS, $\widetilde{u} \in SE(\widetilde{U})$ and $\widetilde{r} \in \mathbb{R}(A)^*$. Then, the subset $B(\widetilde{u}, \widetilde{r}) = \{\widetilde{v} \in SE(\widetilde{U}) : d(\widetilde{u}, \widetilde{v}) \widetilde{<} \widetilde{r}\}$ of $SE(\widetilde{U})$ is called an open disc centered at $\widetilde{u}$ with radius $\widetilde{r}$ and SS $B(\widetilde{u}, \widetilde{r})$ is called a soft open disc with the center $\widetilde{u}$ and radius $\widetilde{r}$.*

Let $\mathscr{B} \subset SE(\widetilde{U})$. Then, $\mathscr{B}$ is called open with respect to d if for all $\widetilde{u} \in \mathscr{B}$ there exists $\widetilde{r} \in SE(\widetilde{U})$ such that $B(\widetilde{u}, \widetilde{r}) \subset \mathscr{B}$. SS $F \in S(\widetilde{U})$ is said to be soft open with respect to d if there exist a collection $\mathscr{B}$ of SEs of F such that $\mathscr{B}$ is open with respect to d and $F = SS(\mathscr{B})$.

Proposition 2.12 *[11] If $(\widetilde{U}, d)$ is SMS satisfying the condition (SM4), then for every open disc $B(\widetilde{u}, \widetilde{r})$ in SMS $(\widetilde{U}, d)$, $SS(B(\widetilde{u}, \widetilde{r}))(\gamma) = B(\widetilde{u}(\gamma), \widetilde{r}(\gamma))$ is an open disc in (U, d_γ) for all $\gamma \in A$.*

Proposition 2.13 *[11] If $(\widetilde{U}, d)$ is SMS satisfying the condition (SM4), then $F \in S(\widetilde{U})$ is a soft open set with respect to d if and only if $F(\gamma)$ is open in (U, d_γ) for all $\gamma \in A$.*

Definition 2.14 *[29] Let $\tau \subset S(\widetilde{U})$ be a family of SSs over U. Then, τ is said to be a soft topology on $\widetilde{U}$ if the following axioms are satisfied:*

(ST1) Φ, $\widetilde{U} \in \tau$.

(ST2) If $F, G \in \tau$, then $F \Cap G \in \tau$.

(ST3) If $F_i \in \tau$ for all $i \in I$, then $\Cup_{i \in I} F_i \in \tau$.

The triplet $(\widetilde{U}, \tau, A)$ is called a soft topological space.

Remark 2.15 *[11] Let $(\widetilde{U}, d)$ be SMS satisfying the condition (SM4). Then, the collection τ of all soft open sets with respect to d form a soft topology on $\widetilde{U}$, this topology is called soft metric topology and denoted by τ_d.*

In the following proposition, we note that the condition "$F(\gamma) \cap G(\gamma) \neq \varnothing$" is not necessary unlike [29] when we demand a crisp topology τ_γ on U whenever $(\widetilde{U}, \tau, A)$ is a soft topology.

Proposition 2.16 *If $(\widetilde{U}, \tau, A)$ is a soft topology, then $\tau_\gamma = \{F(\gamma) : F \in \tau\}$, for all $\gamma \in A$, is a crisp topology on U.*

Proof We have $\varnothing, U \in \tau_\gamma$ for all $\gamma \in A$ since Φ, $\widetilde{U} \in \tau$. Let $F_1(\gamma), F_2(\gamma) \in \tau_\gamma$. Then, we have $F_1, F_2 \in \tau$. Also, if $F_1(\gamma) \cap F_2(\gamma) = \varnothing$, then it is clear that $F_1(\gamma) \cap F_2(\gamma) \in \tau_\gamma$. Otherwise, if $F_1(\gamma) \cap F_2(\gamma) \neq \varnothing$, then we have $F_1 \Cap F_2 \neq \Phi$ which follows that $(F_1 \Cap F_2)(\gamma) = \{\widetilde{u_1}(\gamma) : \widetilde{u_1}(\gamma) \in F_1(\gamma)$ *and* $\widetilde{u_1}(\gamma) \in F_2(\gamma)\} = F_1(\gamma) \cap F_2(\gamma)$. Since $F_1 \Cap F_2$, then we obtain $F_1(\gamma) \cap F_2(\gamma) \in \tau_\gamma$. Finally, let $F_i(\gamma) \in \tau_\gamma$ for all $i \in I$ which means that $F_i \in \tau$ for all $i \in I$ and so, $\Cup_{i \in I} F_i \in \tau$. Also, we know that $(\Cup_{i \in I} F_i)(\gamma) = \bigcup_{i \in I} F_i(\gamma)$ which concludes the proof since this implies that $\bigcup_{i \in I} F_i(\gamma) \in \tau_\gamma$. □

