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Research Article

On oscillatory first order nonautonomous functional difference systems

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Abstract

In this work, an illustrative discussion has been made on sufficient conditions under which all vector solutions of first order 2-dim nonautonomous neutral delay difference systems of the form

$$\Delta \begin{bmatrix} u(\theta)+b(\theta)u(\theta-\kappa) \\ v(\theta)+b(\theta)v(\theta-\kappa) \end{bmatrix} = \begin{bmatrix} a_1(\theta) & a_2(\theta) \\ a_3(\theta) & a_4(\theta) \end{bmatrix} \begin{bmatrix} g_1(u(\theta-\gamma)) \\ g_2(v(\theta-\eta)) \end{bmatrix} + \begin{bmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{bmatrix}, \theta \geq \rho$$

are oscillatory, where $\kappa > 0$, $\gamma \geq 0, \eta \geq 0$ are integers, $a_j(\theta), j = 1,2,3,4, b(\theta), \varphi_1(\theta)$, $\varphi_2(\theta)$ are sequences of real numbers for $\theta \in \mathbb{N}(\theta_0)$ and $g_1, g_2 \in \mathcal{C}(\mathbb{R},\mathbb{R})$ are nondecreasing with the properties $\phi g_1(\phi) > 0, \psi g_2(\psi) > 0$ for $\phi \neq 0, \psi \neq 0$. We verify our results with the examples.

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1. Introduction

In this piece of writing, we have explored the oscillatory tendencies of vector solutions of 2-dim first order nonlinear nonautonomous neutral delay difference systems of the form:

$$(NS1) \qquad \Delta \begin{bmatrix} u(\theta)+b(\theta)u(\theta-\kappa) \\ v(\theta)+b(\theta)v(\theta-\kappa) \end{bmatrix} = \begin{bmatrix} a_1(\theta) & a_2(\theta) \\ a_3(\theta) & a_4(\theta) \end{bmatrix} \begin{bmatrix} g_1(u(\theta-\gamma)) \\ g_2(v(\theta-\eta)) \end{bmatrix} + \begin{bmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{bmatrix},$$

where $\theta \geq \rho$ and $\rho = \max\{\kappa,\gamma,\eta\}$, $\kappa > 0, \gamma \geq 0, \eta \geq 0$ are integers, $a_1(\theta), a_2(\theta), a_3(\theta), a_4(\theta)$, $b(\theta), \varphi_1(\theta)$, $\varphi_2(\theta)$ are real-valued functions, and $g_1, g_2 \in \mathcal{C}(\mathbb{R},\mathbb{R})$ are non-decreasing functions with the properties $\phi g_1(\phi) > 0$ for $\phi \neq 0, \psi g_2(\psi) > 0$ for $\psi \neq 0$. The objective of this study is to shed light on the discussion pertaining to sufficient conditions for the oscillation of all vector solutions $U(\theta) = [u(\theta), v(\theta)]^T$ of $(NS1)$ for various ranges of the neutral coefficient $b(\theta)$ under a suitable choice of the forcing vector.

The impetus for the current study originates from an earlier work [16], where Tripathy investigated the oscillation criteria for two-dimensional linear neutral delay difference systems of the form:

$$(NS2) \qquad \Delta \begin{bmatrix} u(\theta)-b(\theta)u(\theta-\kappa) \\ v(\theta)-b(\theta)v(\theta-\kappa) \end{bmatrix} = \begin{bmatrix} a_1(\theta) & a_2(\theta) \\ a_3(\theta) & a_4(\theta) \end{bmatrix} \begin{bmatrix} u(\theta-\gamma) \\ v(\theta-\eta) \end{bmatrix}, \theta \geq \rho.$$

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The system $(NS2)$ affirms the necessary and sufficient conditions under which all bounded vector solutions of $(NS2)$ either oscillate or converge to zero as $\theta \to \infty$. Secondly, we are motivated by the population model described in [4], where $u(\theta)$ and $v(\theta)$ are the population size of matured males and females and $u(\theta-\kappa)$ and $v(\theta-\kappa)$ are the unmatured males and females size at time θ and $(\theta-\kappa)$ respectively. After κ period, the growth of the population size depends on matured males and females along with the converted unmatured males and females which can be modeled as $(NS1)$.

Motivated by the work of [16], Tripathy and Das in [17] have discussed the necessary and sufficient conditions for oscillation of 2-dim nonlinear neutral delay difference systems of the form:

$$(NS3) \qquad \Delta \begin{bmatrix} u(\theta) - b(\theta)u(\theta-\kappa) \\ v(\theta) - b(\theta)v(\theta-\kappa) \end{bmatrix} = \begin{bmatrix} a_1(\theta) & a_2(\theta) \\ a_3(\theta) & a_4(\theta) \end{bmatrix} \begin{bmatrix} g_1(u(\theta-\gamma)) \\ g_2(v(\theta-\eta)) \end{bmatrix}, \theta \geq \rho.$$

To simplify the problems raised between the works [16] and [17], the present authors have discussed the neutral autonomous system

$$(NS4) \qquad \Delta \begin{bmatrix} u(\theta) - bu(\theta-\kappa) \\ v(\theta) - bv(\theta-\kappa) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} u(\theta-\gamma) \\ v(\theta-\eta) \end{bmatrix}, \theta \geq \rho,$$

where $a_1, a_2, a_3, a_4, b \in \mathbb{R}$, $b > 1, \gamma, \eta \in \mathbb{N}$ in [18].

In [19], the authors have studied the oscillatory behaviour of all vector solutions of two-dimensional nonlinear neutral delay difference system $(NS3)$. Here, we are interested in discussing the same for the system $(NS1)$ which is the nonhomogeneous counterpart of $(NS3)$. $(NS1)$ has been studied to establish sufficient conditions for the oscillation of all vector solutions in the presence of a suitable forcing vector. However, we take into account of the work [19] for the existence of a nonoscillatory vector solution. Regarding difference equations and system of difference equations, we refer to monographs by Agarwal et al. [1–3] and by Elyadi [7]. On the qualitative behaviour of vector solutions of a system of difference equations, we refer to some of the works [5, 6, 8–18] in the literature.

Definition 1.1. By a solution of $(NS1)$ we mean a vector $U(\theta) = [u(\theta), v(\theta)]^T$ which satisfies $(NS1)$ for $\theta \in \mathbb{N}(-\rho) = \{-\rho, -\rho+1, ...0, 1, 2, ...\}$. We say that the solution $U(\theta)$ oscillates componentwise or simply oscillates or strongly oscillates, if each component oscillates. Otherwise, the solution $U(\theta)$ is called non-oscillatory. Therefore, a solution of $(NS1)$ is non-oscillatory, if it has a component which is eventually positive or eventually negative and strongly non-oscillatory if both components of $U(\theta)$ are non-oscillatory. A vector solution $U(\theta)$ of $(NS1)$ has the property oscillates, if each component of $U(\theta)$ has the property.

2. Some oscillation criteria

In this section, sufficient conditions for the oscillation of all vector solutions of the system $(NS1)$ are established. We need the following assumptions for our use in the sequel:

(A_1) $\sum_{\theta=0}^{\infty} a_2(\theta) < \infty$, $\sum_{n=0}^{\infty} a_3(\theta) < \infty$;

(A_2) $\sum_{\theta=0}^{\infty} a_1(\theta) = -\infty$, $\sum_{\theta=0}^{\infty} a_4(\theta) = -\infty$;

(A_3) There exists sequences $H_1(\theta)$, $H_2(\theta)$ of real numbers such that $H_1(\theta)$ changes the sign with $\liminf_{\theta\to\infty} H_1(\theta) = \sigma_1$, and $\limsup_{\theta\to\infty} H_1(\theta) = \tau_1$, whereas $H_2(\theta)$ changes the sign with $\liminf_{\theta\to\infty} H_2(\theta) = \sigma_2$ and $\limsup_{\theta\to\infty} H_2(\theta) = \tau_2$, where $-\infty < \sigma_1 < 0$, $-\infty < \sigma_2 < 0$, $0 < \tau_1 < \infty$, $0 < \tau_2 < \infty$, and $\Delta H_1(\theta) = \varphi_1(\theta)$, $\Delta H_2(\theta) = \varphi_2(\theta)$;

(A_4) There exists a subsequences $\{\theta_j\} \subset \{\theta\}$ such that
$\sum_{j=0}^{\infty} a_1(\theta_j) = -\infty, \quad \sum_{j=0}^{\infty} a_4(\theta_j) = -\infty;$

