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Two New General Integral Results Related to the Hilbert Integral Inequality

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Abstract

In this article, we generalize two integral results from the literature. The first result concerns a flexible double integral inequality, considering a specific form for the integrated function and a double integral as a lower or upper bound. Several examples are discussed, as well as some of its indirect connections with the Hilbert integral inequality. The second result also gives a double integral inequality, but with the product of the square root of simple integrals, following the spirit of the Hilbert integral inequality. Several theoretical and numerical examples are discussed. Both of our results have the property of being dependent on several adjustable functions and parameters, thus offering a wide range of applications.

1. Introduction

Historically, integral inequalities have attracted attention in almost all areas of mathematics. Some of the most famous are the Cauchy-Schwarz integral inequality, the Jensen integral inequality, the Hölder integral inequality, the Minkowski integral inequality, the Hardy-Littlewood-Sobolev integral inequality, the Hilbert integral inequality, the Sobolev integral inequality, the Gagliardo-Nirenberg integral inequality, the Poincaré integral inequality, the Grönwall integral inequality, the Young integral inequality, the logarithmic Sobolev integral inequality, the Chebyshev integral inequality, the Steffensen integral inequality and the Grüss integral inequality. They are widely used in fields as diverse as calculus, functional analysis, probability theory, numerical analysis, mathematical physics, and partial differential equations. For a comprehensive introduction to these inequalities, see [1, 2, 3, 4, 5]. In recent research, the study of integral inequalities has taken on considerable importance. For some contemporary references, i.e., in 2024 at the time of writing, see, for example, [6, 7, 8, 9].

In this article, we focus on the framework of the Hilbert integral inequality. It plays an important role in applications involving double integrals, where certain types of product and ratio functions are present. This is particularly the case in analysis, approximation theory, probability theory and partial differential equations. Mathematically, the Hilbert integral inequality is expressed as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{x+y}dxdy \leq \pi\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(x)dx}, \tag{1.1}$$

where $f,g:[0,+\infty)\to[0,+\infty)$ are quadratic integrable functions. The upper bound is thus of the form constant multiplied by the L_2 norms of f and g. The constant π is optimal and cannot be improved, as shown in [1, 5]. Note that, in the special case $g=f$, the Hilbert integral inequality reduces to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x+y}dxdy \leq \pi\int_0^{+\infty} f^2(x)dx. \tag{1.2}$$

This simplified version will have some focus for the purposes of this article. The importance of the Hilbert integral inequality has led to numerous variants and extensions, with applications in both pure and applied mathematics. These variants have been the subject of extensive research, as can be seen in the studies in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. In

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addition, the survey in [25] provides a comprehensive overview of these developments, including the various techniques used to improve or generalize the inequality. It also gives examples of how these inequalities are used in different contexts. For some recent references on the topic, i.e., in 2024 at the time of writing, see [26, 27, 28, 29].

In this article, we demonstrate two general integral inequalities that extend some results established in [17], which in turn extend those in [16]. In particular, the following formula is discussed in [17, Lemma 2.1]:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy, \tag{1.3}$$

where $h:[0,+\infty)^2\mapsto[0,+\infty)$ is a symmetric bivariate function, and $F:[0,+\infty)^2\mapsto\mathbb{R}$ is a bivariate function depending on an intermediate univariate function $k:[0,+\infty)\mapsto[0,+\infty)$, of the form $F(x,y)=1+k(x)-k(y)$ (or, without loss of generality, $F(x,y)=1+k(y)-k(x)$). In the first result of this article, we show how to extend Equation (1.3), with a more general function F depending on two intermediate univariate functions. In particular, inequalities come naturally depending on the monotonicity of these functions. It is worth noting that the lower or upper bound obtained is expressed as a double integral, similar to the right term in Equation (1.3).

