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Dynamical Behavior of Solutions to Higher-Order System of Fuzzy Difference Equations

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Abstract

In this paper, we concentrate on the global behavior of the fuzzy difference equations system with higher order

$$\alpha_{n+1} = \tau_1 + \frac{\alpha_n}{\sum_{i=1}^{m} \beta_{n-i}}, \beta_{n+1} = \tau_2 + \frac{\beta_n}{\sum_{i=1}^{m} \alpha_{n-i}}, \quad n \in \mathbb{N}_0,$$

where α_n, β_n are positive fuzzy number sequences, parameters τ_1, τ_2 and the initial values $\alpha_{-i}, \beta_{-i}, i \in \{0,1,\dots,m\}$, are positive fuzzy numbers. Firstly, we show the existence and uniqueness of the positive fuzzy solution to the mentioned system. Furthermore, we are searching for the boundedness, persistence and convergence of the positive solution to the given system. Finally, we give some numerical examples to show the efficiency of our results.

1. Introduction

Difference equations has many applications in the real world to many areas such as economics, biology, psychology, sociology, computer sciences etc. That's why, much more attention is given to this area. There are many data in our natural world. Collecting and establishing discrete mathematical models to figure out their behaviors is crucial. A discrete dynamical models of systems are generally established by using difference equations approach. These difference equation models can be seen simple. But, it is really important to comprehend the behaviors of their solutions in the cases generating general solution expressions is difficult.

DeVault et al., in [1], showed that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n \in \mathbb{N}_0,$$

where the parameter $A \in (0,\infty)$, converges to a period of two solutions. Later, Abu-Saris et al., in [2], studied the global asymptotic stability of the unique equilibrium point $\tilde{y} = 1 + A$ of the following equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n \in \mathbb{N}_0,$$

where the parameter A and the initial conditions $y_0, y_{-1}, \dots, y_{-k}$, are positive real numbers.

Papaschinopoulos and Schinas, in [3], studied the oscillatory behavior, the boundedness of the solutions and global asymptotic stability of the positive equilibrium point of the difference equations system

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n \in \mathbb{N}_0,$$

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where p,q are positive integers, the parameter A and the initial conditions $x_{-p},\dots,x_0,y_{-q},\dots,y_0$, are positive real numbers. Also, Zhang et al., in [4], investigated the boundedness, persistence, and global asymptotic behavior of positive solution for the rational difference equations system

$$x_{n+1}=A+\frac{x_n}{\sum_{i=1}^{k}y_{n-i}},y_{n+1}=B+\frac{y_n}{\sum_{i=1}^{k}x_{n-i}},\quad n\in\mathbb{N}_0, \tag{1.1}$$

where the parameters A,B and the initial conditions $x_{-i},y_{-i},i\in\{0,1,\dots,k\}$, are positive real numbers. For more information, see [5, 6, 7]. Further studies about difference equations or difference equation systems can be found in [1, 2, 3, 8, 9, 10, 11, 12, 13] and references therein.

Fuzzy set theory is a mathematical paradigm that deals with sets with indefinite or uncertain bounds. It provides for partial membership of an element in a set. This ambiguous or uncertain data idea is important in modern analytics. The data is frequently incomplete, unclear, or subject to change. Fuzzy set theory allows analysts to model and manipulate such data effectively. It leads to more educated decision-making and improved analytics outcomes.

Zadeh, in [14], introduced the concept of fuzzy sets as a technique of dealing with unclear or imprecise data in engineering and computer science in 1965. Since then, the fuzzy set theory has grown significantly, and its applications have spread across a variety of disciplines such as decision-making, pattern recognition, image processing, natural language processing, and control systems. There are more information about fuzzy set theory at [7, 15, 16, 17].

Deeba et al., in [18], studied fuzzy analog of the first order difference equation

$$x_{n+1}=wx_n+q,\quad n\in\mathbb{N}_0,$$

where x_n is a fuzzy number sequence and the initials w,q,x_0 are fuzzy numbers. Deeba and Korvin [19] considered a model

$$C_{n+1}=C_n-c_1C_{n-1}+c_2,\quad n\in\mathbb{N}_0,$$

where c_1,c_2, are the fuzzy parameters, C_0,C_1, are the fuzzy initial conditions which determines the level of CO_2 in blood. There are also many researches which study qualitative behaviors of positive solutions of fuzzy difference equations and FDEs. For example, Papaschinopoulos and Papadopoulos, in [20], investigated the existence, boundedness, oscillatory and asymptotic behaviors of the positive solutions of the fuzzy difference equation

$$x_{n+1}=A+\frac{B}{x_n},\quad n\in\mathbb{N}_0,$$

with positive fuzzy parameters A,B and positive fuzzy initial condition x_0. They also studied the fuzzy difference equation

$$x_{n+1}=A+\frac{x_n}{x_{n-m}},\quad n\in\mathbb{N}_0,$$

where x_n is a positive fuzzy number sequence and $A,x_0,x_{-1},\dots,x_{-m}$ are positive fuzzy numbers. Yalcinkaya et al., in [21], investigated qualitative behavior of the fuzzy difference equation

$$z_{n+1}=\frac{Az_{n-s}}{B+C\prod_{i=0}^{s}z_{n-i}^{p}},\quad n\in\mathbb{N}_0,$$

with positive integer s, positive parameters A,B,C and positive initial conditions $z_{-i},i\in\{0,1,\dots,s\}$. Zhang et al., in [22], investigated dynamical behavior of the second-order exponential type fuzzy difference equation

$$x_{n+1}=\frac{A+Be^{-x_n}}{C+x_{n-1}},\quad n\in\mathbb{N}_0,$$

with positive fuzzy parameters A,B,C and positive fuzzy initial conditions x_{-1},x_0. Moreover, Atpinar and Yazlik, in [23], analyzed the existence, uniqueness and the qualitative behavior of the two-dimensional exponential FDEs

$$x_{n+1}=\frac{\alpha_1+\beta_1e^{-x_{n-1}}}{\gamma_1+y_n},\quad y_{n+1}=\frac{\alpha_2+\beta_2e^{-y_{n-1}}}{\gamma_2+x_n},\quad n\in\mathbb{N}_0,$$

where the parameters $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ and the initial conditions x_{-1},x_0,y_{-1},y_0 are positive fuzzy numbers. There are more studies about fuzzy difference equations [22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and references therein. The fuzzy difference equations and fuzzy difference equations system, briefly FDEs, have not been studied extensively, yet. Inspired by the aforementioned studies, we concentrate on the FDEs

$$\alpha_{n+1}=\tau_1+\frac{\alpha_n}{\sum_{i=1}^{m}\beta_{n-i}},\beta_{n+1}=\tau_2+\frac{\beta_n}{\sum_{i=1}^{m}\alpha_{n-i}},\quad n\in\mathbb{N}_0, \tag{1.2}$$

where α_n,β_n are positive fuzzy number sequences, the parameters τ_1,τ_2 and the initial values $\alpha_{-i},\beta_{-i},i\in\{0,1,\dots,m\}$, are positive fuzzy numbers.

2. Preliminaries

In this section, we briefly give some definitions, lemmas and theorems which are used throughout the paper. For more information and details can be found in [7, 15, 16, 34].