3. Soft Metric Preserving Functions

In this part, we introduce the notion of SMPF which let us obtain a new SMS from the existing SMS. Then, we obtain some features of this type of soft function and so, we give some characterizations of these functions. Also, we present that the topology generated by soft metric was not preserved under SMPFs.

Definition 3.1 *Let $(\widetilde{U}, d)$ be SMS and $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be a soft function. Define a mapping $d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ by $d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2}))$ for all $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$. Then, the function f is said to be SMPF if the mapping d_f is a soft metric on $SE(\widetilde{U})$.*
For example, we can obtain a bounded SMS from a given SMS $(\widetilde{U}, d)$ such as $d(\widetilde{u_1}, \widetilde{u_2}) \to \frac{d(\widetilde{u_1}, \widetilde{u_2})}{\bar{1} + d(\widetilde{u_1}, \widetilde{u_2})}$ for all $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$. So, the function $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ defined by $f(\widetilde{u_1}) = \frac{\widetilde{u_1}}{\bar{1} + \widetilde{u_1}}$ is SMPF.

Let us denote by $\widetilde{\mathscr{O}}$ the set of all soft functions $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ satisfy the condition

$$f(\widetilde{u_1}) = \bar{0} \Leftrightarrow \widetilde{u_1} = \bar{0}$$

whenever $\widetilde{u_1} \in \mathbb{R}(A)^*$, i.e., $\widetilde{\mathscr{O}} = \{f : \mathbb{R}(A)^* \to \mathbb{R}(A)^* | f^{-1}(\bar{0}) = \bar{0}\}$. The elements of $\widetilde{\mathscr{O}}$ are called soft amenable functions.

Proposition 3.2 *If $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ is SMPF, then $f \in \widetilde{\mathscr{O}}$.*

Proof Let $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be SMPF and consider SMS $(\widetilde{U}, d)$ where

$$d(\widetilde{u_1}, \widetilde{u_2})(\gamma) = e(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))$$

for all $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$. To show that $f(\widetilde{u_1}) = \bar{0} \Leftrightarrow \widetilde{u_1} = \bar{0}$, first assume $f(\widetilde{u_1}) = \bar{0}$. Then, we can write

$$f(\widetilde{u_1}) = f(d(\widetilde{u_1}, \bar{0})) = d_f(\widetilde{u_1}, \bar{0}) = \bar{0} \Rightarrow \widetilde{u_1} = \bar{0}$$

since d_f is a soft metric on $\widetilde{U}$. The other side is obvious. □

Definition 3.3 *A function $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ is called soft subadditive if f satisfies the following inequality:*

$$f(\widetilde{u_1} + \widetilde{u_2}) \leq f(\widetilde{u_1}) + f(\widetilde{u_2})$$

for all $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$.

Proposition 3.4 *If $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ is SMPF, then f is subadditive.*

Proof Let $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be SMPF and consider SMS $(\widetilde{U}, d)$ where

$$d(\widetilde{u_1}, \widetilde{u_2})(\gamma) = e(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))$$

for all $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$. Then, we have

$$\begin{aligned} f(\widetilde{u_1} + \widetilde{u_2}) &= f(d(\widetilde{u_1} + \widetilde{u_2}, \bar{0})) = d_f(\widetilde{u_1} + \widetilde{u_2}, \bar{0}) \leq d_f(\widetilde{u_1} + \widetilde{u_2}, \widetilde{u_2}) + d_f(\widetilde{u_2}, \bar{0}) \\ &= f(d(\widetilde{u_1} + \widetilde{u_2}, \widetilde{u_2})) + f(d(\widetilde{u_2}, \bar{0})) = f(\widetilde{u_1}) + f(\widetilde{u_2}). \end{aligned}$$