In the second result, still based on our extended function F, we generalize [17, Part of the proof of Theorem 3.1] by demonstrating a new variant of the Hilbert integral inequality. It is innovative in its use of two adjustable univariate functions and parameters. More specifically, we demonstrate an integral inequality of the following form:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\le\sqrt{\int_0^{+\infty}p(x)f^2(x)dx}\sqrt{\int_0^{+\infty}q(x)f^2(x)dx},$$

where $p,q:[0,+\infty)\mapsto[0,+\infty)$ are explicitly determined. In a sense, it extends the special Hilbert integral inequality presented in Equation (1.2); when F reduces to the constant 1, it is expected that p and q reduce to the constant π. Some consequences of this result are discussed and a new precise variant of the Hilbert integral inequality is established.

The rest of the article is divided into three sections: Section 2 presents the first general integral inequality result, including the detailed proofs and some examples. A connection with the Hilbert integral inequality is also made. Section 3 deals with the second general integral inequality result. It also gives detailed proof, discussion and some examples. Section 4 contains a conclusion.

2. First general integral inequality result

The proposition below is our first general result on integral inequalities, which significantly extends the scope of [17, Lemma 2.1]. A double integral is obtained as a lower or upper bound.

Proposition 2.1. *Let $f:[0,+\infty)\mapsto[0,+\infty)$ and $u,v:[0,+\infty)\mapsto\mathbb{R}$ be univariate functions, and $h:[0,+\infty)^2\mapsto[0,+\infty)$ be a bivariate function. We suppose that h is symmetric, i.e., $h(x,y)=h(y,x)$ for any $(x,y)\in[0,+\infty)^2$. Based on u and v, let $F:[0,+\infty)^2\mapsto\mathbb{R}$ be the bivariate function defined by*

$$F(x,y)=1+u(x)[v(x)-v(y)].$$

Then, distinguishing four cases of assumptions on u and v, the results below hold.

Case 1: *If v is constant and u is an arbitrary function, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge.

Case 2: *If u is constant and v is an arbitrary function, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}|v(y)|dydx<+\infty. \tag{2.1}$$

Case 3: *If u and v are both increasing, or both decreasing, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy \geq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}|u(y)||v(x)-v(y)|dydx < +\infty. \tag{2.2}$$

Case 4: *If u is increasing and v is decreasing, or if u is decreasing and v is increasing, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy \leq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy,$$

provided that the integrals involved converge and the assumption in Equation (2.2) holds.

Proof. Let us prove the four cases, one after the other.

Case 1: If v is constant and u is an arbitrary function, we have $F(x,y)=1$, so that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.$$

Case 2: If u is constant, say $u(x)=c$ for any $x\in[0,+\infty)$ and v is an arbitrary function, we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}\left\{1+c[v(x)-v(y)]\right\}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy\\ &+ c\left[\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(x)dxdy-\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(y)dxdy\right].\end{aligned}$$

Let us focus on the last integral term (without the constant factor). Changing the notations x and y, using the symmetry of h and the Fubini theorem thanks to Equation (2.1) to justify the change of the order of integration, it can be expressed as

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(y)f(x)}{h(y,x)}v(x)dydx\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}v(x)dxdy.\end{aligned}$$

So we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy+c\times 0\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

The desired result is obtained.

Case 3: Let us now suppose that u and v are both increasing, or both decreasing. The following decomposition holds:

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}\left\{1+u(x)[v(x)-v(y)]\right\}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(x)[v(x)-v(y)]dxdy\\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(x)[v(x)-v(y)]dxdy.\end{aligned}$$

Let us focus on the last integral term (without the constant factor). Changing the notations x and y, using the symmetry of h and the Fubini theorem thanks to Equation (2.2) to justify the change of the order of integration, it can be expressed as

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(x)[v(x)-v(y)]dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(y)f(x)}{h(y,x)}u(y)[v(y)-v(x)]dydx \\ &= -\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}u(y)[v(x)-v(y)]dxdy.\end{aligned}$$

We therefore have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy. \qquad (2.3)\end{aligned}$$

If u and v are both increasing, for any $x\geq y$, we have $u(x)\geq u(y)$ and $v(x)\geq v(y)$, implying that $[u(x)-u(y)][v(x)-v(y)]\geq 0$, and, for any $y\geq x$, we have $u(y)\geq u(x)$ and $v(y)\geq v(x)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\geq 0$.