Let $\mathbb{R}_f$ represent the space of all fuzzy numbers and $w \in \mathbb{R}_f$. For all $\gamma \in (0,1]$ $[w]^\gamma = \{x \in \mathbb{R} : w(x) \geq \gamma\}$ and $[w]^0 = \overline{\bigcup_{\gamma\in(0,1]}[w]^\gamma} = \overline{\{x \in \mathbb{R} : w(x) > 0\}}$. Here, we say that $[w]^0$ is the support of the fuzzy number w and show it by $supp(w)$. w is called a positive fuzzy number if $supp(w) \subset (0,\infty)$. $\mathbb{R}_f^+$ denotes the space of all fuzzy numbers. Let $x, y \in \mathbb{R}_f$, $\lambda \in \mathbb{R}$ and $[x]^\gamma = [L^\gamma_{x_n}, R^\gamma_{x_n}]$, $[y]^\gamma = [L^\gamma_{y_n}, R^\gamma_{y_n}]$. For $\gamma \in (0,1]$ the operations scalar multiplication, addition, multiplication and division on fuzzy numbers are defined as follows:

$$
\begin{aligned}
[\lambda x]^\gamma &= \lambda[x]^\gamma, \\
[x+y]^\gamma &= [x]^\gamma + [y]^\gamma, \\
[xy]^\gamma &= \left[\min\{L^\gamma_x L^\gamma_y, L^\gamma_x R^\gamma_y, R^\gamma_x L^\gamma_y, R^\gamma_x R^\gamma_y\}, \max\{L^\gamma_x L^\gamma_y, L^\gamma_x R^\gamma_y, R^\gamma_x L^\gamma_y, R^\gamma_x R^\gamma_y\}\right], \\
[\frac{x}{y}]^\gamma &= \left[\min\{\frac{L^\gamma_x}{L^\gamma_y}, \frac{L^\gamma_x}{R^\gamma_y}, \frac{R^\gamma_x}{L^\gamma_y}, \frac{R^\gamma_x}{R^\gamma_y}\}, \max\{\frac{L^\gamma_x}{L^\gamma_y}, \frac{L^\gamma_x}{R^\gamma_y}, \frac{R^\gamma_x}{L^\gamma_y}, \frac{R^\gamma_x}{R^\gamma_y}\}\right], 0 \notin [y]^\gamma,
\end{aligned}
$$

respectively.

Definition 2.1. *Consider a fuzzy subset of the real line $w : \mathbb{R} \to (0,1]$ and suppose that the following properties hold:*

(a) w is normal, i. e., there exists $x_0 \in \mathbb{R}$ such that $w(x_0) = 1$,
(b) w is convex, i. e., $\forall \lambda \in (0,1]$ and $x_1, x_2 \in \mathbb{R}$, $w(\lambda x_1 + (1-\lambda)x_2) \geq \min\{w(x_1), w(x_2)\}$
(c) w is upper semi-continuous on $\mathbb{R}$,
(d) w is compactly supported, i. e., $\overline{\bigcup_{\gamma\in(0,1]}[w]^\gamma} = \overline{\{x \in \mathbb{R} : w(x) > 0\}}$ is compact,

we say that w is a fuzzy number.

Lemma 2.2. *Let $x \in R_f^+$ and $[x]^\gamma = [L^\gamma_x, R^\gamma_x]$ for $\gamma \in (0,1]$. Then, for $[L^\gamma_x, R^\gamma_x]$ the following conditions hold:*

1. *L^γ_x is non-decreasing and left continuous,*
2. *R^γ_x is non-increasing and right continuous,*
3. *$L^\gamma_x \leq R^\gamma_x$.*

Lemma 2.3. *Let f be a continuous function from $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}^+$. For any $x, y, z, t \in \mathbb{R}_f^+$ and $\gamma \in (0,1]$,*

$$[f(x,y,z,t)]^\gamma = f([x]^\gamma, [y]^\gamma, [z]^\gamma, [t]^\gamma).$$

Definition 2.4. *Let $\{x_n\}$ be a positive fuzzy number sequence. If there exist positive real numbers m, M such that $supp(x_n) \subset [m,M]$, then we say that positive fuzzy sequence (x_n) is bounded and persistent.*

Theorem 2.5. *Let $[x]^\gamma \in R_f^+$ be a fuzzy number. Then,*

1. *$[x]^\gamma$ is a closed interval $\forall \gamma \in (0,1]$,*
2. *For $\gamma_1, \gamma_2 \in (0,1]$, if $\gamma_1 \leq \gamma_2$, then $x^{\gamma_2} \subseteq x^{\gamma_1}$,*
3. *For any sequence γ_n converging to $\gamma \in (0,1]$ from below, $\cap_{n=1}^\infty x^{\gamma_n} = x^\gamma$,*
4. *For any sequence γ_n converging to 0 from above, $\overline{\cup_{n=1}^\infty [x]^{\gamma_n}} = [x]^0$.*

Definition 2.6. *Let x, y be fuzzy numbers with $[x]^\gamma = [L^\gamma_x, R^\gamma_x]$ and $[y]^\gamma = [L^\gamma_y, R^\gamma_y]$ for $\gamma \in (0,1]$. Then, the metric on fuzzy number space is defined as follows:*

$$D(x,y) = \sup_{\gamma\in(0,1]} \max\{|L^\gamma_x - L^\gamma_y|, |R^\gamma_x - R^\gamma_y|\}. \tag{2.1}$$

Moreover, the norm on fuzzy number space is defined by

$$||X|| = \sup_{\gamma\in(0,1]} \max\{|L^\gamma_x|, |R^\gamma_x|\}.$$

3. Main Results

In this section, we study FDEs (1.2) for positive initial fuzzy numbers. Firstly, we investigate the existence and uniqueness of positive solutions of (1.2) in the following theorem.

Theorem 3.1. *Consider the system (1.2) for positive fuzzy numbers τ_1, τ_2. Then, for given any positive fuzzy numbers $\alpha_{-i}, \beta_{-i}, i \in \{0,1,\dots,m\}$, the system has a unique positive solution.*

Proof. Let the parameters τ_1, τ_2 and the initial conditions $\alpha_{-i}, \beta_{-i}, i \in \{0,1,\dots,m\}$, be positive fuzzy numbers. Suppose that there exist fuzzy number sequences which satisfy (1.2). Consider their γ-cuts for $\gamma \in (0,1]$;

$$\begin{cases} [\alpha_n]^\gamma & = [L^\gamma_{\alpha_n}, R^\gamma_{\alpha_n}], [\beta_n]^\gamma = [L^\gamma_{\beta_n}, R^\gamma_{\beta_n}], \\ [\alpha_{n-i}]^\gamma & = [L^\gamma_{\alpha_{n-i}}, R^\gamma_{\alpha_{n-i}}], [\beta_{n-i}]^\gamma = [L^\gamma_{\beta_{n-i}}, R^\gamma_{\beta_{n-i}}], \\ [\tau_1]^\gamma & = [\tau^\gamma_{1,l}, \tau^\gamma_{1,r}], [\tau_2]^\gamma = [\tau^\gamma_{2,l}, \tau^\gamma_{2,r}]. \end{cases} \tag{3.1}$$

By using (1.2), (3.1) and Lemma (2.3),

$$\begin{aligned} [\alpha_{n+1}]^\gamma &= [L^\gamma_{\alpha_{n+1}}, R^\gamma_{\alpha_{n+1}}], \\ &= \left[\tau_1 + \frac{\alpha_n}{\sum_{i=1}^m \beta_{n-i}}\right]^\gamma, \\ &= [\tau_1]^\gamma + \frac{[\alpha_n]^\gamma}{\sum_{i=1}^m [\beta_{n-i}]^\gamma}, \\ &= [\tau^\gamma_{1,l}, \tau^\gamma_{1,r}] + \frac{[L^\gamma_{\alpha_n}, R^\gamma_{\alpha_n}]}{\sum_{i=1}^m [L^\gamma_{\beta_{n-i}}, R^\gamma_{\beta_{n-i}}]}, \\ &= \left[\tau^\gamma_{1,l} + \frac{L^\gamma_{\alpha_n}}{\sum_{i=1}^m R^\gamma_{\beta_{n-i}}}, \tau^\gamma_{1,r} + \frac{R^\gamma_{\alpha_n}}{\sum_{i=1}^m L^\gamma_{\beta_{n-i}}}\right], \end{aligned} \tag{3.2}$$

and similarly

$$\begin{aligned} [\beta_{n+1}]^\gamma &= [L^\gamma_{\beta_{n+1}}, R^\gamma_{\beta_{n+1}}], \\ &= \left[\tau_2 + \frac{\beta_n}{\sum_{i=1}^m \alpha_{n-i}}\right]^\gamma, \\ &= [\tau_2]^\gamma + \frac{[\beta_n]^\gamma}{\sum_{i=1}^m [\alpha_{n-i}]^\gamma}, \\ &= [\tau^\gamma_{2,l}, \tau^\gamma_{2,r}] + \frac{[L^\gamma_{\beta_n}, R^\gamma_{\beta_n}]}{\sum_{i=1}^m [L^\gamma_{\alpha_{n-i}}, R^\gamma_{\alpha_{n-i}}]}, \\ &= \left[\tau^\gamma_{2,l} + \frac{L^\gamma_{\beta_n}}{\sum_{i=1}^m R^\gamma_{\alpha_{n-i}}}, \tau^\gamma_{2,r} + \frac{R^\gamma_{\beta_n}}{\sum_{i=1}^m L^\gamma_{\alpha_{n-i}}}\right]. \end{aligned} \tag{3.3}$$