(A_5) $\liminf_{\theta\to\infty} H_1(\theta) = -\infty, \quad \limsup_{\theta\to\infty} H_1(\theta) = +\infty,$
$\liminf_{\theta\to\infty} H_2(\theta) = -\infty, \quad \limsup_{\theta\to\infty} H_2(\theta) = +\infty,$
where $\Delta H_1(\theta) = \varphi_1(\theta)$ and $\Delta H_2(\theta) = \varphi_2(\theta)$;

(A_6) $g_1, g_2 \in \mathcal{BC}(\mathbb{R}, \mathbb{R})$;

(A_7) There exists an ϵ with $0 < \epsilon < 1$ such that
$\sum_{\theta=\gamma}^{\infty} a_1(\theta) g_1(H_1^+(\theta-\gamma)-\epsilon) = -\infty = \sum_{\theta=\gamma}^{\infty} a_1(\theta) g_1(H_1^-(\theta+\kappa-\gamma)),$
$\sum_{\theta=\eta}^{\infty} a_2(\theta) g_2(H_2^+(\theta-\eta)-\epsilon) = \infty = \sum_{\theta=\eta}^{\infty} a_2(\theta) g_2(H_2^-(\theta+\kappa-\eta)),$
$\sum_{\theta=\gamma}^{\infty} a_3(\theta) g_1(H_1^+(\theta-\gamma)-\epsilon) = \infty = \sum_{\theta=\gamma}^{\infty} a_3(\theta) g_1(H_1^-(\theta+\kappa-\gamma)),$
$\sum_{\theta=\eta}^{\infty} a_4(\theta) g_2(H_2^+(\theta-\eta)-\epsilon) = -\infty = \sum_{\theta=\eta}^{\infty} a_4(\theta) g_2(H_2^-(\theta+\kappa-\eta));$

(A_8) $\sum_{\theta=\gamma}^{\infty} a_1(\theta) g_1(-H_1^+(\theta+\kappa-\gamma)) = \infty = \sum_{\theta=\gamma}^{\infty} a_1(\theta) g_1(-H_1^-(\theta-\gamma)-\epsilon),$
$\sum_{\theta=\eta}^{\infty} a_2(\theta) g_2(-H_2^+(\theta+\kappa-\eta)) = -\infty = \sum_{\theta=\eta}^{\infty} a_2(\theta) g_2(-H_2^-(\theta-\gamma)-\epsilon),$
$\sum_{\theta=\gamma}^{\infty} a_3(\theta) g_1(-H_1^+(\theta+\kappa-\gamma)) = -\infty = \sum_{\theta=\gamma}^{\infty} a_3(\theta) g_1(-H_1^-(\theta-\gamma)-\epsilon),$
$\sum_{\theta=\eta}^{\infty} a_4(\theta) g_2(-H_2^+(\theta+\kappa-\eta)) = \infty = \sum_{\theta=\eta}^{\infty} a_4(\theta) g_2(-H_2^-(\theta-\eta)-\epsilon),$
where
$H_1^+(\theta) = \max\{H_1(\theta), 0\}$, $H_2^+(\theta) = \max\{H_2(\theta), 0\}$, $H_1^-(\theta) = \max\{-H_1(\theta), 0\}$, $H_2^-(\theta) = \max\{-H_2(\theta), 0\}$.

Theorem 2.1. *Let $0 \le b(\theta) < \infty$. Assume that $a_1(\theta) < 0, a_2(\theta) > 0, a_3(\theta) > 0, a_4(\theta) < 0$ for large θ. If $(A_1), (A_3)$, (A_4) and (A_6) hold, then every vector solution of $(NS1)$ strongly oscillates.*

Proof. If possible, let $U(\theta) = [u(\theta), v(\theta)]^T$ be a strongly nonoscillatory vector solution of $(NS1)$ for any large $\theta \ge \theta_0 > \rho$. Without loss of generality, we undertake the following four possible cases:
Case − 1 : $u(\theta) > 0, u(\theta-\kappa) > 0, u(\theta-\gamma) > 0$ and $v(\theta) > 0, v(\theta-\kappa) > 0, v(\theta-\eta) > 0$ for $\theta \ge \theta_1$.
Case − 2 : $u(\theta) < 0, u(\theta-\kappa) < 0, u(\theta-\gamma) < 0$ and $v(\theta) < 0, v(\theta-\kappa) < 0, v(\theta-\eta) < 0$ for $\theta \ge \theta_1$.
Case − 3 : $u(\theta) > 0, u(\theta-\kappa) > 0, r(\theta-\gamma) > 0$ and $v(\theta) < 0, v(\theta-\kappa) < 0, v(\theta-\eta) < 0$ for $\theta \ge \theta_1$.
Case − 4 : $u(\theta) < 0, u(\theta-\kappa) < 0, u(\theta-\gamma) < 0$ and $v(\theta) > 0, v(\theta-\kappa) > 0, v(\theta-\eta) > 0$ for $\theta \ge \theta_1$.
Considering the system $(NS1)$, we set

$$Q_1(\theta) = \sum_{i=\theta}^{\infty} a_2(i) g_2(v(i-\eta)), \quad Q_2(\theta) = \sum_{i=\theta}^{\infty} a_3(i) g_1(u(i-\gamma));$$
$$r_1(\theta) = u(\theta) + b(\theta)u(\theta-\kappa), \quad r_2(\theta) = v(\theta) + b(\theta)v(\theta-\kappa).$$

If we define

$$M_1(\theta) = r_1(\theta) + Q_1(\theta), \quad M_2(\theta) = r_2(\theta) + Q_2(\theta),$$

then $(NS1)$ reduces to

$$\Delta[M_1(\theta) - H_1(\theta)] = a_1(\theta)g_1(u(\theta-\gamma)) \le 0, \tag{2.1}$$
$$\Delta[M_2(\theta) - H_2(\theta)] = a_4(\theta)g_2(v(\theta-\eta)) \le 0 \tag{2.2}$$

for $\theta \ge \theta_1 > \theta_0$. Indeed, $[M_1(\theta)-H_1(\theta)]$ and $[M_2(\theta)-H_2(\theta)]$ are monotonic for $\theta \ge \theta_2 > \theta_1$. For **Case − 1**, $M_1(\theta) > 0$. If $[M_1(\theta) - H_1(\theta)] < 0$ for $\theta \ge \theta_2$, then $0 < M_1(\theta) < H_1(\theta)$, a contradiction due to (A_3). Ultimately, $[M_1(\theta) - H_1(\theta)] \ge 0$ for $\theta \ge \theta_2$. Therefore, $\lim_{\theta\to\infty}[M_1(\theta) - H_1(\theta)]$ exists and so also, $\lim_{\theta\to\infty} M_1(\theta)$ due to (A_6). Indeed,

$$M_1(\theta) = u(\theta) + b(\theta)u(\theta-\kappa) + Q_1(\theta) \ge u(\theta)$$

implies that we can find $l > 0$ and $\theta_3 > \theta_2$ such that $0 \le u(\theta) \le l$ for $\theta \ge \theta_3$. So, there exists a subsequence $\{\theta_j\} \subset \{\theta\}$ and $L < l$ such that $\liminf_{j\to\infty} u(\theta_j - \gamma) \ge L$. Rewriting (2.1) in $\{\theta_j\}$, we have

$$\Delta[M_1(\theta_j) - H_1(\theta_j)] \le a_1(\theta_j)g_1(L), \ \ \theta_j \ge \theta_3,$$

that is,

$$g_1(L)\sum_{j=1}^{\infty} a_1(\theta_j) \ge \sum_{j=1}^{\infty} \Delta[M_1(\theta_j) - H_1(\theta_j)] \ge -[M_1(\theta_1) - H_1(\theta_1)] > -\infty$$

gives a contradiction to (A_4). The argument is similar for $[M_2(\theta) - H_2(\theta)]$. In **Case − 2**, we set $\lambda(\theta) = -u(\theta)$ and $\chi(\theta) = -v(\theta)$ for $\theta \ge \theta_1$. So, $(NS1)$ can be written as

$$(NS5) \qquad \Delta\begin{bmatrix}\lambda(\theta)+b(\theta)\lambda(\theta-\kappa)\\ \chi(\theta)+b(\theta)\chi(\theta-\kappa)\end{bmatrix} = \begin{bmatrix}a_1(\theta)\ a_2(\theta)\\ a_3(\theta)\ a_4(\theta)\end{bmatrix}\begin{bmatrix}g_1^*(\lambda(\theta-\gamma))\\ g_2^*(\chi(\theta-\eta))\end{bmatrix} + \begin{bmatrix}\varphi_1^*(\theta)\\ \varphi_2^*(\theta)\end{bmatrix}, \theta \ge \rho,$$