On the other hand, if u and v are both decreasing, for any $x\geq y$, we have $u(y)\geq u(x)$ and $v(y)\geq v(x)$, implying again that $[u(x)-u(y)][v(x)-v(y)]\geq 0$, and, for any $y\geq x$, we have $u(x)\geq u(y)$ and $v(x)\geq v(y)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\geq 0$. Since f and h are positive, we have

$$\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy\geq 0.$$

This and Equation (2.3) imply that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\geq\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.$$

The desired result is obtained.

Case 4: Let us now suppose that u is increasing and v is decreasing, or u is decreasing and v is increasing. Applying Equation (2.3), we still can write

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy \\ &+ \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy. \qquad (2.4)\end{aligned}$$

If u is increasing and v is decreasing, for any $x\geq y$, we have $u(x)\geq u(y)$ and $v(y)\geq v(x)$, implying that $[u(x)-u(y)][v(x)-v(y)]\leq 0$, and, for any $y\geq x$, we have $u(y)\geq u(x)$ and $v(x)\geq v(y)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\leq 0$.

On the other hand, if u is decreasing and v is increasing, for any $x\geq y$, we have $u(y)\geq u(x)$ and $v(x)\geq v(y)$, implying again that $[u(x)-u(y)][v(x)-v(y)]\leq 0$, and, for any $y\geq x$, we have $u(x)\geq u(y)$ and $v(y)\geq v(x)$, still implying that $[u(x)-u(y)][v(x)-v(y)]\leq 0$. Since f and h are positive, we have

$$\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}[u(x)-u(y)][v(x)-v(y)]dxdy\leq 0.$$

The combination of this with Equation (2.4) gives

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\leq\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.$$

The desired result is obtained.

This concludes the proof of Proposition 2.1. □

The interest of Proposition 2.1 is that the double integral under consideration is very general in form, and lower and upper bounds can be derived under a simple monotonicity analysis of only two intermediate functions. However, if there is no monotonicity (or no constant constant function), it cannot be applied.

Taking u as the constant equal to 1 (and v arbitrary), Case 2 in Proposition 2.1 becomes [17, Lemma 2.1], recalled in Equation (1.3) (with $k = v$). It also extends [16, Lemma 1.3], which considers u as the constant equal to 1 and $v(x) = 1/(1+x)$. The other cases give new perspectives of applications.

As a direct consequence, if u is increasing and v is decreasing, or if u is decreasing and v is increasing, applying Case 4 of Proposition 2.1 with $h(x,y) = x+y$ and the Hilbert integral inequality, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x+y}F(x,y)dxdy \leq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x+y}dxdy \leq \pi\int_0^{+\infty}f^2(x)dx,$$

provided that the integrals involved converge and the assumption in Equation (2.2) holds. Some numerical examples are now proposed to illustrate the results in Proposition 2.1, starting with Case 2. We take $f(x) = e^{-x}$ and $h(x,y) = x+y$, so that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy = 1,$$

to work with a manageable benchmark.

Illustration of Case 2: Taking $u(x) = 4$ and $v(x) = \log(x)$, so that u is constant and v is an arbitrary selected function, we have $F(x,y) = 1+u(x)[v(x)-v(y)] = 1+4\log(x/y)$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+4\log\left(\frac{x}{y}\right)\right]dxdy\\ &= 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As expected, the desired double integrals are equal.

Illustration of Case 3: Taking $u(x) = x$ and $v(x) = x^2$, so that u and v are both increasing, we have $F(x,y) = 1+u(x)[v(x)-v(y)] = 1+x(x^2-y^2)$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+x(x^2-y^2)\right]dxdy\\ &= 2\\ &\geq 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As another example for this case, taking $u(x) = e^{-x}$ and $v(x) = 1/(1+x)$, so that u and v are both decreasing, we have $F(x,y) = 1+u(x)[v(x)-v(y)] = 1+e^{-x}[1/(1+x)-1/(1+y)]$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+e^{-x}\left(\frac{1}{1+x}-\frac{1}{1+y}\right)\right]dxdy\\ &\approx 1.03772\\ &\geq 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy = \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As expected, the desired inequality is obtained for both examples.