□

Remark 3.5 *The converse implication of the above proposition may not be satisfied. Consider the soft mapping $f(\widetilde{u_1}) = \bar{1}$ for all $\widetilde{u_1} \in \mathbb{R}(A)^*$. Then, it is obvious that f is soft subadditive but f is not SMPF since $f \notin \widetilde{\mathscr{O}}$.*

Proposition 3.6 *If $f:\mathbb{R}(A)^* \to \mathbb{R}(A)^*$ is soft subadditive, non-decreasing and $f \in \widetilde{\mathscr{O}}$, then f is SMPF.*

Proof Let $(\widetilde{U}, d)$ be SMS. Now, we need to show that $d_f: SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ defined by $d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2}))$ is a soft metric on $\widetilde{U}$. Since f is non-decreasing and $d(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\leq} \bar{0}$ for all $\widetilde{u_1}, \widetilde{u_2} \in \mathbb{R}(A)^*$, we obtain $d_f(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\geq} f(\bar{0}) = \bar{0}$. Assume that $d_f(\widetilde{u_1}, \widetilde{u_2}) = \bar{0}$. Then, we have $d(\widetilde{u_1}, \widetilde{u_2}) = \bar{0}$ which means that $\widetilde{u_1} = \widetilde{u_2}$ since d is a soft metric on $\widetilde{U}$. Let $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3} \in \mathbb{R}(A)^*$.

$$\begin{aligned} d_f(\widetilde{u_1}, \widetilde{u_2}) &= f(d(\widetilde{u_1}, \widetilde{u_2})) \widetilde{\leq} f(d(\widetilde{u_1}, \widetilde{u_3}) + d(\widetilde{u_3}, \widetilde{u_2})) \ (since\ f\ is\ non-decreasing) \\ &\widetilde{\leq} f(d(\widetilde{u_1}, \widetilde{u_3})) + f(d(\widetilde{u_3}, \widetilde{u_2})) = d_f(\widetilde{u_1}, \widetilde{u_3}) + d_f(\widetilde{u_3}, \widetilde{u_2}) \ (since\ f\ is\ subadditive). \end{aligned}$$

As a result, d_f is a soft metric on $\widetilde{U}$ and so, f is SMPF. □

Example 3.7 *Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ defined by*

$$f(\widetilde{u_1}) = \begin{cases} \bar{0}, & \widetilde{u_1} = \bar{0} \\ \bar{1}, & \widetilde{u_1} \neq \bar{0} \end{cases}.$$

Then, f is SMPF since f is soft subadditive, non-decreasing and $f \in \widetilde{\mathscr{O}}$.

Remark 3.8 *The transferred SMS $(\widetilde{U}, d_f)$ may not satisfy the condition (SM4) even if SMS $(\widetilde{U}, d)$ satisfies the condition (SM4) when f is SPMF.*

Example 3.9 *Let $U = \{a, b\}$, $A = \{\gamma, \mu\}$ and $SE(\widetilde{U}) = \{\widetilde{v_1}, \widetilde{v_2}, \widetilde{v_3}, \widetilde{v_4}\}$ where $\widetilde{v_1}(\gamma) = a$, $\widetilde{v_1}(\mu) = a$, $\widetilde{v_2}(\gamma) = a$, $\widetilde{v_2}(\mu) = b$, $\widetilde{v_3}(\gamma) = b$, $\widetilde{v_3}(\mu) = a$, $\widetilde{v_4}(\gamma) = b$ and $\widetilde{v_4}(\mu) = b$. Consider the soft metric $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ given by $d(\widetilde{u_1}, \widetilde{u_2})(\gamma) = |\widetilde{u_1}(\gamma) - \widetilde{u_2}(\gamma)|$. Then, it is easily seen that $(\widetilde{U}, d)$ satisfies the condition (SM4). Take SMPF $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ defined by*

$$f(\widetilde{u_1}) = \begin{cases} \bar{0}, & \widetilde{u_1} = \bar{0} \\ \bar{1}, & \widetilde{u_1} \neq \bar{0} \end{cases}.$$

Now, we obtain the mapping $d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ as

$$d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2})) = \begin{cases} \bar{0}, & \widetilde{u_1} = \widetilde{u_2} \\ \bar{1}, & \widetilde{u_1} \neq \widetilde{u_2} \end{cases}$$

which is a discrete soft metric on $\widetilde{U}$. However, for $(a, a) \in U \times U$ and $\gamma \in A$, we have $\{d(\widetilde{u_1}, \widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = a\} = \{0, 1\}$ which is not a singelton set. Hence, $(\widetilde{U}, d_f)$ does not satisfy the condition (SM4).