where $\varphi_1^*(\theta) = -\varphi_1(\theta), \varphi_2^*(\theta) = -\varphi_2(\theta)$, $g_1^*(\lambda) = -g_1(-\lambda), g_2^*(\chi) = -g_2(-\chi)$. Let $H_1^*(\theta) = -H_1(\theta), H_2^*(\theta) = -H_2(\theta)$. Then $\Delta H_1^*(\theta) = \varphi_1^*(\theta)$ and $\Delta H_2^*(\theta) = \varphi_2^*(\theta)$. So, **Case − 2** is all about similar as **Case − 1**. For **Case − 3**, (2.1) and (2.2) resolve into

$$\Delta[M_1(\theta) - H_1(\theta)] = a_1(\theta)g_1(u(\theta-\gamma)) \le 0, \tag{2.3}$$
$$\Delta[M_2(\theta) - H_2(\theta)] = a_4(\theta)g_2(v(\theta-\eta)) \ge 0 \tag{2.4}$$

for which $[M_1(\theta) - H_1(\theta)]$ and $[M_2(\theta) - H_2(\theta)]$ are monotonic. If $[M_1(\theta) - H_1(\theta)] > 0$, then $\lim_{k\to\infty}[M_1(\theta) - H_1(\theta)]$ exists and we are done in **Case − 1**. Suppose that $-v(\theta) = t(\theta)$ and $H_2^*(\theta) = -H_2(\theta)$ in (2.4), then

$$\Delta[t(\theta) + b(\theta)t(\theta-\kappa) - \sum_{j=\theta}^{\infty} a_3(j)g_1(u(j-\gamma)) - H_2^*(\theta)] = a_4(\theta)g_2^*(t(\theta-\eta))$$

looks very much alike to (2.3) and the argument follows immediately. **Case − 4** is analogous to **Case − 3**. This completes the proof of the theorem. □

Theorem 2.2. *Let* $-1 < \alpha_1 \le b(\theta) \le 0$ *for large* θ. *If all conditions of Theorem 2.1 hold, then every vector solution of* $(NS1)$ *strongly oscillates.*

Proof. On the contrary, we proceed as in Theorem 2.1 and we have four cases. So, we can find $\theta_2 > \theta_1 + \rho$ such that $[M_1(\theta) - H_1(\theta)]$ and $[M_2(\theta) - H_2(\theta)]$ are monotonic for $\theta \ge \theta_2$. For **Case − 1**, if $[M_1(\theta) - H_1(\theta)] > 0$, then $\lim_{k\to\infty}[M_1(\theta) - H_1(\theta)]$ exists and hence $\lim_{\theta\to\infty} M_1(\theta) = \lim_{\theta\to\infty}[Q_1(\theta) + r_1(\theta)]$ exists, that is, $\lim_{\theta\to\infty} r_1(\theta)$ exists. If $M_1(\theta) > 0$, then we have done in Theorem 2.1. If $M_1(\theta) < 0$, then $r_1(\theta) + Q_1(\theta) < 0$. We have $Q_1(\theta) > 0$ that implies $r_1(\theta) < 0$, that is, $u(\theta) + b(\theta)u(\theta-\kappa) < 0$ which implies

$$u(\theta) < -b(\theta)u(\theta-\kappa) < u(\theta-\kappa) < u(\theta-2\kappa) < u(\theta-3\kappa) < \cdots < u(\theta_2) < \infty$$

and hence $r_1(\theta)$ is bounded. Therefore, proceeding as in Theorem 2.1, we get a contradiction to (A_4). Next, we let $[M_1(\theta) - H_1(\theta)] \le 0$ for $\theta \ge \theta_2$. We assert that $u(\theta)$ is bounded.

Suppose that $u(\theta)$ is unbounded. So, there exists a subsequence $\{\delta_j\}$ of $\{\theta\}$ such that $\delta_j \to \infty$ and $u(\delta_j) \to \infty$ as $j \to \infty$ and $u(\delta_j) = \max\{u(\theta) : \theta_2 \leq \theta \leq \delta_j\}$. We can choose δ_j sufficiently large such that $\delta_j - \kappa > \theta_2$ and hence

$$\begin{aligned}[M_1(\delta_j) - H_1(\delta_j)] &= u(\delta_j) + b(\delta_j)u(\delta_j - \kappa) + \sum_{i=\delta_j}^{\infty} a_2(i)g_2(v(i-\eta)) - H_1(\delta_j) \\ &\geq (1+\alpha_1)u(\delta_j - \kappa) + \sum_{i=\delta_j}^{\infty} a_2(i)g_2(v(i-\eta)) - H_1(\delta_j) \\ &\to \infty \ \ as \ \ j \to \infty,\end{aligned}$$

a contradiction. Thus, our assertion holds and consequently, $\lim_{\theta\to\infty}[M_1(\theta) - H_1(\theta)]$ exists and $\lim_{\theta\to\infty} M_1(\theta) = \lim_{\theta\to\infty}[Q_1(\theta) + r_1(\theta)]$ exists. As a result, $\lim_{\theta\to\infty} r_1(\theta)$ exists. Hence, the above conclusion follows.

Case $-$ **2** follows from Theorem 2.1. For **Case** $-$ **3**, we have (2.3) and (2.4) for which $[M_1(\theta) - H_1(\theta)]$ and $[M_2(\theta) - H_2(\theta)]$ are monotonic. The rest of the proof is similar to Theorem 2.1. This completes the proof of the theorem. ☐

Theorem 2.3. *Let* $-\infty < \alpha_2 < b(\theta) < -1$ *for large* θ. *If all conditions of Theorem 2.1 hold, then every vector solution of* (S_1) *strongly oscillates.*

Proof. On the contrary, the proof follows from the proof of Theorem 2.2. But, we have to show that $u(\theta)$ is bounded in each case. If $M_1(\theta) > 0$ for $\theta \geq \theta_2$, then $r_1(\theta) > 0$ which implies $u(\theta) > -b(\theta)u(\theta-\kappa) > b(\theta)u(\theta-\kappa) > u(\theta-2\kappa) > u(\theta-3\kappa) > \cdots > u(\theta_2)$, that is, $\liminf_{\theta\to\infty} u(\theta) > 0$. We claim that $u(\theta)$ is bounded. If not, then there exists a subsequence $\{\theta_j\} \subset \{\theta\}$ such that $u(\theta_j - \gamma) > B$. Hence, (2.1) becomes

$$\Delta[M_1(\theta_j) - H_1(\theta_j)] = a_1(\theta_j)g_1(u(\theta_j - \gamma)).$$

Summing from $j = 1$ to ∞, we get

$$\sum_{j=1}^{\infty} a_1(\theta_j)g_1(u(\theta_j - \gamma)) = \sum_{j=1}^{\infty} \Delta[M_1(\theta_j) - H_1(\theta_j)]$$

which implies

$$g_1(B)\sum_{j=1}^{\infty} a_1(\theta_j) > \sum_{j=1}^{\infty} \Delta[M_1(\theta_j) - H_1(\theta_j)],$$

that is,

$$\sum_{j=1}^{\infty} a_1(\theta_j) > \frac{\sum_{j=1}^{\infty} \Delta[M_1(\theta_j) - H_1(\theta_j)]}{g_1(B)} > -\infty,$$

which is a contradiction to (A_4). Hence, $M_1(\theta) < 0$ for $\theta \geq \theta_2$, that is, $r_1(\theta) < 0$ for given $Q_1(\theta) > 0$. If $r_1(\theta) > 0$, then we may also write

$$r_1(\theta) = u(\theta) + b(\theta)u(\theta-\kappa) \geq b(\theta)u(\theta-\kappa) \geq \alpha_2 u(\theta-\kappa)$$

this implies

$$r_1(\theta) \geq \alpha_2 u(\theta-\kappa),$$

that is,

$$u(\theta) \geq \frac{1}{\alpha_2} r_1(\theta+\kappa)$$

and hence $\liminf_{\theta\to\infty} u(\theta) > 0$. Therefore, proceeding as in Theorem 2.1, we get a contradiction to (A_4). Hence, $u(\theta)$ is bounded, which completes the proof of the theorem. ☐

Example 2.4. Consider a 2-dimensional nonlinear nonautonomous neutral delay difference system of the form:

$$(NS6)\ \ \Delta\begin{bmatrix} u(\theta)+(1+e^{1-\theta})u(\theta-2)\\ v(\theta)+(1+e^{1-\theta})v(\theta-2)\end{bmatrix}$$

$$=\begin{bmatrix} -4(2+\frac{e}{2}+\frac{e^{-\theta}}{2}+e^{1-\theta}) & \frac{10}{3}e^{1-\theta}\\ 6e^{-\theta} & -10(4+\frac{e^2}{3}+2e^{-\theta}+e^{1-\theta})\end{bmatrix}\begin{bmatrix}\frac{u(\theta-4)}{1+u^2(\theta-4)}\\ \frac{v(\theta-6)}{1+v^2(\theta-6)}\end{bmatrix}+\begin{bmatrix}(-1)^\theta e\\ (-1)^\theta e^2\end{bmatrix}$$

for $\theta>6$. Clearly, (A_1) and (A_2) hold. If we set $H_1(\theta)=\frac{e}{2}(-1)^{\theta+1}$ and $H_2(\theta)=\frac{e^2}{2}(-1)^{\theta+1}$, then (A_3) is satisfied. By Theorem-2.1, every vector solution of $(NS6)$ is strongly oscillatory. In particular, $U(\theta)=[(-1)^\theta,3(-1)^\theta]^T$ is such a solution of the given system.