Illustration of Case 4: Taking $u(x)=\sqrt{x}$ and $v(x)=1/(1+x^2)$, so that u is increasing and v is decreasing, we have $F(x,y)=1+u(x)[v(x)-v(y)]=1+\sqrt{x}[1/(1+x^2)-1/(1+y^2)]$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+\sqrt{x}\left(\frac{1}{1+x^2}-\frac{1}{1+y^2}\right)\right]dxdy\\ &\approx 0.931516\\ &\leq 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As another example for this case, taking $u(x)=e^{-x^2}$ and $v(x)=\log(x)$, so that u is decreasing and v is increasing, we have $F(x,y)=1+u(x)[v(x)-v(y)]=1+e^{-x^2}\log(x/y)$, and

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}\left[1+e^{-x^2}\log\left(\frac{x}{y}\right)\right]dxdy\\ &\approx 0.752483\\ &\leq 1\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{e^{-x-y}}{x+y}dxdy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy.\end{aligned}$$

As expected, the desired inequality is obtained for both examples.

The next section is devoted to a general variant of the Hilbert integral inequality, with some connection to the main double integral in Proposition 2.1. Additional assumptions are made on F and h, including the positivity of F.

3. Second general integral inequality result

Inspired by [17, Theorem 3.1] and in the light of the functional configuration in Proposition 2.1, the result below shows a generalized variant of the Hilbert integral inequality. Upper bounds are obtained through various weighted L_2 norms of f.

Proposition 3.1. *Let $f:[0,+\infty)\mapsto[0,+\infty)$ and $u,v:[0,+\infty)\mapsto\mathbb{R}$ be univariate functions, and $h:[0,+\infty)^2\mapsto[0,+\infty)$ be a bivariate function. Based on u and v, let $F:[0,+\infty)^2\mapsto\mathbb{R}$ be the bivariate function defined by*

$$F(x,y)=1+u(x)[v(x)-v(y)].$$

The assumptions below are made for F and h.

A1: *F is positive, i.e., for any $(x,y)\in[0,+\infty)^2$, $F(x,y)\geq 0$.*

A2: *h is symmetric, i.e., $h(x,y)=h(y,x)$ for any $(x,y)\in[0,+\infty)^2$, and homogeneous in the sense that there exists $\lambda\in\mathbb{R}$ satisfying, for any $(x,y,z)\in[0,+\infty)^3$,*

$$h(zx,zy)=z^{\lambda}h(x,y).$$

Then, for any $\alpha\in\mathbb{R}$, the following integral inequality holds:

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy &\leq \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha-u(x)T_\alpha[v](x)\right\}f^2(x)dx}\\ &\times\sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{c_\alpha+T_\alpha[uv](x)-v(x)T_\alpha[u](x)\right\}f^2(x)dx},\end{aligned}$$

where

$$c_\alpha=\int_0^{+\infty}\frac{r^\alpha}{h(1,r)}dr$$

and, for any function $k : [0,+\infty) \mapsto \mathbb{R}$, $T_\alpha[k]$ is the following integral operator:

$$T_\alpha[k](x) = \int_0^{+\infty} \frac{r^\alpha}{h(1,r)} k(rx)dr,$$

provided that the integrals involved converge. Taking k as the constant equal to 1*, we can note that $T_\alpha[k](x) = c_\alpha$.*

Proof. Using the positivity of F described in A1, the decomposition $(y/x)^{\alpha/2}(x/y)^{\alpha/2} = 1$ and applying the Cauchy-Schwarz integral inequality according to the variables x and y, we get