Proposition 3.10 *If SPMF $f : SE(\widetilde{U}) \to SE(\widetilde{U})$ is surjective, then the transferred SMS $(\widetilde{U}, d_f)$ satisfies the condition (SM4) when the soft metric $(\widetilde{U}, d)$ satisfies the condition (SM4).*

Proof Assume that there exists a point $(a,b) \in U \times U$ and a parameter $\gamma \in A$ such that $\{d_f(\widetilde{u_1},\widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$ is not a singelton set. Hence, there are k_1, k_2 $(k_1 \neq k_2)$ such that $k_1, k_2 \in \{d_f(\widetilde{u_1},\widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$. This means that $f_\gamma(d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))) = k_1$ and $f_\gamma(d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))) = k_2$ whenever $\widetilde{u_1}(\gamma) = a$ and $\widetilde{u_2}(\gamma) = b$. We know that f_γ is surjective for all $\gamma \in A$ since f is a soft surjective function. Hence, there exists a point m_1, m_2 such that $d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma)) = m_1$ and $d_\gamma(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma)) = m_2$. Therefore, we have that there exists SE $\widetilde{v_1} \neq \widetilde{u_1}$ such that $\widetilde{v_1}(\gamma) = \widetilde{u_1}(\gamma) = a$ or $\widetilde{y_1} \neq \widetilde{u_2}$ such that $\widetilde{y_1}(\gamma) = \widetilde{u_2}(\gamma) = b$ since d_γ is a crisp metric on U. This follows that $\{d(\widetilde{u_1},\widetilde{u_2})(\gamma) : \widetilde{u_1}(\gamma) = a, \widetilde{u_2}(\gamma) = b\}$ is not a singelton set and so, we obtain a contradiction. As a result, $(\widetilde{U}, d_f)$ satisfies the condition (SM4). □

Definition 3.11 *Let $\widetilde{k},\widetilde{l},\widetilde{m} \in \mathbb{R}(A)^*$. A triplet $(\widetilde{k},\widetilde{l},\widetilde{m})$ is said to be soft triangular if*

$$\widetilde{k} \widetilde{\leq} \widetilde{l} + \widetilde{m}, \quad \widetilde{l} \widetilde{\leq} \widetilde{k} + \widetilde{m} \quad \text{and} \quad \widetilde{m} \widetilde{\leq} \widetilde{k} + \widetilde{l}.$$

Proposition 3.12 *If $(\widetilde{U}, d)$ is SMS and $\widetilde{u_1},\widetilde{u_2},\widetilde{u_3} \in \widetilde{U}$, then $(d(\widetilde{u_1},\widetilde{u_2}), d(\widetilde{u_2},\widetilde{u_3}), d(\widetilde{u_1},\widetilde{u_3}))$ is a soft triangular triplet.*

Theorem 3.13 *Let $f \in \widetilde{\mathcal{O}}$ and $\widetilde{k},\widetilde{l},\widetilde{m} \in SE(\widetilde{U})$. Then, the followings are equivalent:*

(i) f is SMPF.

(ii) If $(\widetilde{k},\widetilde{l},\widetilde{m})$ is a soft triangular triplet, then $(f(\widetilde{k}), f(\widetilde{l}), f(\widetilde{m}))$ is soft triangular triplet.