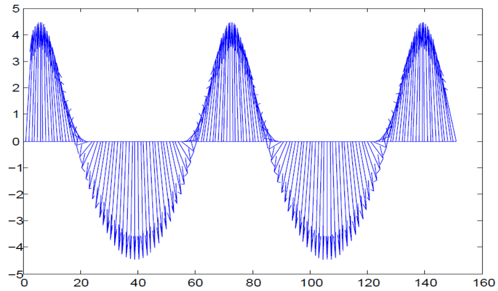

Fig. 1

Remark 2.5. It is observed that, Theorem 2.1 holds if we replace (A_3) by (A_5). However, Theorem 2.2 and Theorem 2.3 do not work for (A_5). Hence, we have the following result without proof:

Theorem 2.6. *Let* $0\le b(\theta)<\infty$. *Assume that* $a_1(\theta)<0, a_2(\theta)>0, a_3(\theta)>0, a_4(\theta)<0$ *for large* θ. *If* $(A_1),(A_4)$, (A_5) *and* (A_6) *hold, then every vector solution of* $(NS1)$ *strongly oscillates.*

Example 2.7. Consider a 2-dimensional nonlinear nonautonomous neutral delay difference system of the form:

$$(NS7)\ \ \Delta\begin{bmatrix} u(\theta)+(-2+e^{-\theta})u(\theta-3)\\ v(\theta)+(-2+e^{-\theta})v(\theta-3)\end{bmatrix}$$

$$=\begin{bmatrix} -10(6+\frac{e^\theta}{3}-e^{-\theta}) & \frac{78}{5}e^{1-\theta}\\ \frac{50}{3}e^{-\theta} & -26(6+\frac{2e^\theta}{5}-e^{1-\theta})\end{bmatrix}\begin{bmatrix}\frac{u(\theta-4)}{1+u^2(\theta-4)}\\ \frac{v(\theta-4)}{1+v^2(\theta-4)}\end{bmatrix}+\begin{bmatrix}(-1)^\theta e^\theta\\ 2(-1)^\theta e^\theta\end{bmatrix}$$

for $\theta>4$. Clearly, (A_1) and (A_2) hold. If we set $H_1(\theta)=(e+1)^{-1}(-1)^{\theta+1}e^\theta$ and $H_2(\theta)=2(e+1)^{-1}(-1)^{\theta+1}e^\theta$, then (A_5) is satisfied. We note that $(NS7)$ has an oscillatory vector solution $U=[3(-1)^\theta,5(-1)^\theta]^T$ aside to the Remark 2.5.

Example 2.8. Consider a 2-dimensional nonlinear nonautonomous neutral delay difference system of the form:

$$(NS8)\ \ \Delta\begin{bmatrix} u(\theta)+(1+e^{-5\theta})u(\theta-2)\\ v(\theta)+(1+e^{-5\theta})v(\theta-2)\end{bmatrix}$$

$$=\begin{bmatrix} a_1(\theta) & a_2(\theta)\\ a_3(\theta) & a_4(\theta)\end{bmatrix}\begin{bmatrix}\frac{u(\theta-2)}{1+u^2(\theta-2)}\\ \frac{v(\theta-2)}{1+v^2(\theta-2)}\end{bmatrix}+\begin{bmatrix}(-1)^{\theta+1}e^\theta\\ 2(-1)^{\theta+1}e^\theta\end{bmatrix}$$

for $\theta > 2$. Here, we take $a_1(\theta) = -(1+e^{2\theta-4})(1+e+e^3+2e^{-5\theta-4}+e^{-5\theta})$, $a_2(\theta) = \frac{1}{2}(1+4e^{2\theta-4})e^{-5\theta-4}$, $a_3(\theta) = 2(1+e^{2\theta-4})e^{-5\theta}$, $a_4(\theta) = -(1+4e^{2\theta-4})(1+e+e^3+e^{-5\theta-4}+2e^{-5\theta})$. Clearly, (A_1) and (A_2) hold. If we set $H_1(\theta) = (e+1)^{-1}(-1)^\theta e^\theta$ and $H_2(\theta) = 2(e+1)^{-1}(-1)^\theta e^\theta$, then (A_5) is satisfied. By Theorem 2.6, every vector solution of $(NS8)$ is strongly oscillatory. In particular, $U = [(-1)^\theta e^\theta, 2(-1)^\theta e^\theta]^T$ is such a solution of the given system.

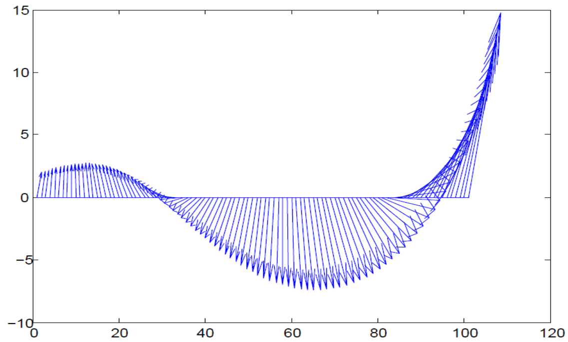

Fig. 2

Theorem 2.9. *Let $-1 \le \beta_1 \le b(\theta) \le 0$. Assume that $(A_3), (A_4), (A_6), (A_7)$ and (A_8) hold. Then every vector solution of $(NS1)$ strongly oscillates.*

Proof. If not, let $[u(\theta), v(\theta)]^T$ be a strongly nonoscillatory vector solution of $(NS1)$. So, we consider the four cases of Theorem 2.1.
Case − 1 : If $u(\theta) > 0, u(\theta-\kappa) > 0, u(\theta-\gamma) > 0$ for $\theta \ge \theta_1$ and $v(\theta) > 0, v(\theta-\kappa) > 0, v(\theta-\eta) > 0$ for $\theta \ge \theta_1$. Setting $Q_1(\theta), Q_2(\theta), r_1(\theta), r_2(\theta)$ as in Theorem 2.1 for $\theta \ge \theta_2 > \theta_1$, let we define $S(\theta) = Q_1(\theta) + r_1(\theta) - H_1(\theta)$, $W(\theta) = Q_2(\theta) + r_2(\theta) - H_2(\theta)$. Therefore, $(NS1)$ can be written as

$$\Delta S(\theta) = a_1(\theta) g_1(u(\theta-\gamma)) \le 0, \tag{2.5}$$
$$\Delta W(\theta) = a_4(\theta) g_2(v(\theta-\eta)) \le 0 \tag{2.6}$$

for $\theta \ge \theta_2$, that is, $\{S(\theta)\}$ and $\{W(\theta)\}$ are monotonically decreasing real valued sequences for $\theta \ge \theta_2$. If $S(\theta) > 0$ for $\theta \ge \theta_2$, then

$$Q_1(\theta) + u(\theta) + b(\theta)u(\theta-\kappa) - H_1(\theta) > 0$$

implies that

$$\begin{aligned} u(\theta) &> H_1(\theta) - Q_1(\theta) - b(\theta)u(\theta-\kappa) \\ &> H_1(\theta) - Q_1(\theta). \end{aligned}$$

Because, $Q_1(\theta) > 0$ and $\Delta Q_1(\theta) < 0$, and then of course $\lim_{\theta\to\infty} Q_1(\theta) = 0$. So, we can find $0 < \epsilon < 1$ and $\theta_3 > \theta_2$ such that $Q_1(\theta) \le \epsilon$ for $\theta \ge \theta_3$. As a result,

$$u(\theta) > H_1(\theta) - \epsilon, \ \theta \ge \theta_3.$$

Now, $u(\theta) + \epsilon > 0$ and $u(\theta) + \epsilon > H_1(\theta)$ for $\theta \ge \theta_3$ implies that

$$u(\theta-\gamma) \ge H_1^+(\theta-\gamma) - \epsilon, \ \theta \ge \theta_4 > \theta_3. \tag{2.7}$$