So, we obtained the following equations system:

$$\begin{aligned} L^\gamma_{\alpha_{n+1}} &= \tau^\gamma_{1,l} + \frac{L^\gamma_{\alpha_n}}{\sum_{i=1}^m R^\gamma_{\beta_{n-i}}}, \ R^\gamma_{\alpha_{n+1}} = \tau^\gamma_{1,r} + \frac{R^\gamma_{\alpha_n}}{\sum_{i=1}^m L^\gamma_{\beta_{n-i}}}, \\ L^\gamma_{\beta_{n+1}} &= \tau^\gamma_{2,l} + \frac{L^\gamma_{\beta_n}}{\sum_{i=1}^m R^\gamma_{\alpha_{n-i}}}, \ R^\gamma_{\beta_{n+1}} = \tau^\gamma_{2,r} + \frac{R^\gamma_{\beta_n}}{\sum_{i=1}^m L^\gamma_{\alpha_{n-i}}}. \end{aligned} \tag{3.4}$$

Let $0 \le \gamma_1 \le \gamma_2 \le 1$. From Lemma (2.2),

$$\begin{aligned} 0 < \tau^{\gamma_1}_{1,l} \le \tau^{\gamma_2}_{1,l} \le \tau^{\gamma_2}_{1,r} \le \tau^{\gamma_1}_{1,r}, \\ 0 < \tau^{\gamma_1}_{2,l} \le \tau^{\gamma_2}_{2,l} \le \tau^{\gamma_2}_{2,r} \le \tau^{\gamma_1}_{2,r}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 < L^{\gamma_1}_{\alpha_{n-i}} \le L^{\gamma_2}_{\alpha_{n-i}} \le R^{\gamma_2}_{\alpha_{n-i}} \le R^{\gamma_1}_{\alpha_{n-i}}, \\ 0 < L^{\gamma_1}_{\beta_{n-i}} \le L^{\gamma_2}_{\beta_{n-i}} \le R^{\gamma_2}_{\beta_{n-i}} \le R^{\gamma_1}_{\beta_{n-i}}, \end{aligned} \tag{3.6}$$

for $i \in \{0,1,\dots,m\}$ and

$$\begin{aligned} 0 < L^{\gamma_1}_{\alpha_n} \le L^{\gamma_2}_{\alpha_n} \le R^{\gamma_2}_{\alpha_n} \le R^{\gamma_1}_{\alpha_n}, \\ 0 < L^{\gamma_1}_{\beta_n} \le L^{\gamma_2}_{\beta_n} \le R^{\gamma_2}_{\beta_n} \le R^{\gamma_1}_{\beta_n}, \end{aligned} \tag{3.7}$$

where $n \in \mathbb{N}_0$. By using inequalities in (3.5), (3.6) and (3.7) and keeping in mind that $\gamma_1 \leq \gamma_2$,

$$L^{\gamma_1}_{\alpha_{n+1}} = \tau^{\gamma_1}_{1,l} + \frac{L^{\gamma_1}_{\alpha_n}}{\sum_{i=1}^m R^{\gamma_1}_{\beta_{n-i}}} \leq \tau^{\gamma_2}_{1,l} + \frac{L^{\gamma_2}_{\alpha_n}}{\sum_{i=1}^m R^{\gamma_2}_{\beta_{n-i}}} \leq L^{\gamma_2}_{\alpha_{n+1}}$$
$$\leq \tau^{\gamma_2}_{1,r} + \frac{R^{\gamma_2}_{\alpha_n}}{\sum_{i=1}^m L^{\gamma_2}_{\beta_{n-i}}} \leq R^{\gamma_2}_{\alpha_{n+1}} \leq \tau^{\gamma_1}_{1,r} + \frac{R^{\gamma_1}_{\alpha_n}}{\sum_{i=1}^m L^{\gamma_1}_{\beta_{n-i}}} = R^{\gamma_1}_{\alpha_{n+1}}$$

and

$$L^{\gamma_1}_{\beta_{n+1}} = \tau^{\gamma_1}_{2,l} + \frac{L^{\gamma_1}_{\beta_n}}{\sum_{i=1}^m R^{\gamma_1}_{\alpha_{n-i}}} \leq \tau^{\gamma_2}_{2,l} + \frac{L^{\gamma_1}_{\beta_n}}{\sum_{i=1}^m R^{\gamma_1}_{\alpha_{n-i}}} \leq L^{\gamma_2}_{\beta_{n+1}}$$
$$\leq \tau^{\gamma_2}_{2,r} + \frac{R^{\gamma_2}_{\beta_n}}{\sum_{i=1}^m L^{\gamma_2}_{\alpha_{n-i}}} \leq R^{\gamma_2}_{\beta_{n+1}} \leq \tau^{\gamma_1}_{2,r} + \frac{R^{\gamma_1}_{\beta_n}}{\sum_{i=1}^m L^{\gamma_1}_{\alpha_{n-i}}} = R^{\gamma_1}_{\beta_{n+1}}.$$

Next, by using induction, we will show positive fuzzy solution of the FDEs (1.2) exists. For $n = 0$,

$$[L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}] = \left[\tau^{\gamma}_{1,l} + \frac{L^{\gamma}_{\alpha_0}}{\sum_{i=1}^m R^{\gamma}_{\beta_{-i}}}, \tau^{\gamma}_{1,r} + \frac{R^{\gamma}_{\alpha_0}}{\sum_{i=1}^m L^{\gamma}_{\beta_{-i}}}\right],$$
$$[L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}] = \left[\tau^{\gamma}_{2,l} + \frac{L^{\gamma}_{\beta_0}}{\sum_{i=1}^m R^{\gamma}_{\alpha_{-i}}}, \tau^{\gamma}_{2,r} + \frac{R^{\gamma}_{\beta_0}}{\sum_{i=1}^m L^{\gamma}_{\alpha_{-i}}}\right].$$

Since, τ_1, τ_2 and α_i, β_i for $i \in \{0, 1, \ldots, m\}$ are positive fuzzy numbers, for $\gamma \in (0,1]$, $[L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}]$ and $[L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}]$ are γ-cuts of $\alpha_1 = \tau^{\gamma}_1 + \frac{\alpha^{\gamma}_0}{\sum_{i=1}^m \beta^{\gamma}_{-i}}$ and $\beta_1 = \tau^{\gamma}_2 + \frac{\beta^{\gamma}_0}{\sum_{i=1}^m \alpha^{\gamma}_{-i}}$. Moreover, $\tau_{1,l}, \tau_{1,r}, \tau_{2,l}, \tau_{2,r}$ and for $i \in \{0, 1, \ldots, m\}$ $L^{\gamma}_{\alpha_{-i}}, R^{\gamma}_{\alpha_{-i}}, L^{\gamma}_{\beta_{-i}}, R^{\gamma}_{\beta_{-i}}$ are left continuous, then so are $L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}, L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}$.
Now, assume that for $j \in \{1, 2, \ldots, k\}$, $[L^{\gamma}_{\alpha_j}, R^{\gamma}_{\alpha_j}]$ and $[L^{\gamma}_{\beta_j}, R^{\gamma}_{\beta_j}]$ are the γ-cuts of $\alpha_j = \tau_1 + \frac{\alpha_{j-1}}{\sum_{i=1}^m \beta^{\gamma}_{j-i-1}}$ and $\beta_j = \tau_1 + \frac{\beta_{j-1}}{\sum_{i=1}^m \alpha^{\gamma}_{j-i-1}}$.
For $n = k+1$, we have

$$[\alpha_{k+1}]^{\gamma} = \left[\tau^{\gamma}_{1,l} + \frac{L^{\gamma}_{\alpha_k}}{\sum_{i=1}^m R^{\gamma}_{\beta_{k-i}}}, \tau^{\gamma}_{1,r} + \frac{R^{\gamma}_{\alpha_k}}{\sum_{i=1}^m L^{\gamma}_{\beta_{k-i}}}\right] = \left[\tau_1 + \frac{\alpha_k}{\sum_{i=1}^m \beta_{k-i}}\right]^{\gamma},$$
$$[\beta_{k+1}]^{\gamma} = \left[\tau^{\gamma}_{2,l} + \frac{L^{\gamma}_{\beta_k}}{\sum_{i=1}^m R^{\gamma}_{\alpha_{k-i}}}, \tau^{\gamma}_{2,r} + \frac{R^{\gamma}_{\beta_k}}{\sum_{i=1}^m L^{\gamma}_{\alpha_{k-i}}}\right] = \left[\tau_2 + \frac{\beta_k}{\sum_{i=1}^m \alpha_{k-i}}\right]^{\gamma}.$$