Proof $(i) \Rightarrow (ii)$: Let f be SMPF and $(\widetilde{k},\widetilde{l},\widetilde{m})$ be a soft triangular triplet. Consider SMS $(\widetilde{U}, d)$ where

$$d(\widetilde{u_1},\widetilde{u_2})(\gamma) = e(\widetilde{u_1}(\gamma),\widetilde{u_2}(\gamma))$$

for all $\widetilde{u_1},\widetilde{u_2} \in SE(\widetilde{U})$. Since $(\widetilde{k},\widetilde{l},\widetilde{m})$ is a soft triangular triplet, then we can find $\widetilde{u},\widetilde{v},\widetilde{w} \in \mathbb{R}(A)^*$ such that $d(\widetilde{u},\widetilde{v}) = \widetilde{k}$, $d(\widetilde{v},\widetilde{w}) = \widetilde{l}$ and $d(\widetilde{u},\widetilde{w}) = \widetilde{m}$. Then, we obtain

$$f(\widetilde{k}) = f(d(\widetilde{u},\widetilde{v})) = d_f(\widetilde{u},\widetilde{v}) \widetilde{\leq} d_f(\widetilde{u},\widetilde{w}) + d_f(\widetilde{w},\widetilde{v}) = f(\widetilde{l}) + f(\widetilde{m})$$

and, with similar way, $f(\widetilde{l}) \widetilde{\leq} f(\widetilde{k}) + f(\widetilde{m})$ and $f(\widetilde{m}) \widetilde{\leq} f(\widetilde{k}) + f(\widetilde{l})$. Hence, we conclude that $(f(\widetilde{k}), f(\widetilde{l}), f(\widetilde{m}))$ is soft triangular triplet.

$(ii) \Rightarrow (i)$: Let $f \in \widetilde{\mathcal{O}}$ and $(\widetilde{U}, d)$ be SMS. Since $(d(\widetilde{u_1},\widetilde{u_2}), d(\widetilde{u_1},\widetilde{u_2}), \bar{0})$ is a soft triangular triplet, we obtain $(f(d(\widetilde{u_1},\widetilde{u_2})), f(d(\widetilde{u_1},\widetilde{u_2})), f(\bar{0}))$ is soft triangular triplet which means that $d_f(\widetilde{u_1},\widetilde{u_2}) \widetilde{\geq} \bar{0}$ for all $\widetilde{u_1},\widetilde{u_2} \in \mathbb{R}(A)^*$. Also, it is clear that $d_f(\widetilde{u_1},\widetilde{u_2}) = \bar{0} \Leftrightarrow \widetilde{u_1} = \widetilde{u_2}$ since $f \in \widetilde{\mathcal{O}}$. Finally, since $(d(\widetilde{u_1},\widetilde{u_2}), d(\widetilde{u_2},\widetilde{u_3}), d(\widetilde{u_1},\widetilde{u_3}))$ is a soft triangular triplet, from hypothesis, we also obtain that $d_f(\widetilde{u_1},\widetilde{u_2}) \widetilde{\leq} d_f(\widetilde{u_1},\widetilde{u_3}) + d_f(\widetilde{u_3},\widetilde{u_2})$ which concludes that f is SMPF. □

Proposition 3.14 *Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be a soft function associated with the family of functions $\{f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+, \gamma \in A\}$. If f is SMPF, then f_γ is a MPF for all $\gamma \in A$.*

Proof Let us consider SMS $(\overline{\mathbb{R}(A)^*}, d)$ where $d(\bar{x},\bar{y}) = |\bar{x} - \bar{y}|$ for all $\bar{x},\bar{y} \in \overline{\mathbb{R}(A)^*}$. We will show that f_γ is amenable, non-decreasing and subadditive function for all $\gamma \in A$ (We refer to [7] for the notion of metric preserving function).

Let us take $\gamma \in A$.

(Amenable:) Let $x \in \mathbb{R}^+$. Then, we have the followings:

$$\begin{aligned} f_\gamma(x) = 0 \quad &\Leftrightarrow \quad 0 = f_\gamma(\bar{x}(\gamma)) = f_\gamma(d(\bar{x},\bar{0})(\gamma)) = f(d(\bar{x},\bar{0}))(\gamma) \\ &\Leftrightarrow \quad f(d(\bar{x},\bar{0})) = \bar{0} \Leftrightarrow d(\bar{x},\bar{0}) = \bar{0} \Leftrightarrow \bar{x} = \bar{0} \Leftrightarrow x = 0. \end{aligned}$$