Using (2.7) and summing (2.5) from θ_4 to $(s_1 - 1)$, we obtain

$$\sum_{\theta=\theta_4}^{s_1-1} a_1(\theta) g_1(u(\theta-\gamma)) = -S(\theta_4) + S(s_1) \ge -S(\theta_4)$$

and therefore,

$$\sum_{\theta=\theta_4}^{\infty} a_1(\theta)g_1(H_1^+(\theta-\gamma)-\epsilon) > -\infty,$$

a contradiction to (A_7). Hence, $S(\theta) < 0$ for $\theta \geq \theta_2$, that is,

$$Q_1(\theta) + u(\theta) + b(\theta)u(\theta-\kappa) < H_1(\theta).$$

Therefore,

$$\begin{aligned} u(\theta) &< H_1(\theta) - Q_1(\theta) - b(\theta)u(\theta-\kappa) \\ &< H_1(\theta) - Q_1(\theta) + u(\theta-\kappa) \end{aligned}$$

implies that

$$\begin{aligned} u(\theta) &> u(\theta+\kappa) - H_1(\theta+\kappa) + Q_1(\theta+\kappa) \\ &\geq -H_1(\theta+\kappa) \end{aligned}$$

for $\theta \geq \theta_3 > \theta_2$. So, we can find $\theta_4 > \theta_3$ such that

$$u(\theta-\gamma) \geq H_1^-(\theta+\kappa-\gamma), \ \theta \geq \theta_4. \tag{2.8}$$

Let $\lim_{\theta\to\infty} S(\theta) = -\infty$, $-\infty \leq \mu < 0$. If $\mu = -\infty$, then $\lim_{\theta\to\infty}[u(\theta) + b(\theta)u(\theta-\kappa)] = -\infty$. Hence, we obtain $u(\theta) + b(\theta)u(\theta-\kappa) < 0$, that is,

$$u(\theta) < -b(\theta)u(\theta-\kappa) \leq u(\theta-\kappa) \leq u(\theta-2\kappa) \leq \cdots \leq u(\theta_4) < \infty$$

shows that $u(\theta)$ is bounded, a contradiction to our supposition. Now, using (2.8) and summing (2.5) from θ_4 to (s_1-1), we obtain

$$\sum_{\theta=\theta_4}^{s_1-1} a_1(\theta)g_1(u(\theta-\gamma)) = -S(\theta_4) + S(s_1) \geq S(s_1)$$

for which

$$\sum_{\theta=\theta_4}^{\infty} a_1(\theta)g_1(H_1^-(\theta+\kappa-\gamma)) > -\infty,$$

a contradiction to (A_7). The above argument can be seen for (2.6) as well.

Case − 2 : If we take $u(\theta) < 0, u(\theta-\kappa) < 0, u(\theta-\gamma) < 0$ for $\theta \geq \theta_1$ and $v(\theta) < 0, v(\theta-\kappa) < 0, v(\theta-\eta) < 0$ for $\theta \geq \theta_1$, and then proceeding as in Theorem 2.1, we can find the same system as $(NS5)$. Here, we note that

$$\begin{aligned} &\liminf_{\theta\to\infty} H_1^*(\theta) = -\limsup_{\theta\to\infty} H_1(\theta) = -\tau_1, \ \limsup_{\theta\to\infty} H_1^*(\theta) = -\liminf_{\theta\to\infty} H_1(\theta) = -\sigma_1, \\ &\liminf_{\theta\to\infty} H_2^*(\theta) = -\limsup_{\theta\to\infty} H_1(\theta) = -\tau_2, \ \limsup_{\theta\to\infty} H_2^*(\theta) = -\liminf_{\theta\to\infty} H_2(\theta) = -\sigma_2, \end{aligned}$$

and also,

$$\sum_{\theta=\gamma}^{\infty} a_1(\theta)g_1^*({H_1^*}^+(\theta-\gamma)-\epsilon) = -\sum_{\theta=\gamma}^{\infty} a_1(\theta)g_1(-H_1^-(\theta-\gamma)-\epsilon) = -\infty,$$

$$\sum_{\theta=\gamma}^{\infty} a_1(\theta)g_1^*({H_1^*}^-(\theta+\kappa-\gamma)) = -\sum_{\theta=\gamma}^{\infty} a_1(\theta)g_1(-H_1^+(\theta+\kappa-\gamma)) = -\infty,$$

$$\sum_{\theta=\eta}^{\infty} a_2(\theta)g_2^*({H_2^*}^+(\theta-\eta)-\epsilon) = -\sum_{\theta=\eta}^{\infty} a_2(\theta)g_2(-H_2^-(\theta-\eta)-\epsilon) = \infty,$$

$$\sum_{\theta=\eta}^{\infty} a_2(\theta)g_2^*({H_2^*}^-(\theta+\kappa-\eta)) = -\sum_{\theta=\eta}^{\infty} a_2(\theta)g_2(-H_2^+(\theta+\kappa-\eta)) = \infty,$$

$$\sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1^*(H_1^{*^+}(\theta-\gamma)-\epsilon) = -\sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1(-H_1^-(\theta-\gamma)-\epsilon) = \infty,$$
$$\sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1^*(H_1^{*^-}(\theta+\kappa-\gamma)) = -\sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1(-H_1^+(\theta+\kappa-\gamma)) = \infty,$$
$$\sum_{\theta=\eta}^{\infty} a_4(\theta)g_2^*(H_2^{*^+}(\theta-\eta)-\epsilon) = -\sum_{\theta=\eta}^{\infty} a_4(\theta)g_2(-H_2^-(\theta-\eta)-\epsilon) = -\infty,$$
$$\sum_{\theta=\eta}^{\infty} a_4(\theta)g_2^*(H_2^{*^-}(\theta+\kappa-\eta)) = -\sum_{\theta=\eta}^{\infty} a_4(\theta)g_2(-H_2^+(\theta+\kappa-\eta)) - \infty.$$

Hence, proceeding as in **Case – 1**, we get a contradiction to (A_8).
Case – 3 : In this case, (2.5) and (2.6) are of the form

$$\Delta S(\theta) = a_1(\theta)g_1(u(\theta-\gamma)) \leq 0, \tag{2.9}$$
$$\Delta W(\theta) = a_4(\theta)g_2(v(\theta-\eta)) \geq 0. \tag{2.10}$$

If we put $-v(\theta) = l(\theta)$ and $H_2^*(\theta) = -H_2(\theta)$ in (2.10), we get

$$\Delta[l(\theta) + b(\theta)l(\theta-\kappa) - \sum_{i=\theta}^{\infty} a_3(i)g_1(u(i-\gamma)) - H_2^*(\theta)] = a_4(\theta)g_2(l(\theta-\eta)),$$

that is,

$$\Delta W^*(\theta) = a_4(\theta)g_2(l(\theta-\eta)) \leq 0 \tag{2.11}$$

which is similar to (2.9) and all we have discussed in **Case – 1**. **Case – 4** is similar to **Case – 3**. Hence, the details are omitted. □

Theorem 2.10. *Let $0 \leq b(\theta) \leq \beta_2 < \infty$. Assume that $(A_3), (A_7), (A_8)$*

(A_9) *there exists $\delta_1 > 0, \delta_2 > 0$ such that*
$g_1(u) + g_1(v) \geq \delta_1 g_1(u+v),\ g_2(u) + g_2(v) \geq \delta_2 g_2(u+v),\ u, v \in \mathbb{R}^+$, *and*

(A_{10}) $g_1(uv) = g_1(u)g_1(v),\ g_2(uv) = g_2(u)g_2(v),\ u, v \in \mathbb{R}$

hold. Furthermore, assume that one of following conditions:

(A_{11}) $\sum_{\theta=\gamma}^{\infty} a_1^*(\theta)g_1(H_1^+(\theta-\gamma)-\epsilon) = -\infty = \sum_{\theta=\gamma}^{\infty} a_1^*(\theta)g_1(H_1^-(\theta-\gamma))$,
$\sum_{\theta=\eta}^{\infty} a_2^*(\theta)g_2(H_2^+(\theta-\eta)-\epsilon) = \infty = \sum_{\theta=\eta}^{\infty} a_2^*(\theta)g_2(H_2^-(\theta-\eta))$,
$\sum_{\theta=\gamma}^{\infty} a_3^*(\theta)g_1(H_1^+(\theta-\gamma)-\epsilon) = \infty = \sum_{\theta=\gamma}^{\infty} a_3^*(\theta)g_1(H_1^-(\theta-\gamma))$,
$\sum_{\theta=\eta}^{\infty} a_4^*(\theta)g_2(H_2^+(\theta-\eta)-\epsilon) = -\infty = \sum_{\theta=\eta}^{\infty} a_4^*(\theta)g_2(H_2^-(\theta-\eta))$;