Therefore, $[L^{\gamma}_{\alpha_{k+1}}, R^{\gamma}_{\alpha_{k+1}}]$ and $[L^{\gamma}_{\beta_{k+1}}, R^{\gamma}_{\beta_{k+1}}]$ are the γ-cuts of the fuzzy numbers $\alpha_{k+1} = \tau_1 + \frac{\alpha_k}{\sum_{i=1}^m \beta_{k-i}}$ and $\beta_{k+1} = \tau_2 + \frac{\beta_k}{\sum_{i=1}^m \alpha_{k-i}}$.
Hence, for $\forall n \in \mathbb{N}$ and $\forall \gamma \in (0,1]$, $[L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}]$ and $[L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}]$ are the γ-cuts of the fuzzy numbers α_n and β_n, by induction.

Now, we claim that supports of both α_n and β_n, $\mathrm{supp}\alpha_n = \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}]}$ and $\mathrm{supp}\beta_n = \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}]}$ are compact by induction. For $n = 1$, since τ_1, τ_2 and $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$ are positive fuzzy numbers, there exist $M_{\tau_1}, N_{\tau_1}, M_{\tau_2}, N_{\tau_2}, M_{\alpha_{-i}}, N_{\alpha_{-i}}, M_{\beta_{-i}}, N_{\beta_{-i}} \in \{0, 1, \ldots, m\}$ such that for all $\gamma \in (0,1]$,

$$\begin{cases} [\tau^{\gamma}_{1,l}, \tau^{\gamma}_{1,r}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [\tau^{\gamma}_{1,l}, \tau^{\gamma}_{1,r}]} \subseteq [M_{\tau_1}, N_{\tau_1}] \\ [\tau^{\gamma}_{2,l}, \tau^{\gamma}_{2,r}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [\tau^{\gamma}_{2,l}, \tau^{\gamma}_{2,r}]} \subseteq [M_{\tau_2}, N_{\tau_2}] \\ [L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\alpha_1}, R^{\gamma}_{\alpha_1}]} \subseteq [M_{\alpha_1}, N_{\alpha_1}] \\ [L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}] & \subseteq \overline{\bigcup_{\gamma \in (0,1]} [L^{\gamma}_{\beta_1}, R^{\gamma}_{\beta_1}]} \subseteq [M_{\beta_1}, N_{\beta_1}]. \end{cases}$$

By using induction, we obtain $\overline{\bigcup_{\gamma \in (0,1]} [\alpha^{\gamma}_{n,l}, \alpha^{\gamma}_{n,r}]}$ and $\overline{\bigcup_{\gamma \in (0,1]} [\beta^{\gamma}_{n,l}, \beta^{\gamma}_{n,r}]}$ are compact and $\overline{\bigcup_{\gamma \in (0,1]} [\alpha^{\gamma}_{n,l}, \alpha^{\gamma}_{n,r}]}$ and $\overline{\bigcup_{\gamma \in (0,1]} [\beta^{\gamma}_{n,l}, \beta^{\gamma}_{n,r}]} \subseteq (0, +\infty)$ for $n \in \mathbb{N}_0$. Hence, $\alpha_n = [\alpha_n]^{\gamma} = [L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}]$ and $\beta_n = [\beta_n]^{\gamma} = [L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}]$ are also positive fuzzy number sequences.

Finally, we will show uniqueness of positive solutions of FDEs (1.2) by using contradiction method. Assume that there exist other solutions α'_n and β'_n to the given system (1.2) with the same initial initial values τ_1, τ_2 and $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$.

Then, for $\alpha \in (0,1]$;

$$[\alpha_n]^{\gamma} = [\alpha'_n]^{\gamma}, [\beta_n]^{\gamma} = [\beta'_n]^{\gamma}.$$

Hence, there exists a unique solution of (1.2) for given initial conditions τ_1, τ_2 and $\alpha_{-i}, \beta_{-i}, i \in \{0, 1, \ldots, m\}$, which is desired. □

Now, we will investigate boundedness and persistence of positive solutions of (1.2).

Let u_n, v_n, w_n, t_n represent $L^{\gamma}_{\alpha_n}, R^{\gamma}_{\alpha_n}, L^{\gamma}_{\beta_n}, R^{\gamma}_{\beta_n}$ respectively. Then, from (3.4), we can write the following system as

$$\begin{cases} L_{\alpha_{n+1}} &= u_{n+1} = \lambda_1 + \frac{u_n}{\sum_{i=1}^{m} t_{n-i}}, \\ R_{\alpha_{n+1}} &= v_{n+1} = \lambda_2 + \frac{v_n}{\sum_{i=1}^{m} w_{n-i}}, \\ L_{\beta_{n+1}} &= w_{n+1} = \lambda_3 + \frac{w_n}{\sum_{i=1}^{m} v_{n-i}}, \\ R_{\beta_{n+1}} &= t_{n+1} = \lambda_4 + \frac{t_n}{\sum_{i=1}^{m} u_{n-i}}, \end{cases} \quad n \in \mathbb{N}_0, \tag{3.8}$$

where the parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive real numbers.

Theorem 3.2. *Consider system* (3.8) *and suppose that*

$$\frac{1}{m} < \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}. \tag{3.9}$$

If (3.9) *is satisfied, then for every positive solutions* (u_n, v_n, w_n, t_n) *of* (3.8) *for* $n > m$ *the following inequalities hold:*

$$\begin{aligned} \lambda_1 \le u_n \le \frac{1}{(m\lambda_4)^{n-m}}\left(u_m - \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1}\right) + \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1}, \\ \lambda_2 \le v_n \le \frac{1}{(m\lambda_3)^{n-m}}\left(v_m - \frac{m\lambda_2\lambda_3}{m\lambda_3 - 1}\right) + \frac{m\lambda_2\lambda_3}{m\lambda_3 - 1}, \\ \lambda_3 \le w_n \le \frac{1}{(m\lambda_2)^{n-m}}\left(w_m - \frac{m\lambda_2\lambda_3}{m\lambda_2 - 1}\right) + \frac{m\lambda_2\lambda_3}{m\lambda_2 - 1}, \\ \lambda_4 \le t_n \le \frac{1}{(m\lambda_1)^{n-m}}\left(t_m - \frac{m\lambda_1\lambda_4}{m\lambda_1 - 1}\right) + \frac{m\lambda_1\lambda_4}{m\lambda_1 - 1}, \end{aligned} \tag{3.10}$$

which shows the boundedness and persistence of (u_n, v_n, w_n, t_n).