(Non-decreasing:) Let $x \le y$ for any $x, y \in \mathbb{R}^+$.

$$\begin{aligned} f_\gamma(x) \quad &= \quad f_\gamma(\bar{x}(\gamma)) = f_\gamma(d(\bar{x},\bar{0})(\gamma)) = f(d(\bar{x},\bar{0}))(\gamma) \\ &\le \quad f(d(\bar{y},\bar{0}))(\gamma) = f_\gamma(d(\bar{y},\bar{0})(\gamma)) = f_\gamma(\bar{y}(\gamma)) = f_\gamma(y). \end{aligned}$$

(Subadditive:) Let $x, y \in \mathbb{R}^+$.

$$\begin{aligned} f_\gamma(x+y) \quad &= \quad f_\gamma(\overline{x+y}(\gamma)) = f_\gamma(d(\overline{x+y},\bar{0})(\gamma)) = f(d(\overline{x+y},\bar{0}))(\gamma) \\ &\le \quad f(d(\overline{x+y},\bar{y}))(\gamma) + f(d(\bar{y},\bar{0}))(\gamma) \\ &= \quad f_\gamma(d(\overline{x+y},\bar{y})(\gamma)) + f_\gamma(d(\bar{y},\bar{0})(\gamma)) \\ &= \quad f_\gamma(\bar{x}(\gamma)) + f_\gamma(\bar{y}(\gamma)) = f_\gamma(x) + f_\gamma(y). \end{aligned}$$

□

Remark 3.15 *The converse of the above proposition may not be true. Let us consider the example given in [11] as follows:*

Let $U = \{a,b\}$, $A = \{\gamma,\mu\}$ and $SE(\widetilde{U}) = \{\widetilde{v_1},\widetilde{v_2},\widetilde{v_3},\widetilde{v_4}\}$ where $\widetilde{v_1}(\gamma) = a$, $\widetilde{v_1}(\mu) = a$, $\widetilde{v_2}(\gamma) = a$, $\widetilde{v_2}(\mu) = b$, $\widetilde{v_3}(\gamma) = b$, $\widetilde{v_3}(\mu) = a$, $\widetilde{v_4}(\gamma) = b$ and $\widetilde{v_4}(\mu) = b$. Consider the discrete metric $d : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ given by

$$d(\widetilde{u_1},\widetilde{u_2}) = \begin{cases} \bar{0}, & \widetilde{u_1} = \widetilde{u_2} \\ \bar{1}, & \widetilde{u_1} \ne \widetilde{u_2} \end{cases}.$$

Take the function $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f_\gamma(x) = \frac{x}{1+x}$ for all $x \in \mathbb{R}^+$ and $\gamma \in A$ and the soft function $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ given by $f(\widetilde{u_1}) = \frac{\widetilde{u_1}}{\bar{1}+\widetilde{u_1}}$. Here, it is obvious that $f(\widetilde{u_1})(\gamma) = f_\gamma(\widetilde{u_1}(\gamma))$ for all $\widetilde{u_1} \in SE(\widetilde{U})$ and $\gamma \in A$. Also, f_γ is a MPF and f is SMPF. However, the mapping

$d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ *defined by* $d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2}))$ *for all* $\widetilde{u_1}, \widetilde{u_2} \in SE(\widetilde{U})$*, is not a soft metric on* $\widetilde{U}$ *since* $d_f(\widetilde{v_1}, \widetilde{v_1})(\gamma) = f_\gamma(0)$, $d_f(\widetilde{v_1}, \widetilde{v_1})(\gamma) = f_\gamma(1)$ *and* $f_\gamma(0) \neq f_\gamma(1)$.