(A_{12}) $\sum_{\theta=\gamma}^{\infty} a_1^*(\theta)g_1(-H_1^+(\theta+\kappa-\gamma)) = \infty = \sum_{\theta=\gamma}^{\infty} a_1^*(\theta)g_1(-H_1^-(\theta-\gamma)-\epsilon)$,
$\sum_{\theta=\eta}^{\infty} a_2^*(\theta)g_2(-H_2^+(\theta+\kappa-\eta)) = -\infty = \sum_{\theta=\eta}^{\infty} a_2^*(\theta)g_2(-H_2^-(\theta-\gamma)-\epsilon)$,
$\sum_{\theta=\gamma}^{\infty} a_3^*(\theta)g_1(-H_1^+(\theta+\kappa-\gamma)) = -\infty = \sum_{\theta=\gamma}^{\infty} a_3^*(\theta)g_1(-H_1^-(\theta-\gamma)-\epsilon)$,
$\sum_{\theta=\eta}^{\infty} a_4^*(\theta)g_2(-H_2^+(\theta+\kappa-\eta)) = \infty = \sum_{\theta=\eta}^{\infty} a_4^*(\theta)g_2(-H_2^-(\theta-\eta)-\epsilon)$,

where
$a_1^*(\theta) = \min\{a_1(\theta), a_1(\theta-\kappa)\}, a_2^*(\theta) = \min\{a_2(\theta), a_2(\theta-\kappa)\}, a_3^*(\theta) = \min\{a_3(\theta), a_3(\theta-\kappa)\}, a_4^*(\theta) = \min\{a_4(\theta), a_4(\theta-\kappa)\}$, *and $H_1^+, H_1^-, H_2^+, H_2^-$ are defined in (A_8)*

hold. Then every vector solution of $(NS1)$ oscillates.

Proof. On the contrary, let $[u(\theta), v(\theta)]^T$ be a strongly nonoscillatory vector solution of $(NS1)$. Proceeding as in Theorem 2.9, it follows that $S(\theta)$ and $W(\theta)$ are monotonically decreasing real valued functions for $\theta \geq \theta_1$. If $S(\theta) < 0$ for $\theta \geq \theta_2$, then $Q_1(\theta) + r_1(\theta) - H_1(\theta) < 0$ implies $r_1(\theta) - H_1(\theta) < 0$. As a result, $0 < r_1(\theta) < H_1(\theta)$ shows that $\liminf_{\theta\to\infty} H_1(\theta) > 0$, a contradiction. Ultimately, $S(\theta) > 0$ for $\theta \geq \theta_2$, that is, $Q_1(\theta) + r_1(\theta) - H_1(\theta) > 0$. Therefore, $r_1(\theta) > H_1(\theta) - Q_1(\theta)$ and hence $r_1(\theta) > H_1(\theta) - \epsilon$ for $\theta \geq \theta_3 > \theta_2$. Consequently, $r_1(\theta) + \epsilon > H_1(\theta)$ implies that $r_1(\theta) + \epsilon \geq \max\{H_1(\theta), 0\}$ for $\theta \geq \theta_4 > \theta_3$. So, we have

$$r_1(\theta) > H_1^+(\theta) - \epsilon, \; \theta \geq \theta_4. \tag{2.12}$$

If we look at $(NS1)$, then

$$\Delta S(\theta) - a_1(\theta) g_1(u(\theta - \gamma)) = 0 \tag{2.13}$$

which can also be written as

$$\Delta S(\theta - \kappa) - a_1(\theta - \kappa) g_1(u(\theta - \kappa - \gamma)) = 0. \tag{2.14}$$

Clearly, (2.14) and (2.13) can be embedded in the following nonlinear equation

$$\Delta S(\theta) - a_1(\theta) g_1(u(\theta - \gamma)) + g_1(\beta_2)\Big[\Delta S(\theta - \kappa) - a_1(\theta - \kappa) g_1(u(\theta - \kappa - \gamma))\Big] = 0.$$

According to our hypothesis, we find

$$\Delta[S(\theta) + g_1(\beta_2) S(\theta - \kappa)] \leq a_1^*(\theta)\Big[g_1(u(\theta - \gamma)) + g_1(\beta_2 u(\theta - \kappa - \gamma))\Big]$$

for $\theta \geq \theta_5 > \theta_4$. Again by (A_9),

$$\begin{aligned}\Delta[S(\theta) + g_1(\beta_2) S(\theta - \kappa)] &\leq \delta_1 a_1^*(\theta)[g_1\{u(\theta - \gamma) + \beta_2 u(\theta - \kappa - \gamma)\}] \\ &\leq \delta_1 a_1^*(\theta)\,[g_1(r_1(\theta - \gamma))]\,,\end{aligned} \tag{2.15}$$

where $r_1(\theta - \gamma) \leq u(\theta - \gamma) + \beta_2 u(\theta - \kappa - \gamma)$ for $\theta \geq \theta_5$. Summing (2.15) from θ_5 to $(\theta - 1)$, we get

$$-\delta_1\Big[\sum_{s=\theta_5}^{\theta-1} a_1^*(s) g_1(r_1(s - \gamma))\Big] \leq S(\theta_5) - S(\theta) + g_1(\beta_2) S(\theta_5 - \kappa) - g_1(\beta_2) S(\theta - \kappa).$$

Using (2.12), it follows that

$$-\delta_1\Big[\sum_{s=\theta_5}^{\infty} a_1^*(s) g_1(H_1^+(s - \gamma) - \epsilon)\Big] \leq S(\theta_5) + g_1(\beta_2) S(\theta_5 - \kappa) < \infty$$

which is a contradiction to (A_{11}). The above argument we mean for $W(\theta)$ as well. The rest of the proof analogous to Theorem 2.9. Hence, the theorem is proved. □

Theorem 2.11. *Let $-\infty \leq \beta_3 \leq b(\theta) < -1$ for any large θ. If $(A_3), (A_4), (A_7), (A_8)$ and (A_{13}) there exists an ϵ with $0 < \epsilon < 1$ such that*

$$\sum_{\theta=\gamma}^{\infty} a_1(\theta) g_1\Big(\frac{H_1^+(\theta-\gamma)-\epsilon}{-\beta_3}\Big) = -\infty = \sum_{\theta=\gamma}^{\infty} a_1(\theta) g_1\Big(\frac{H_1^-(\theta+\kappa-\gamma)}{-\beta_3}\Big),$$

$$\sum_{\theta=\eta}^{\infty} a_2(\theta) g_2\Big(\frac{H_2^+(\theta-\eta)-\epsilon}{-\beta_3}\Big) = \infty = \sum_{\theta=\eta}^{\infty} a_2(\theta) g_2\Big(\frac{H_2^-(\theta+\kappa-\eta)}{-\beta_3}\Big),$$

$$\sum_{\theta=\gamma}^{\infty} a_3(\theta) g_1\Big(\frac{H_1^+(\theta-\gamma)-\epsilon}{-\beta_3}\Big) = \infty = \sum_{\theta=\gamma}^{\infty} a_3(\theta) g_1\Big(\frac{H_1^-(\theta+\kappa-\gamma)}{-\beta_3}\Big),$$

$$\sum_{\theta=\eta}^{\infty} a_4(\theta) g_2\Big(\frac{H_2^+(\theta-\eta)-\epsilon}{-\beta_3}\Big) = -\infty = \sum_{\theta=\eta}^{\infty} a_4(\theta) g_2\Big(\frac{H_2^-(\theta+\kappa-\eta)}{-\beta_3}\Big);$$

$$(A_{14})\ \sum_{\theta=\gamma}^{\infty} a_1(\theta)g_1\left(\frac{-H_1^+(\theta+\kappa-\gamma)}{-\beta_3}\right) = \infty = \sum_{\theta=\gamma}^{\infty} a_1(\theta)g_1\left(\frac{-H_1^-(\theta-\gamma)-\epsilon}{-\beta_3}\right),$$
$$\sum_{\theta=\eta}^{\infty} a_2(\theta)g_2\left(\frac{-H_2^+(\theta+\kappa-\eta)}{-\beta_3}\right) = -\infty = \sum_{\theta=\eta}^{\infty} a_2(\theta)g_2\left(\frac{-H_2^-(\theta-\gamma)-\epsilon}{-\beta_3}\right),$$
$$\sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1\left(\frac{-H_1^+(\theta+\kappa-\gamma)}{-\beta_3}\right) = -\infty = \sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1\left(\frac{-H_1^-(\theta-\gamma)-\epsilon}{-\beta_3}\right),$$
$$\sum_{\theta=\eta}^{\infty} a_4(\theta)g_2\left(\frac{-H_2^+(\theta+\kappa-\eta)}{-\beta_3}\right) = \infty = \sum_{\theta=\eta}^{\infty} a_4(\theta)g_2\left(\frac{-H_2^-(\theta-\eta)-\epsilon}{-\beta_3}\right)$$