Proof. Let (u_n, v_n, w_n, t_n) be positive solution of system (3.8). Since, u_n, v_n, w_n, t_n, for all $n \ge 1$, are positive,

$$\lambda_1 \le u_n, \quad \lambda_2 \le v_n, \quad \lambda_3 \le w_n, \quad \lambda_4 \le t_n. \tag{3.11}$$

Furthermore, by (3.8) and (3.11), we get

$$\begin{cases} u_n = \lambda_1 + \frac{u_n}{\sum_{i=2}^{m+1} t_{n-i}} \le \lambda_1 + \frac{1}{m\lambda_4} u_{n-1}, \\ v_n = \lambda_2 + \frac{v_n}{\sum_{i=2}^{m+1} w_{n-i}} \le \lambda_2 + \frac{1}{m\lambda_3} v_{n-1}, \\ w_n = \lambda_3 + \frac{w_n}{\sum_{i=2}^{m+1} v_{n-i}} \le \lambda_3 + \frac{1}{m\lambda_2} w_{n-1}, \\ t_n = \lambda_4 + \frac{t_n}{\sum_{i=2}^{m+1} u_{n-i}} \le \lambda_4 + \frac{1}{m\lambda_1} t_{n-1}, \end{cases} \tag{3.12}$$

for $n > m$. On this part of the proof, we just show boundedness for u_n. Since, proofs for v_n, w_n, t_n are similar, we omit them. Define $\tilde{u}_n = \lambda_1 + \frac{1}{m\lambda_4}\tilde{u}_{n-1}$ for $n > m$ and $\tilde{u}_n = u_n$ for $n = 1, 2, \ldots, m$. Our claim is

$$u_n \le \tilde{u}_n, n \in \mathbb{N}. \tag{3.13}$$

We show satisfying the inequality in (3.13) by induction. It is obvious that $u_n \le \tilde{u}_n$ for $n \in \{1, 2, \ldots, m\}$. Suppose that (3.13) holds for any $k = n \ge m+1$. Then, from (3.12), we have

$$u_{n+1} \le \lambda_1 + \frac{1}{m\lambda_4} u_n \le \lambda_1 + \frac{1}{m\lambda_4}\tilde{u}_n = \tilde{u}_{n+1}. \tag{3.14}$$

Therefore, $u_n \le \tilde{u}_n$ for $n \in \mathbb{N}$, by induction. Then,

$$\begin{aligned} \tilde{u}_n &= \lambda_1 + \frac{1}{m\lambda_4}\tilde{u}_{n-1} \\ \tilde{u}_n &= \lambda_1 + \frac{1}{m\lambda_4}\left(\lambda_1 + \frac{1}{m\lambda_4}\tilde{u}_{n-2}\right) \\ &\vdots \\ \tilde{u}_n &= \frac{1}{(m\lambda_4)^{n-m}}\left(u_m - \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1}\right) + \frac{m\lambda_1\lambda_4}{m\lambda_4 - 1}. \end{aligned}$$

Hence, u_n is bounded. So proof for u_n is finished. Similarly, it can be shown that v_n, w_n, t_n are also bounded. □

Theorem 3.3. *Consider system* (3.8)*. If the condition* (3.10) *holds, then* (3.8) *has a unique equilibrium point* $(\bar{u},\bar{v},\bar{w},\bar{t})$ *given by*

$$\bar{u}=\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_4-1)},\ \bar{v}=\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_3-1)},\ \bar{w}=\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_2-1)},\ \bar{t}=\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_1-1)}, \tag{3.15}$$

and every positive solution tends to given equilibrium point as $n\to\infty$.

Proof. From equilibrium point definition, we can simply obtain the equilibrium point given as

$$\Gamma=\left(\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_4-1)},\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_3-1)},\frac{m^2\lambda_2\lambda_3-1}{m(m\lambda_2-1)},\frac{m^2\lambda_1\lambda_4-1}{m(m\lambda_1-1)}\right). \tag{3.16}$$

Since every positive solution of system (3.8) is bounded and persistent from Theorem (3.2), it can be written that

$$\begin{aligned}
\lim_{n\to\infty}\inf u_n=l_1,\ \lim_{n\to\infty}\sup u_n=L_1,\\
\lim_{n\to\infty}\inf v_n=l_2,\ \lim_{n\to\infty}\sup v_n=L_2,\\
\lim_{n\to\infty}\inf w_n=l_3,\ \lim_{n\to\infty}\sup w_n=L_3,\\
\lim_{n\to\infty}\inf t_n=l_4,\ \lim_{n\to\infty}\sup t_n=L_4,
\end{aligned} \tag{3.17}$$

where $l_i,L_i\in(0,\infty)$, for $i\in\{1,2,3,4\}$. Then, by using Theorem (3.2) and (3.17),

$$\begin{aligned}
\lambda_1+\frac{l_1}{mL_4}\le l_1,\ L_1\le\lambda_1+\frac{L_1}{ml_4},\\
\lambda_2+\frac{l_2}{mL_3}\le l_2,\ L_2\le\lambda_2+\frac{L_2}{ml_3},\\
\lambda_3+\frac{l_3}{mL_2}\le l_3,\ L_3\le\lambda_3+\frac{L_3}{ml_2},\\
\lambda_4+\frac{l_4}{mL_1}\le l_4,\ L_4\le\lambda_4+\frac{L_4}{ml_1}.
\end{aligned}$$

Next, after arranging the inequalities, we obtain

$$\begin{aligned}
\frac{m\lambda_4L_1+l_4}{m}\le L_1l_4\le\frac{m\lambda_1l_4+L_1}{m},\\
\frac{m\lambda_3L_2+l_3}{m}\le L_2l_3\le\frac{m\lambda_2l_3+L_2}{m},\\
\frac{m\lambda_2L_3+l_2}{m}\le L_3l_2\le\frac{m\lambda_3l_2+L_3}{m},\\
\frac{m\lambda_1L_4+l_1}{m}\le L_4l_1\le\frac{m\lambda_4l_1+L_4}{m},
\end{aligned} \tag{3.18}$$

from which it follows that

$$\begin{aligned}
L_1(m\lambda_4-1)\le l_4(m\lambda_1-1),\\
L_2(m\lambda_3-1)\le l_3(m\lambda_2-1),\\
L_3(m\lambda_2-1)\le l_2(m\lambda_3-1),\\
L_4(m\lambda_1-1)\le l_1(m\lambda_4-1).
\end{aligned} \tag{3.19}$$

It is obvious from hypothesis of the Theorem (3.2) that $1<m\lambda_i$ for $i\in\{1,2,3,4\}$. Multiplying the first and the fourth inequalities in (3.19) and the second and the third inequalities in (3.19) gives

$$L_1L_4\le l_1l_4,\quad L_2L_3\le l_2l_3. \tag{3.20}$$

So,

$$L_1L_4=l_1l_4,\quad L_1L_4=l_1l_4. \tag{3.21}$$

Our claim is

$$L_1=l_1,\quad L_2=l_2,\quad L_3=l_3,\quad L_4=l_4.$$

Assume that $l_1 < L_1, l_2 < L_2, l_3 < L_3, l_4 < L_4$. Then by using (3.21), we get

$$\begin{aligned} L_1L_4 &= l_1l_4 < l_1L_4, \\ L_1L_4 &= l_1l_4 < L_1l_4, \\ L_2L_3 &= l_2l_3 < l_2L_3, \\ L_2L_3 &= l_2l_3 < L_2l_3, \end{aligned}$$

gives us

$$\begin{aligned} L_1 &< l_1, \\ L_2 &< l_2, \\ L_3 &< l_3, \\ L_4 &< l_4, \end{aligned}$$

which is a contradiction. Therefore,

$$l_1 = L_1,\ l_2 = L_2,\ l_3 = L_3,\ l_4 = L_4. \tag{3.22}$$

Hence, by using (3.8) and (3.22), it follows that

$$\lim_{n\to\infty} u_n = \tilde{u},\ \lim_{n\to\infty} v_n = \tilde{v},\ \lim_{n\to\infty} w_n = \tilde{w},\ \lim_{n\to\infty} t_n = \tilde{t}.$$

Thus, proof is completed. □

Theorem 3.4. *Consider system (3.8). If both (3.9) and the following inequalities*

$$\frac{m^2\lambda_1\lambda_4-1}{m\lambda_1-1}+\frac{m^2\lambda_1\lambda_4-1}{m\lambda_4-1}<1, \quad \frac{m^2\lambda_2\lambda_3-1}{m\lambda_2-1}+\frac{m^2\lambda_2\lambda_3-1}{m\lambda_3-1}<1, \tag{3.23}$$

are satisfied, then the unique positive equilibrium point given in (3.15) is locally asymptotically stable.