Proposition 3.16 *If* $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is a MPF for all* $\gamma \in A$*, then the soft function* f *is SMPF when SMS* $(\widetilde{U}, d)$ *satisfies the condition (SM4).*

Proof Let $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a MPF for all $\gamma \in A$. Suppose that $(\widetilde{U}, d)$ is SMS satisfying the condition (SM4). Then, we know that the mapping $d_\gamma : U \times U \to \mathbb{R}^+$ defined by $d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma)) = d(\widetilde{u_1}, \widetilde{u_2})(\gamma)$, for all $\gamma \in A$, is a metric on U. Now, we show that $d_f : SE(\widetilde{U}) \times SE(\widetilde{U}) \to \mathbb{R}(A)^*$ defined by $d_f(\widetilde{u_1}, \widetilde{u_2}) = f(d(\widetilde{u_1}, \widetilde{u_2}))$ is a soft metric on $\widetilde{U}$.
Let $\gamma \in A$.

$$d_f(\widetilde{u_1}, \widetilde{u_2})(\gamma) = f(d(\widetilde{u_1}, \widetilde{u_2}))(\gamma) = f_\gamma(d(\widetilde{u_1}, \widetilde{u_2})(\gamma)) = f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))) \geq f_\gamma(0) = 0$$

. So, we have $d_f(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\geq} \overline{0}$ since γ is an arbitrary chosen parameter.
(SM1) Let $\gamma \in A$ and $d_f(\widetilde{u_1}, \widetilde{u_2}) = \overline{0}$. Then, we have

$$0 = f(d(\widetilde{u_1}, \widetilde{u_2}))(\gamma) = f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))) \Rightarrow d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma)) = 0 \Rightarrow \widetilde{u_1}(\gamma) = \widetilde{u_2}(\gamma)$$

which means that $\widetilde{u_1} = \widetilde{u_2}$ since γ is an arbitrary parameter. It is clear that $d_f(\widetilde{u_1}, \widetilde{u_2}) = \overline{0}$ when $\widetilde{u_1} = \widetilde{u_2}$.
(SM2) It is obvious from the definitions.
(SM3) Let $\widetilde{u_1}, \widetilde{u_2}, \widetilde{u_3} \in \mathbb{R}(A)^*$. Then, we obtain

$$\begin{aligned} d_f(\widetilde{u_1}, \widetilde{u_2})(\gamma) &= f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_2}(\gamma))) \leq f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_3}(\gamma)) + d_\gamma(\widetilde{u_3}(\gamma), \widetilde{u_2}(\gamma))) \\ &\leq f_\gamma(d_\gamma(\widetilde{u_1}(\gamma), \widetilde{u_3}(\gamma))) + f_\gamma(d_\gamma(\widetilde{u_3}(\gamma), \widetilde{u_2}(\gamma))) \\ &= f(d(\widetilde{u_1}, \widetilde{u_3}))(\gamma) + f(d(\widetilde{u_3}, \widetilde{u_2}))(\gamma) = d_f(\widetilde{u_1}, \widetilde{u_3})(\gamma) + d_f(\widetilde{u_3}, \widetilde{u_2})(\gamma) \end{aligned}$$

which follows that $d_f(\widetilde{u_1}, \widetilde{u_2}) \widetilde{\leq} d_f(\widetilde{u_1}, \widetilde{u_3}) + d_f(\widetilde{u_3}, \widetilde{u_2})$ as required. □

In the following example, we notice that the topology generated by the transformed SMS may not be equivalent to the topology generated by SMS to be transformed.

Example 3.17 *Consider the soft metric* $d : \overline{\mathbb{R}(A)^*} \times \overline{\mathbb{R}(A)^*} \to \mathbb{R}(A)^*$ *given by* $d(\overline{x}, \overline{y})(\gamma) = |\overline{x}(\gamma) - \overline{y}(\gamma)|$ *where* A *is a non-empty parameter set. Then, it is easily seen that* $(\widetilde{U}, d)$ *satisfies the condition (SM4). Take SMPF* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *defined by*

$$f(\widetilde{u_1}) = \begin{cases} \overline{0}, & \widetilde{u_1} = \overline{0} \\ \overline{1}, & \widetilde{u_1} \neq \overline{0} \end{cases}.$$