hold, then every bounded vector solution of $(NS1)$ *oscillates.*

Proof. If possible, let $[u(\theta), v(\theta)]^T$ be a bounded strongly nonoscillatory vector solution of $(NS1)$. Proceeding as in Theorem 2.9, it follows that $S(\theta)$ and $W(\theta)$ are monotonically decreasing real valued functions for $\theta \geq \theta_1$ and the case $S(\theta) > 0$ is similar. If $S(\theta) < 0$ for $\theta \geq \theta_1$, then $\lim_{\theta\to\infty} S(\theta)$ exists since $u(\theta)$ is bounded. Equivalently, $S(\theta) < 0$ gives

$$u(\theta) + b(\theta)u(\theta-\kappa) < H_1(\theta),$$

that is,

$$\begin{aligned} u(\theta) &< H_1(\theta) - b(\theta)u(\theta-\kappa) \\ &< H_1(\theta) - \beta_3 u(\theta-\kappa). \end{aligned}$$

As a result,

$$H_1(\theta+\kappa) - \beta_3 u(\theta) > u(\theta+\kappa)$$

implies

$$-\beta_3 u(\theta-\gamma) > -H_1(\theta+\kappa-\gamma), \quad \theta \geq \theta_2 > \theta_1$$

and we have $u(\theta-\gamma) > 0$. Therefore, $u(\theta-\gamma) > \frac{H_1^-(\theta+\kappa-\gamma)}{-\beta_3}$ for $\theta \geq \theta_3 > \theta_2$. Now, we can read (2.1) as

$$\Delta S(\theta) - a_1(\theta)g_1\Big(\frac{H_1^-(\theta+\kappa-\gamma)}{-\beta_3}\Big) \leq 0.$$

Summing the preceding inequality from θ_3 to $(\theta-1)$, we get

$$-\sum_{\theta=\theta_3}^{\theta-1} a_1(\theta)g_1\Big(\frac{H_1^-(\theta+\kappa-\gamma)}{-\beta_3}\Big) \leq -S(\theta) + S(\theta_3) < S(\theta).$$

Therefore,

$$\sum_{\theta=\theta_3}^{\infty} a_1(\theta)g_1\Big(\frac{H_1^-(\theta+\kappa-\gamma)}{-\beta_3}\Big) > -\infty,$$

a contradiction to (A_{13}). Similar analysis holds for $W(\theta)$. The rest of the part is similar to Theorem 2.9. Hence, the details are omitted. This completes the proof of the theorem. □

Example 2.12. Consider a 2-dimensional nonlinear nonautonomous neutral delay difference system of the form:

$$(NS9)\ \ \Delta \begin{bmatrix} u(\theta) - (\frac{1}{3} + \frac{1}{5}(-1)^\theta)u(\theta-1) \\ v(\theta) - (\frac{1}{3} + \frac{1}{5}(-1)^\theta)v(\theta-1) \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{2}{3} & \frac{3}{64} \\ \frac{3}{4} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} u^3(\theta-2) \\ v^3(\theta-2) \end{bmatrix} + \begin{bmatrix} 3(-1)^{\theta+1} \\ 6(-1)^{\theta+1} \end{bmatrix}$$

for $\theta > 2$. Clearly, $(A_3), (A_4), (A_6), (A_7)$ and (A_8) are satisfied for $(NS9)$ and if we set $H_1(\theta) = \frac{3}{2}(-1)^\theta$ and $H_2(\theta) = 3(-1)^\theta$, then

$$\Delta\ \begin{bmatrix} H_1(\theta) \\ H_2(\theta) \end{bmatrix} = \begin{bmatrix} 3(-1)^{\theta+1} \\ 6(-1)^{\theta+1} \end{bmatrix}.$$

Indeed,

$$H_1^+(\theta)=\begin{cases}\frac{3}{2},\ \theta\ even\\ 0\ ,\theta\ odd,\end{cases}\qquad H_1^-(\theta)=\begin{cases}0,\ \theta\ even\\ \frac{3}{2},\ \theta\ odd,\end{cases}$$

$$H_2^+(\theta)=\begin{cases}3,\ \theta\ even\\ 0\ ,\theta\ odd,\end{cases}\qquad H_2^-(\theta)=\begin{cases}0,\ \theta\ even\\ 3,\ \theta\ odd\end{cases}\quad .$$

Choose $\epsilon=\frac{1}{2}$. Therefore,

$$\begin{aligned}\sum_{\theta=\gamma}^{\infty}a_1(\theta)g_1(H_1^+(\theta-\gamma)-\epsilon) &= -\frac{2}{3}\sum_{\theta=\gamma}^{\infty}g_1(H_1^+(\theta-\gamma)-\epsilon)\\ &= -\frac{2}{3}\sum_{\theta=\gamma}^{\infty}[(H_1^+(\theta-\gamma)-\epsilon)]^3\\ &= -\frac{2}{3}\sum_{\theta=2}^{\infty}[(H_1^+(\theta-2)-\epsilon)]^3\\ &= -\frac{2}{3}\sum_{2k+2=2}^{\infty}[(H_1^+(2k+2-2)-\epsilon)]^3\\ &= -\frac{2}{3}\sum_{k=0}^{\infty}[(H_1^+(2k)-\epsilon)]^3=-\frac{2}{3}\sum_{k=0}^{\infty}(1)^3=-\infty\end{aligned}$$

and

$$\begin{aligned}\sum_{\theta=\gamma}^{\infty}a_1(\theta)g_1(H_1^-(\theta+\kappa-\gamma)) &= -\frac{2}{3}\sum_{\theta=\gamma}^{\infty}g_1(H_1^-(\theta+\kappa-\gamma))\\ &= -\frac{2}{3}\sum_{\theta=\gamma}^{\infty}[(H_1^-(\theta+\kappa-\gamma))]^3\\ &= -\frac{2}{3}\sum_{\theta=2}^{\infty}[(H_1^-(\theta-1))]^3\\ &= -\frac{2}{3}\sum_{2k+2=2}^{\infty}[(H_1^-(2k+2-1))]^3\\ &= -\frac{2}{3}\sum_{k=0}^{\infty}[(H_1^-(2k+1))]^3=-\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^3=-\infty.\end{aligned}$$

Also, we can verify

$$\begin{aligned}\sum_{\theta=\eta}^{\infty}a_2(\theta)g_2(H_2^+(\theta-\eta)-\epsilon)&=\infty=\sum_{\theta=\eta}^{\infty}a_2(\theta)g_2(H_2^-(\theta-\eta))\\ \sum_{\theta=\gamma}^{\infty}a_3(\theta)g_1(H_1^+(\theta-\gamma)-\epsilon)&=\infty=\sum_{\theta=\gamma}^{\infty}a_2(\theta)g_1(H_1^-(\theta-\eta))\\ \sum_{\theta=\eta}^{\infty}a_4(\theta)g_2(H_2^+(\theta-\eta)-\epsilon)&=-\infty=\sum_{\theta=\eta}^{\infty}a_4(\theta)g_2(H_2^-(\theta-\eta)).\end{aligned}$$

Hence, (A_7) is satisfied. Also, we can verified (A_8). By Theorem-2.9, every vector solution of the system $(NS9)$ oscillates. In particular, $U(\theta)=[2(-1)^\theta,4(-1)^\theta]^T$ is such a vector solution of the system.