Proof. From Theorem (3.3), the system (3.8) has a unique equilibrium point (3.15). The linearized equation of system (3.8) about the equilibrium point is

$$\Omega_{n+1} = \mathcal{P}\Omega_n$$

where $\Omega_n = (u_n, u_{n-1}, \ldots, u_{n-m}, v_n, v_{n-1}, \ldots, v_{n-m}, w_n, w_{n-1}, , \ldots, w_{n-m}, t_n, t_{n-1}, \ldots, t_{n-m})^T$ and $\mathcal{P} = (\rho_{ij}), 1 \le i, j \le 4m+4$ is a $(4m+4)\times(4m+4)$ matrix such that

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{\tilde{t}} & \mathcal{P}_0 & \mathcal{P}_0 & \mathcal{P}_1 \\ \mathcal{P}_0 & \mathcal{P}_{\tilde{w}} & \mathcal{P}_2 & \mathcal{P}_0 \\ \mathcal{P}_0 & \mathcal{P}_3 & \mathcal{P}_{\tilde{v}} & \mathcal{P}_0 \\ \mathcal{P}_4 & \mathcal{P}_0 & \mathcal{P}_0 & \mathcal{P}_{\tilde{u}} \end{bmatrix}_{(4m+4)\times(4m+4)}, \tag{3.24}$$

where $\mathcal{P}_{\tilde{u}}, \mathcal{P}_{\tilde{v}}, \mathcal{P}_{\tilde{w}}, \mathcal{P}_{\tilde{t}}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ are $(m+1)\times(m+1)$ matrices are defined as follows:

$$\mathcal{P}_{\tilde{u}} = \begin{bmatrix} \frac{1}{m\tilde{u}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathcal{P}_{\tilde{v}} = \begin{bmatrix} \frac{1}{m\tilde{v}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathcal{P}_{\tilde{w}} = \begin{bmatrix} \frac{1}{m\tilde{w}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\mathcal{P}_{\tilde{t}} = \begin{bmatrix} \frac{1}{m\tilde{t}} & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \mathcal{P}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \mathcal{P}_1 = \begin{bmatrix} 0 & -\frac{\tilde{u}}{m^2\tilde{t}^2} & \cdots & -\frac{\tilde{u}}{m^2\tilde{t}^2} & -\frac{\tilde{u}}{m^2\tilde{t}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}_2 = \begin{bmatrix} 0 & -\frac{\tilde{v}}{m^2\tilde{w}^2} & \cdots & -\frac{\tilde{v}}{m^2\tilde{w}^2} & -\frac{\tilde{v}}{m^2\tilde{w}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \mathcal{P}_3 = \begin{bmatrix} 0 & -\frac{\tilde{w}}{m^2\tilde{v}^2} & \cdots & -\frac{\tilde{w}}{m^2\tilde{v}^2} & -\frac{\tilde{w}}{m^2\tilde{v}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\mathcal{P}_4 = \begin{bmatrix} 0 & -\frac{\tilde{t}}{m^2\tilde{u}^2} & \cdots & -\frac{\tilde{t}}{m^2\tilde{u}^2} & -\frac{\tilde{t}}{m^2\tilde{u}^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Let $\sigma_1,\sigma_2,\dots,\sigma_{4m+4}$ be the eigenvalues of the matrix $\mathcal{P}$ and D be the diagonal matrix $(d_1,d_2,\dots,d_{4m+4})$ such that $d_1 = d_{m+2} = d_{2m+3} = d_{3m+4} = 1$ and $d_j = d_{m+1+j} = d_{2m+2+j} = d_{3m+3+j} = 1-j\varepsilon$, for $j\in\{2,3,\dots,m+1\}$, where

$$0<\varepsilon<\frac{1}{m+1}min\big\{(1-\frac{\bar{u}+\bar{t}}{m\bar{u}^2}),(1-\frac{\bar{u}+\bar{t}}{m\bar{t}^2}),(1-\frac{\bar{v}+\bar{w}}{m\bar{v}^2}),(1-\frac{\bar{v}+\bar{w}}{m\bar{w}^2})\big\}. \tag{3.25}$$

It is obvious that D is invertible. Computing $D\mathcal{P}D^{-1}$ gives us the matrix

$$\mathcal{P}^{(1)}=\begin{bmatrix}\mathcal{P}^{(1)}_{\bar{t}} & \mathcal{P}^{(1)}_{0} & \mathcal{P}^{(1)}_{0} & \mathcal{P}^{(1)}_{1}\\ \mathcal{P}^{(1)}_{0} & \mathcal{P}^{(1)}_{\bar{w}} & \mathcal{P}^{(1)}_{2} & \mathcal{P}^{(1)}_{0}\\ \mathcal{P}^{(1)}_{0} & \mathcal{P}^{(1)}_{3} & \mathcal{P}^{(1)}_{\bar{v}} & \mathcal{P}^{(1)}_{0}\\ \mathcal{P}^{(1)}_{4} & \mathcal{P}^{(1)}_{0} & \mathcal{P}^{(1)}_{0} & \mathcal{P}^{(1)}_{\bar{u}}\end{bmatrix}_{(4m+4)\times(4m+4)}, \tag{3.26}$$

where

$$\mathcal{P}^{(1)}_{\bar{u}}=\begin{bmatrix}\frac{1}{m\bar{u}} & 0 & \cdots & 0 & 0\\ d_{3m+5}d^{-1}_{3m+4} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & d_{4m+4}d^{-1}_{4m+3} & 0\end{bmatrix}, \mathcal{P}^{(1)}_{\bar{v}}=\begin{bmatrix}\frac{1}{m\bar{v}} & 0 & \cdots & 0 & 0\\ d_{2m+4}d^{-1}_{2m+3} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & d_{3m+3}d^{-1}_{3m+2} & 0\end{bmatrix},$$

$$\mathcal{P}^{(1)}_{\bar{w}}=\begin{bmatrix}\frac{1}{m\bar{w}} & 0 & \cdots & 0 & 0\\ d_{m+3}d^{-1}_{m+2} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & d_{2m+2}d^{-1}_{2m+1} & 0\end{bmatrix}, \mathcal{P}^{(1)}_{\bar{t}}=\begin{bmatrix}\frac{1}{m\bar{t}} & 0 & \cdots & 0 & 0\\ d_{2}d^{-1}_{1} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & d_{m+1}d^{-1}_{m} & 0\end{bmatrix},$$

$$\mathcal{P}^{(1)}_{0}=\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},$$

$$\mathcal{P}^{(1)}_{1}=\begin{bmatrix}0 & \frac{-\bar{u}}{m^2\bar{t}^2}d_1d^{-1}_{3m+5} & \cdots & \frac{-\bar{u}}{m^2\bar{t}^2}d_1d^{-1}_{4m+3} & \frac{-\bar{u}}{m^2\bar{t}^2}d_1d^{-1}_{4m+4}\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},$$

$$\mathcal{P}^{(1)}_{2}=\begin{bmatrix}0 & \frac{-\bar{v}}{m^2\bar{w}^2}d_{m+2}d^{-1}_{2m+4} & \cdots & \frac{\bar{v}}{m^2\bar{w}^2}d_{m+2}d^{-1}_{3m+2} & \frac{-\bar{v}}{m^2\bar{w}^2}d_{m+2}d^{-1}_{3m+3}\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},$$

$$\mathcal{P}^{(1)}_{3}=\begin{bmatrix}0 & \frac{-\bar{w}}{m^2\bar{v}^2}d_{2m+3}d^{-1}_{m+3} & \cdots & \frac{-\bar{w}}{m^2\bar{v}^2}d_{2m+3}d^{-1}_{2m+1} & \frac{-\bar{w}}{m^2\bar{v}^2}d_{2m+3}d^{-1}_{2m+2}\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},$$