Now, we obtain the mapping $d_f : \overline{\mathbb{R}(A)^*} \times \overline{\mathbb{R}(A)^*} \to \mathbb{R}(A)^*$ *as*

$$d_f(\overline{x},\overline{y}) = f(d(\overline{x},\overline{y})) = \begin{cases} \overline{0}, & \overline{x} = \overline{y} \\ \overline{1}, & \overline{x} \neq \overline{y} \end{cases}$$

which is a discrete soft metric on $\overline{\mathbb{R}(A)^*}$ *satisfying the condition (SM4). Then, we have*

$$B_d(\widetilde{k},\widetilde{r}) = \{\overline{x} : d(\overline{x},\widetilde{k}) \widetilde{<} \widetilde{r}\} \Rightarrow SS(B_d(\widetilde{k},\widetilde{r}))(\gamma) = B_{d_\gamma}(a,r) = (a-r,a+r)$$

and

$$B_{d_f}(\widetilde{k},\widetilde{r}) = \{\overline{x} : d_f(\overline{x},\widetilde{k}) \widetilde{<} \widetilde{r}\} \Rightarrow SS(B_{d_f}(\widetilde{k},\widetilde{r}))(\gamma) = B_{d_{f_\gamma}}(\overline{x}(\gamma),\overline{y}(\gamma)) = \begin{cases} \{a\}, & \widetilde{r}(\gamma) \leq 1 \\ \mathbb{R}^+, & \widetilde{r}(\gamma) > 1 \end{cases}$$

which means that $(\tau_d)_\gamma \neq (\tau_{d_f})_\gamma$ *and so, we have that* $\tau_d \neq \tau_{d_f}$.

Definition 3.18 *Let* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *be a surjective SMPF.* f *is called strong SMPF (S-SMPF) if for each soft metric space* $(\widetilde{U},d)$ *satisfying the condition (SM4), the soft metrics* d *and* d_f *are topologically equivalent, i.e.,* $\tau_d = \tau_{d_f}$.

Proposition 3.19 *If* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is S-SMPF, then* $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is S-MPF for all* $\gamma \in A$.

Proof It is obvious from Proposition 3.14 and Definition 3.18. □

Proposition 3.20 *If* $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *is a surjective S-MPF for every* $\gamma \in A$*, then* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *is S-SMPF.*

Theorem 3.21 *Let* $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ *be SMPF. Then, the following assertions are equivalent:*

(i) f *is S-SMPF.*

(ii) f *is surjective and continuous.*

(iii) f *is surjective and continuous at* $\overline{0}$.

Proof $(i) \Rightarrow (ii)$ Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be S-SMPF, then from Proposition 3.19 $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is S-MPF for all $\gamma \in A$. Since f_γ, for all $\gamma \in A$, is continuous, then we have that f is continuous over $\mathbb{R}(A)^*$.

$(ii) \Rightarrow (i)$ Let $f : \mathbb{R}(A)^* \to \mathbb{R}(A)^*$ be a surjective and continuous SMPF, then we have that $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is surjective and continuous for all $\gamma \in A$ which means that f_γ is S-MPF for all $\gamma \in A$. Hence, we conclude that f is S-SMPF.

$(ii) \Rightarrow (iii)$ This observation is clear.

$(iii) \Rightarrow (ii)$ Let f be surjective and continuous at $\bar{0}$. Then, $f_\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is surjective and continuous at 0 for all $\gamma \in A$. This implies that f_γ is continuous over $\mathbb{R}^+$ and so, we have that f is continuous. □

4. Conclusion

Soft metric spaces provide a significant generalization of classical metric spaces by incorporating SS theory to handle uncertainty and vagueness. They retain the essential properties of metric spaces while allowing for a more flexible and nuanced representation of distances in situations where precise measurements are not possible. This makes them a valuable tool in a wide range of theoretical and practical applications, from decision-making to data analysis and beyond. As research in this area continues, further refinements and applications of soft metric spaces are likely to emerge, broadening their impact across multiple domains. On the other hand, metric preserving functions play a vital role in many areas of mathematics and its applications by ensuring that the metric structure of spaces is maintained under transformations. Whether in geometry, data analysis, or physics, these functions help maintain consistency in distance relationships, enabling meaningful interpretations and reliable results in various domains. Understanding these functions' properties and applications provide deeper insight into the structure and behavior of metric spaces and their transformations. This study includes an introduction to SMPFs and some characterizations of these types of functions by means of some properties of the soft functions. For future work, we plan to investigate soft contraction preserving functions which allow us to find the fixed point of functions on the transferred SMSs and also we research the soft partial metric preserving functions and their relationships with SMPFs.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Elif Güner]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%50).

Author [Halis Aygün]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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