Example 2.13. Consider a 2-dimensional nonlinear nonautonomous neutral delay difference system of the form:

$$(NS10)\ \Delta \begin{bmatrix} u(\theta) + (e^{-\theta}u(\theta-1)) \\ v(\theta) + (e^{-\theta}v(\theta-1)) \end{bmatrix} = \begin{bmatrix} (e^{-\theta}+e^{-\theta-1}-5) & \frac{8}{3} \\ 6 & (e^{-\theta}+e^{-\theta-1}-\frac{14}{3}) \end{bmatrix} \begin{bmatrix} u^{\frac{1}{3}}(\theta-2) \\ v^{\frac{1}{3}}(\theta-2) \end{bmatrix} + \begin{bmatrix} 2(-1)^{\theta+1} \\ 4(-1)^{\theta+1} \end{bmatrix}$$

for $\theta > 2$. Clearly, $(A_3), (A_7), (A_8), (A_9), (A_{10}), (A_{11})$ and (A_{12}) are satisfied for $(NS10)$ and if we set $H_1(\theta) = (-1)^\theta$ and $H_2(\theta) = 2(-1)^\theta$, then

$$\Delta \begin{bmatrix} H_1(\theta) \\ H_2(\theta) \end{bmatrix} = \begin{bmatrix} 2(-1)^{\theta+1} \\ 4(-1)^{\theta+1} \end{bmatrix}.$$

Indeed,

$$H_1^+(\theta) = \begin{cases} 1, \ \theta \ even \\ 0\ , \theta \ odd, \end{cases} \quad H_1^-(\theta) = \begin{cases} 0, \ \theta \ even \\ 1, \ \theta \ odd, \end{cases}$$

$$H_2^+(\theta) = \begin{cases} 2, \ \theta \ even \\ 0\ , \theta \ odd, \end{cases} \quad H_2^-(\theta) = \begin{cases} 0, \ \theta \ even \\ 2, \ \theta \ odd \end{cases} .$$

Choose $\epsilon = \frac{1}{3} < 1$. Again,

$$\begin{aligned} a_1^*(\theta) &= \min\{a_1(\theta), a_1(\theta-2)\} \\ &= \min\{(e^{-\theta}+e^{-\theta-1}-5), (e^{-\theta+1}+e^{2-\theta}-5)\} = (e^{-\theta}+e^{-\theta-1}-5). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{\theta=\gamma}^{\infty} a_1^*(\theta) g_1(H_1^+(\theta-\gamma)-\epsilon) &= \sum_{\theta=\gamma}^{\infty}(e^{-\theta}+e^{-\theta-1}-5)) g_1(H_1^+(\theta-\gamma)-\epsilon) \\ &= \sum_{\theta=\gamma}^{\infty}(e^{-\theta}+e^{-\theta-1}-5))[(H_1^+(\theta-\gamma)-\epsilon)]^{\frac{1}{3}} \\ &= \sum_{\theta=2}^{\infty}(e^{-\theta}+e^{-\theta-1}-5))[(H_1^+(\theta-2)-\epsilon)]^{\frac{1}{3}} \\ &= \sum_{2k+2=2}^{\infty}(e^{-2k-2}+e^{-2k-3}-5)[(H_1^+(2k+2-2)-\epsilon)]^{\frac{1}{3}} \\ &= \sum_{k=0}^{\infty}(e^{-2k-2}+e^{-2k-3}-5)[(H_1^+(2k)-\epsilon)]^{\frac{1}{3}} \\ &= \left(\frac{2}{3}\right)^{\frac{1}{3}} \sum_{k=0}^{\infty}(e^{-2k-2}+e^{-2k-3}-5) = -\infty \end{aligned}$$

and

$$\begin{aligned} \sum_{\theta=\gamma}^{\infty} a_1^*(\theta) g_1(H_1^-(\theta+\kappa-\gamma)) &= \sum_{\theta=\gamma}^{\infty}(5-e^{-\theta+1}-e^{2-\theta}) g_1(H_1^-(\theta+\kappa-\gamma)) \\ &= \sum_{\theta=\gamma}^{\infty}(5-e^{-\theta+1}-e^{2-\theta})[H_1^-(\theta+\kappa-\gamma)]^{\frac{1}{3}} \\ &= \sum_{\theta=2}^{\infty}(5-e^{-\theta+1}-e^{2-\theta})[H_1^-(\theta-1)]^{\frac{1}{3}} \\ &= \sum_{2k+2=2}^{\infty}(e^{-2k-2}+e^{-2k-3}-5)[H_1^-(2k+2-1)]^{\frac{1}{3}} \end{aligned}$$

$$= \sum_{k=0}^{\infty}(e^{-2k-2} + e^{-2k-3} - 5)[H_1^-(2k+1)]^{\frac{1}{3}}$$
$$= (1)^{\frac{1}{3}} \sum_{k=0}^{\infty}(e^{-2k-2} + e^{-2k-3} - 5) = -\infty.$$

Also, we can find

$$\sum_{\theta=\eta}^{\infty} a_2(\theta)g_2(H_2^+(\theta-\eta)-\epsilon) = \infty = \sum_{\theta=\eta}^{\infty} a_2(\theta)g_2(H_2^-(\theta+\kappa-\eta)),$$
$$\sum_{\theta=\gamma}^{\infty} a_3(\theta)g_1(H_1^+(\theta-\gamma)-\epsilon) = \infty = \sum_{\theta=\gamma}^{\infty} a_2(\theta)g_1(H_1^-(\theta+\kappa-\gamma)),$$
$$\sum_{\theta=\eta}^{\infty} a_4(\theta)g_2(H_2^+(\theta-\eta)-\epsilon) = -\infty = \sum_{\theta=\eta}^{\infty} a_4(\theta)g_2(H_2^-(\theta+\kappa-\eta)).$$

Hence, (A_{11}) is satisfied. Similarly, we can verified (A_{12}). By Theorem-2.10, every vector solution of the system $(NS10)$ oscillates. In particular, $U(\theta) = [2(-1)^\theta, 3(-1)^\theta]^T$ is such a vector solution of the system.

3. Discussion

Remark 3.1. We may note that if $g_1, g_2 \in \mathcal{BC}(\mathbb{R}, \mathbb{R})$, then $g_1(u)$ and $g_2(v)$ could be of the form:

$$\begin{bmatrix} g_1(u) \\ g_2(v) \end{bmatrix} = \begin{bmatrix} \frac{|u| sgnu}{\sigma^2+u^2} \\ \frac{|v| sgnv}{\tau^2+v^2} \end{bmatrix},$$

where $\sigma, \tau \in \mathbb{R}\backslash\{0\}$. However, we could undertake

$$\begin{bmatrix} g_1(u) \\ g_2(v) \end{bmatrix} = \begin{bmatrix} |u|^{\gamma_1} sgnu \\ |v|^{\gamma_2} sgnv \end{bmatrix},$$

where γ_1, γ_2 are the ratio of odd positive integers and g_1, g_2 are sublinear satisfying $g_1(u) + g_1(v) \geq \delta_1 g_1(u+v)$, $g_2(u)+g_2(v) \geq \delta_2 g_2(u+v)$. On the other hand, any g_1 and g_2 satisfying the conditions $g_1(uv) = g_1(u)g_1(v)$, $g_2(uv) = g_2(u)g_2(v)$, $u, v \in \mathbb{R}^+$ may be of the form:

$$\begin{bmatrix} g_1(u) \\ g_2(v) \end{bmatrix} = \begin{bmatrix} |u|^{\gamma_1} sgnu \\ |v|^{\gamma_2} sgnv \end{bmatrix},$$

where γ_1, γ_2 are the ratio of odd positive integers.

Remark 3.2. In this work, we have not gone through the existence of nonoscillatory vector solutions of $(NS1)$. To have a look at the existence result, the analysis is similar to [19]. For completeness, we state one of the results without proof:

Theorem 3.3. *Suppose that* $-1 < \beta_4 \leq b(\theta) \leq 0$ *and* $a_1(\theta) < 0, a_2(\theta) > 0, a_3(\theta) > 0, a_4(\theta) < 0$ *for large value of* θ. *Let* $g_1, g_2 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ *such that* $\phi g_1(\phi) > 0, \phi \neq 0$, $\psi g_2(\psi) > 0, \psi \neq 0$. *Assume that* g_1 *and* g_2 *are Lipschitzian in the interval of the form* $[c, d], -\infty < c < d < \infty$. *If* $(A_1), (A_3)$ *and*

$$(A_{15}) \quad \sum_{\theta=0}^{\infty} a_1(\theta) > -\infty, \quad \sum_{\theta=0}^{\infty} a_4(\theta) > -\infty$$

hold, then $(NS1)$ *admits a bounded strongly nonoscillatory vector solution.*

Remark 3.4. To complete our undertaken problem, we have used different types of nonlinear functions g_1 and g_2 as long as the neutral coefficient $b(\theta)$ is concerned. From Theorem 2.1 to Theorem 2.9, we use (A_6). However, we couldn't use this for Theorem 2.10 and Theorem 2.11 rather than the sublinear condition (A_9) and (A_{10}). But in Theorem 2.11, g_1 and g_2 could be either superlinear or sublinear due to only (A_{10}). So, this work predicts more open problems for future work in this direction.

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