$$\mathcal{P}^{(1)}_{4}=\begin{bmatrix}0 & \frac{-\bar{t}}{m^2\bar{u}^2}d_{3m+4}d^{-1}_{2} & \cdots & \frac{-\bar{t}}{m^2\bar{u}^2}d_{3m+4}d^{-1}_{m} & \frac{-\bar{t}}{m^2\bar{u}^2}d_{3m+4}d^{-1}_{m+1}\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix},$$

are $(m+1)\times(m+1)$ matrices. Also,

$$\begin{aligned}&0<d_{m+1}<d_m<\cdots<d_1,\\ &0<d_{2m+2}<d_{2m+1}<\cdots<d_{m+2},\\ &0<d_{3m+3}<d_{3m+2}<\cdots<d_{2m+3},\\ &0<d_{4m+4}<d_{4m+3}<\cdots<d_{3m+4},\end{aligned}$$

implies that

$$
\begin{aligned}
d_2 d_1^{-1} &< 1, \\
d_3 d_2^{-1} &< 1, \\
&\vdots \\
d_{m+1} d_m^{-1} &< 1, \\
d_{m+3} d_{m+2}^{-1} &< 1, \\
&\vdots \\
d_{2m+2} d_{2m+1}^{-1} &< 1, \\
&.
\end{aligned}
$$

and

$$
\begin{aligned}
d_{2m+4} d_{2m+3}^{-1} &< 1, \\
&\vdots \\
d_{3m+3} d_{3m+2}^{-1} &< 1, \\
d_{3m+5} d_{3m+4}^{-1} &< 1, \\
&\vdots \\
d_{4m+4} d_{4m+3}^{-1} &< 1.
\end{aligned}
$$

Moreover, by using (3.9), (3.23) and (3.25), we obtain

$$
\begin{aligned}
\frac{1}{m\bar{t}} + \frac{\bar{u}}{m^2\bar{t}^2} d_1 d_{3m+5}^{-1} + \cdots + \frac{\bar{u}}{m^2\bar{t}^2} d_1 d_{4m+4}^{-1} &= \frac{1}{m\bar{t}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{u}}{m^2\bar{t}^2} \\
&< \frac{1}{m\bar{t}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{u}}{m\bar{t}^2} \\
&< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{t}} + \frac{\bar{u}}{m\bar{t}^2}\Big) \\
&< 1,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{m\bar{w}} + \frac{\bar{v}}{m^2\bar{w}^2} d_{m+2} d_{2m+4}^{-1} + \cdots + \frac{\bar{v}}{m^2\bar{w}^2} d_{m+2} d_{3m+3}^{-1} &= \frac{1}{m\bar{w}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{v}}{m^2\bar{w}^2} \\
&< \frac{1}{m\bar{w}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{v}}{m\bar{w}^2} \\
&< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{w}} + \frac{\bar{v}}{m\bar{w}^2}\Big) \\
&< 1,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{m\bar{v}} + \frac{\bar{w}}{m^2\bar{v}^2} d_{2m+3} d_{m+3}^{-1} + \cdots + \frac{\bar{w}}{m^2\bar{v}^2} d_{2m+3} d_{2m+2}^{-1} &= \frac{1}{m\bar{v}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{w}}{m^2\bar{v}^2} \\
&< \frac{1}{m\bar{v}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{w}}{m\bar{v}^2} \\
&< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{v}} + \frac{\bar{w}}{m\bar{v}^2}\Big) \\
&< 1
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{m\bar{u}} + \frac{\bar{t}}{m^2\bar{u}^2} d_{3m+4} d_2^{-1} + \cdots + \frac{\bar{t}}{m^2\bar{w}^2} d_{3m+4} d_{m+1}^{-1} &= \frac{1}{m\bar{u}} + \Big(\frac{1}{1-2\varepsilon} + \cdots + \frac{1}{1-(m+1)\varepsilon}\Big) \frac{\bar{t}}{m^2\bar{u}^2} \\
&< \frac{1}{m\bar{u}} + \frac{1}{1-(m+1)\varepsilon} \frac{\bar{t}}{m\bar{u}^2} \\
&< \frac{1}{1-(m+1)\varepsilon} \Big(\frac{1}{m\bar{t}} + \frac{\bar{u}}{m\bar{u}^2}\Big) \\
&< 1.
\end{aligned}
$$

Since $\mathcal{P}$ and $D\mathcal{P}D^{-1}$ has the same eigenvalues, for $j \in \{1,2,\dots,4m+4\}$, we can write the following inequality as

$$\max|\mathcal{P}_j| \leq ||D\mathcal{P}D^{-1}||_\infty \quad = \max\left\{\begin{array}{l} d_2d_1^{-1},\dots,d_{m+1}d_m^{-1},d_{m+3}d_{m+2}^{-1},\dots,d_{2m+2}d_{2m+1}^{-1}, \\ d_{2m+4}d_{2m+3}^{-1},\dots,d_{3m+3}d_{3m+2}^{-1},d_{3m+5}d_{3m+4}^{-1},\dots,d_{4m+4}d_{4m+3}^{-1}, \\ \frac{1}{m\bar{t}}+(\frac{1}{1-2\varepsilon}+\dots+\frac{1}{1-(m+1)\varepsilon})\frac{\bar{u}}{m^2\bar{t}^2}, \\ \frac{1}{m\bar{w}}+(\frac{1}{1-2\varepsilon}+\dots+\frac{1}{1-(m+1)\varepsilon})\frac{\bar{v}}{m^2\bar{w}^2}, \\ \frac{1}{m\bar{v}}+(\frac{1}{1-2\varepsilon}+\dots+\frac{1}{1-(m+1)\varepsilon})\frac{\bar{w}}{m^2\bar{v}^2}, \\ \frac{1}{m\bar{u}}+(\frac{1}{1-2\varepsilon}+\dots+\frac{1}{1-(m+1)\varepsilon})\frac{\bar{t}}{m^2\bar{u}^2} \end{array}\right\}$$

$$< 1.$$

Therefore, the equilibrium point given in (3.15) is locally asymptotically stable. □

Theorem 3.5. *If the conditions* (3.9) *and* (3.23) *are satisfied, then the unique equilibrium point given in* (3.15) *of the system* (3.8) *is globally asymptotically stable.*

Theorem 3.6. *Consider the FDEs* (1.2) *for all* $\gamma \in (0,1]$. *If*

$$\frac{1}{m} < \min\{\tau_{1,l},\tau_{1,r},\tau_{2,l},\tau_{2,r}\}, \tag{3.27}$$

then every positive solution (α_n,β_n) *of the FDEs* (1.2) *is bounded and persistent.*

Proof. Let (α_n,β_n) be a solution of (1.2) and satisfy (3.27). Then, we have

$$\begin{cases} [\alpha_n]^\gamma &= [L^\gamma_{\alpha_n},R^\gamma_{\alpha_n}], \quad [\beta_n]^\gamma = [L^\gamma_{\beta_n},R^\gamma_{\beta_n}], \\ [\tau_1]^\gamma &= [\tau^\gamma_{1,l},\tau^\gamma_{1,r}], \quad [\tau_2]^\gamma = [\tau^\gamma_{2,l},\tau^\gamma_{2,r}]. \end{cases} \tag{3.28}$$

From (3.4) and Theorem (3.2), we have

$$\begin{aligned} \tau_{1,l} &\leq L_{\alpha_n} \leq \frac{1}{(m\tau_{2,r})^{n-m}}\left(L_{\alpha_m}-\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{2,r}-1}\right)+\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{2,r}-1}, \\ \tau_{1,r} &\leq R_{\alpha_n} \leq \frac{1}{(m\tau_{2,l})^{n-m}}\left(R_{\alpha_m}-\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{2,l}-1}\right)+\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{2,l}-1}, \\ \tau_{2,l} &\leq L_{\beta_n} \leq \frac{1}{(m\tau_{1,r})^{n-m}}\left(L_{\beta_m}-\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{1,r}-1}\right)+\frac{m\tau_{1,r}\tau_{2,l}}{m\tau_{1,r}-1}, \\ \tau_{2,r} &\leq R_{\beta_n} \leq \frac{1}{(m\tau_{1,l})^{n-m}}\left(R_{\beta_m}-\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{1,l}-1}\right)+\frac{m\tau_{1,l}\tau_{2,r}}{m\tau_{1,l}-1}. \end{aligned} \tag{3.29}$$