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy &= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)}{\sqrt{h(x,y)}}\sqrt{F(x,y)}\left(\frac{y}{x}\right)^{\alpha/2} \times \frac{f(y)}{\sqrt{h(x,y)}}\sqrt{F(x,y)}\left(\frac{x}{y}\right)^{\alpha/2} dxdy \\
&\leq \sqrt{\int_0^{+\infty}\int_0^{+\infty} \frac{f^2(x)}{h(x,y)} F(x,y)\left(\frac{y}{x}\right)^{\alpha} dxdy}\sqrt{\int_0^{+\infty}\int_0^{+\infty} \frac{f^2(y)}{h(x,y)} F(x,y)\left(\frac{x}{y}\right)^{\alpha} dxdy} \\
&= \sqrt{\int_0^{+\infty} p(x)f^2(x)dx}\sqrt{\int_0^{+\infty} q(y)f^2(y)dy}, \qquad (3.1)
\end{aligned}$$

where

$$p(x) = \int_0^{+\infty} \frac{1}{h(x,y)} F(x,y)\left(\frac{y}{x}\right)^{\alpha} dy$$

and

$$q(y) = \int_0^{+\infty} \frac{1}{h(x,y)} F(x,y)\left(\frac{x}{y}\right)^{\alpha} dx.$$

Let us now express $p(x)$ and $q(y)$, one after the other. Using the change of variables $y = rx$ and the homogeneous property of h in A2, we get

$$\begin{aligned}
p(x) &= x\int_0^{+\infty} \frac{1}{h(x,rx)} F(x,rx)r^\alpha dr = x^{1-\lambda}\int_0^{+\infty} \frac{1}{h(1,r)} F(x,rx)r^\alpha dr \\
&= x^{1-\lambda}\int_0^{+\infty} \frac{1}{h(1,r)} \{1+u(x)[v(x)-v(rx)]\} r^\alpha dr \\
&= x^{1-\lambda}\left\{[1+u(x)v(x)]\int_0^{+\infty} \frac{r^\alpha}{h(1,r)} dr - u(x)\int_0^{+\infty} \frac{r^\alpha}{h(1,r)} v(rx)dr\right\} \\
&= x^{1-\lambda}\{[1+u(x)v(x)]c_\alpha - u(x)T_\alpha[v](x)\}. \qquad (3.2)
\end{aligned}$$

On the other hand, for $q(y)$, using the change of variables $x = ry$, the symmetry and the homogeneous property of h in A2, we get

$$\begin{aligned}
q(y) &= y\int_0^{+\infty} \frac{1}{h(ry,y)} F(ry,y)r^\alpha dr = y^{1-\lambda}\int_0^{+\infty} \frac{1}{h(r,1)} F(ry,y)r^\alpha dr \\
&= y^{1-\lambda}\int_0^{+\infty} \frac{1}{h(1,r)} \{1+u(ry)[v(ry)-v(y)]\} r^\alpha dr \\
&= y^{1-\lambda}\left\{\int_0^{+\infty} \frac{r^\alpha}{h(1,r)} dr + \int_0^{+\infty} \frac{r^\alpha}{h(1,r)} u(ry)v(ry)dr - v(y)\int_0^{+\infty} \frac{r^\alpha}{h(1,r)} u(ry)dr\right\} \\
&= y^{1-\lambda}\{c_\alpha + T_\alpha[uv](y) - v(y)T_\alpha[u](y)\}. \qquad (3.3)
\end{aligned}$$

Combining Equations (3.1), (3.2) and (3.3), and standardizing the notation x and y, we obtain

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy &\leq \sqrt{\int_0^{+\infty} x^{1-\lambda}\{[1+u(x)v(x)]c_\alpha - u(x)T_\alpha[v](x)\} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} x^{1-\lambda}\{c_\alpha + T_\alpha[uv](x) - v(x)T_\alpha[u](x)\} f^2(x)dx},
\end{aligned}$$

which is the desired inequality. This concludes the proof. □

The interest of Proposition 3.1 lies in its generality and the form of the upper bound obtained; it is typical of those appearing in some variants of the Hilbert integral inequality, i.e., with the product of two weighted L_2 norms of f.

In fact, if we analyze the proof of Proposition 3.1, it can be easily extended to two functions, $f,g:[0,+\infty)\mapsto[0,+\infty)$, as follows:

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}F(x,y)dxdy &\leq \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha-u(x)T_\alpha[v](x)\right\}f^2(x)dx}\\ &\times\sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{c_\alpha+T_\alpha[uv](x)-v(x)T_\alpha[u](x)\right\}g^2(x)dx}.\end{aligned}$$

We have concentrated on the case $f=g$ mainly to make some connections with Proposition 2.1.

Let now discuss A1. If, for any $x\in[0,+\infty)$, $u(x)\in[0,1]$ and $v(x)\in[0,1]$, then, for any $(x,y)\in[0,+\infty)^2$, we have $u(x)v(x)\geq 0$ and $u(x)v(y)\leq 1$, so that

$$F(x,y)=u(x)v(x)+[1-u(x)v(y)]\geq 0.$$

The assumption A1 is thus satisfied.

In the context of Case 3 in Proposition 2.1, i.e., if u and v are both increasing, or both decreasing, under some integrability assumptions, if A1 and A2 of Proposition 3.1 are satisfied, then this result gives

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy &\leq \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy\\ &\leq \sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha-u(x)T_\alpha[v](x)\right\}f^2(x)dx}\\ &\times\sqrt{\int_0^{+\infty}x^{1-\lambda}\left\{c_\alpha+T_\alpha[uv](x)-v(x)T_\alpha[u](x)\right\}f^2(x)dx}.\end{aligned}\tag{3.4}$$

As noted in [17], the choices $\alpha=-1/2$, $h(x,y)=x+y$, $u(x)=1$ and $v(x)=1/(1+x)$ give the improved Hilbert integral inequality demonstrated in [16, Theorem 2.1].

With this in mind, let us illustrate Proposition 3.1 with a new example activating the function u. We consider $\alpha=-1/2$, $h(x,y)=x+y$, $u(x)=1/(1+x)$ and $v(x)=1/(1+x)$. So we have

$$F(x,y)=1+\frac{1}{1+x}\left(\frac{1}{1+x}-\frac{1}{1+y}\right)=1+\frac{y-x}{(1+x)^2(1+y)}.$$

Since $u(x)\in[0,1]$ and $v(x)\in[0,1]$, the assumption A1 holds. Furthermore, with the selected function h, the assumption A2 is obviously satisfied with $\lambda=1$. Let now remark that

$$\begin{aligned}c_\alpha &= \int_0^{+\infty}\frac{r^\alpha}{h(1,r)}dr\\ &= \int_0^{+\infty}\frac{1}{\sqrt{r}(1+r)}dr\\ &= \left\{2\arctan[\sqrt{r}]\right\}_{r=0}^{r\to+\infty}\\ &= \pi,\end{aligned}$$

$$\begin{aligned}T_\alpha[u](x) &= \int_0^{+\infty}\frac{r^\alpha}{h(1,r)}u(rx)dr=\int_0^{+\infty}\frac{1}{\sqrt{r}(1+r)(1+rx)}dr\\ &= \left\{\frac{2}{x-1}\left[\sqrt{x}\arctan[\sqrt{xr}]-\arctan[\sqrt{r}]\right]\right\}_{r=0}^{r\to+\infty}\\ &= \frac{\pi}{1+\sqrt{x}},\end{aligned}$$

$$T_\alpha[v](x)=T_\alpha[u](x)=\frac{\pi}{1+\sqrt{x}}$$

and

$$\begin{aligned}
T_\alpha[uv](x) &= \int_0^{+\infty} \frac{r^\alpha}{h(1,r)} u(rx)v(rx)dr \\
&= \int_0^{+\infty} \frac{1}{\sqrt{r}(1+r)(1+rx)^2} dr \\
&= \left\{ \frac{1}{(x-1)^2} \left[\frac{(x-1)x\sqrt{r}}{1+rx} + 2\arctan[\sqrt{r}] + (x-3)\sqrt{x}\arctan[\sqrt{xr}] \right] \right\}_{r=0}^{r\to+\infty} \\
&= \frac{[2+\sqrt{x}]\,\pi}{2\,[1+\sqrt{x}]^2}.
\end{aligned}$$

It follows from Proposition 3.1 that

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}\left[1+\frac{y-x}{(1+x)^2(1+y)}\right]dxdy &= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy \\
&\leq \sqrt{\int_0^{+\infty} x^{1-\lambda}\left\{[1+u(x)v(x)]c_\alpha - u(x)T_\alpha[v](x)\right\} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} x^{1-\lambda}\left\{c_\alpha + T_\alpha[uv](x) - v(x)T_\alpha[u](x)\right\} f^2(x)dx} \\
&= \sqrt{\int_0^{+\infty} \left\{\left[1+\frac{1}{(1+x)^2}\right]\pi - \frac{\pi}{(1+x)[1+\sqrt{x}]}\right\} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} \left\{\pi + \frac{[2+\sqrt{x}]\,\pi}{2\,[1+\sqrt{x}]^2} - \frac{\pi}{(1+x)[1+\sqrt{x}]}\right\} f^2(x)dx} \\
&= \pi\sqrt{\int_0^{+\infty} \frac{x^{5/2}+2x^{3/2}+x^2+x+2\sqrt{x}+1}{[1+\sqrt{x}](1+x)^2} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} \frac{5x^{3/2}+2x^2+6x+3\sqrt{x}+2}{2[1+\sqrt{x}]^2(1+x)} f^2(x)dx.}
\end{aligned}$$

Also, since u and v are both decreasing, based on Equation (3.4), we have

$$\begin{aligned}
\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}dxdy &\leq \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{x+y}\left[1+\frac{y-x}{(1+x)^2(1+y)}\right]dxdy \\
&\leq \pi\sqrt{\int_0^{+\infty} \frac{x^{5/2}+2x^{3/2}+x^2+x+2\sqrt{x}+1}{[1+\sqrt{x}](1+x)^2} f^2(x)dx} \\
&\times \sqrt{\int_0^{+\infty} \frac{5x^{3/2}+2x^2+6x+3\sqrt{x}+2}{2[1+\sqrt{x}]^2(1+x)} f^2(x)dx.}
\end{aligned}$$

Let us verify these inequalities with a numerical example. Considering $f(x) = e^{-x}$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{e^{-x-y}}{x+y}dxdy = 1,$$

$$\int_0^{+\infty}\int_0^{+\infty} \frac{e^{-x-y}}{x+y}\left[1+\frac{y-x}{(1+x)^2(1+y)}\right]dxdy \approx 1.02897$$

$$\int_0^{+\infty} \frac{x^{5/2}+2x^{3/2}+x^2+x+2\sqrt{x}+1}{[1+\sqrt{x}](1+x)^2} e^{-2x}dx \approx 0.535435,$$

$$\int_0^{+\infty} \frac{5x^{3/2}+2x^2+6x+3\sqrt{x}+2}{2[1+\sqrt{x}]^2(1+x)} e^{-2x}dx \approx 0.523919,$$

and we check that $1 \leq 1.02897 \leq \pi\sqrt{0.535435}\sqrt{0.523919} \approx 1.66393$. So many more examples can be formulated on a similar basis of analysis.

4. Conclusion

In this article, we have established two new integral inequalities that extend some key results in [17, 16]. Both are centered on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy$$

where the novelty lies in the general definition of F of the following form: $F(x,y)=1+u(x)[v(x)-v(y)]$. The first result is adaptable and gives lower and upper bounds for this double integral. The second result is related to the setting of the Hilbert integral inequality, where some new upper bounds are obtained involving weighted L_2 norms of f. The perspectives of our results make them important in several mathematical areas where challenging double integrals (involving certain product and ratio functions) need to be bounded in order to draw conclusions.

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