Also, for all $\gamma \in (0,1]$, the support sets of τ_1,τ_2 are

$$\begin{aligned} &[\tau^\gamma_{1,l},\tau^\gamma_{1,r}] \subseteq [M_{\tau_1},N_{\tau_1}], \\ &[\tau^\gamma_{2,l},\tau^\gamma_{2,r}] \subseteq [M_{\tau_2},N_{\tau_2}]. \end{aligned} \tag{3.30}$$

Moreover, left and right components of γ - cuts of $M_{\tau_1},N_{\tau_1},M_{\tau_2},N_{\tau_2}$ are positive real numbers. So, by using (3.29) and (3.30), for $\gamma \in (0,1]$, we obtain

$$\begin{aligned} [L^\gamma_{\alpha_n},R^\gamma_{\alpha_n}] &\subseteq \left[M_{\tau_1},\frac{1}{(mN_{\tau_{2,r}})^{n-m}}\left(M_{\alpha_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right], \\ [L^\gamma_{\beta_n},R^\gamma_{\beta_n}] &\subseteq \left[N_{\tau_2},\frac{1}{(mM_{\tau_{1,l}})^{n-m}}\left(N_{\beta_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mM_{\tau_{1,l}}-1}\right]. \end{aligned} \tag{3.31}$$

from which along with there exist m_1,m_2,M_1,M_2 such that $m_1 \leq M_{\tau_1}$,$m_2 \leq N_{\tau_2}$, $\frac{1}{(mN_{\tau_{2,r}})^{n-m}}\left(M_{\alpha_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1} \leq M_1$, and $\frac{1}{(mM_{\tau_{1,l}})^{n-m}}\left(N_{\beta_m}-\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mN_{\tau_{2,r}}-1}\right)+\frac{mM_{\tau_{1,l}}N_{\tau_{2,r}}}{mM_{\tau_{1,l}}-1} \leq M_2$.

Therefore,

$$[L^{\gamma}_{\alpha_{n,l}}, R^{\gamma}_{\alpha_{n,r}}] \subseteq \overline{\bigcup_{\gamma\in(0,1]} [L^{\gamma}_{\alpha_{n,l}}, R^{\gamma}_{\alpha_{n,r}}]} \subseteq [m_1, M_1],$$
$$[L^{\gamma}_{\beta_{n,l}}, R^{\gamma}_{\beta_{n,r}}] \subseteq \overline{\bigcup_{\gamma\in(0,1]} [L^{\gamma}_{\beta_{n,l}}, R^{\gamma}_{\beta_{n,r}}]} \subseteq [m_2, M_2].$$

Hence, every positive solution of FDEs system (1.2) is persistent and bounded. This completes proof. □

Theorem 3.7. *Let us consider the FDEs* (1.2)*. If* (3.27) *holds, then the positive solution* (α_n, β_n) *of* (1.2) *converges to a unique equilibrium point* $(\bar{\alpha}, \bar{\beta})$ *as* $n \to \infty$*, where*

$$\bar{\alpha} = \left[\frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{2,r}-1)}, \frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{2,l}-1)}\right], \bar{\beta} = \left[\frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{1,r}-1)}, \frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{1,l}-1)}\right]. \tag{3.32}$$

Proof. Since (3.2) and (3.3) hold and also from Theorem (3.3), we can write

$$\lim_{n\to\infty} L^{\gamma}_{\alpha_n} = l_{\alpha_n} = \frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{2,r}-1)}, \lim_{n\to\infty} R^{\gamma}_{\alpha_n} = r_{\alpha_n} = \frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{2,l}-1)},$$
$$\lim_{n\to\infty} L^{\gamma}_{\beta_n} = l_{\beta_n} = \frac{m^2\tau_{1,r}\tau_{2,l}-1}{m(m\tau_{1,r}-1)}, \lim_{n\to\infty} R^{\gamma}_{\beta_n} = r_{\beta_n} = \frac{m^2\tau_{1,l}\tau_{2,r}-1}{m(m\tau_{1,l}-1)}.$$

Thus, we get

$$\lim_{n\to\infty} D\big(\alpha_n, \bar{\alpha}\big) = \lim_{n\to\infty} D(\alpha_n - [l_{\alpha_n}, r_{\alpha_n}]) = \lim_{n\to\infty} \sup \max\{|L^{\gamma}_{\alpha_n} - l_{\alpha_n}|, |R^{\gamma}_{\alpha_n} - r_{\alpha_n}|\} = 0,$$
$$\lim_{n\to\infty} D\big(\beta_n, \bar{\beta}\big) = \lim_{n\to\infty} D(\beta_n - [l_{\beta_n}, r_{\beta_n}]) = \lim_{n\to\infty} \sup \max\{|L^{\gamma}_{\beta_n} - l_{\beta_n}|, |R^{\gamma}_{\beta_n} - r_{\beta_n}|\} = 0.$$

So, $\lim_{n\to\infty}\alpha_n = \bar{\alpha}$ and $\lim_{n\to\infty}\beta_n = \bar{\beta}$ means that every positive solution of equation (1.2) converges to equilibrium point $\big(\bar{\alpha}, \bar{\beta}\big)$ as $n \to \infty$. □

4. Numerical Results

In this section we will give some numerical examples in order to verify the efficiency of the results.

Example 4.1. *Consider following system when* $m = 4$ *for system* (1.2)*:*

$$\alpha_{n+1} = \tau_1 + \frac{\alpha_n}{\beta_{n-1}+\beta_{n-2}+\beta_{n-3}+\beta_{n-4}}, \beta_{n+1} = \tau_2 + \frac{\beta_n}{\alpha_{n-1}+\alpha_{n-2}+\alpha_{n-3}+\alpha_{n-4}}.$$

Also the parameters τ_1, τ_2 *and the initial conditions* α_{-i}, β_{-i}*, for* $i = \{0,1,2,3,4\}$*, are triangular fuzzy numbers, respectively,*

$$\tau_1(x) = \begin{cases} \frac{5x}{2}-1, & 0.4 \le x \le 0.8 \\ -\frac{5x}{2}+3, & 0.8 \le x \le 1.2 \end{cases}, \qquad \tau_2(x) = \begin{cases} \frac{10x}{3}-1, & 0.3 \le x \le 0.6 \\ \frac{-10x}{3}+3, & 0.6 \le x \le 0.9 \end{cases}, \tag{4.1}$$

$$\alpha_{-i}(x) = \begin{cases} 5x-\frac{13}{2}, & 1.3 \le x \le 1.5 \\ -5x+\frac{17}{2}, & 1.5 \le x \le 1.7 \end{cases}, \quad \beta_{-i}(x) = \begin{cases} 5x-\frac{13}{2}, & 1.3 \le x \le 1.5 \\ -5x+\frac{17}{2}, & 1.5 \le x \le 1.7 \end{cases}. \tag{4.2}$$

By using (4.1) *and* (4.2)*, the bounded support sets for* $\gamma \in (0,1]$ *are as follows*

$$\begin{cases} supp\tau_1 \subseteq [0.4, 1.2], & supp\tau_2 \subseteq [0.3, 0.9], \\ supp\alpha_{-i} \subseteq [1.3, 1.7], & supp\beta_{-i} \subseteq [1.3, 1.7]. \end{cases} \tag{4.3}$$

This example shows persistence and boundedness of FDEs system (1.2) *if condition* $\frac{1}{m} < \min\{\tau_{1,l}, \tau_{1,r}, \tau_{2,l}, \tau_{2,r}\}$ *is satisfied. Moreover, note that as* $n \to \infty$*, every positive solution of FDEs system* (1.2) *converges to a unique equilibrium point* $(\tilde{\alpha}, \tilde{\beta})$ *in given* (3.32) *as it can be seen in Figure* (1)*. Figure* (2) *shows the attractors of system* (1.2) *for* $\gamma = 0.2$*,* $\gamma = 0.5$*,* $\gamma = 0.8$ *and* $\gamma = 1$*.*

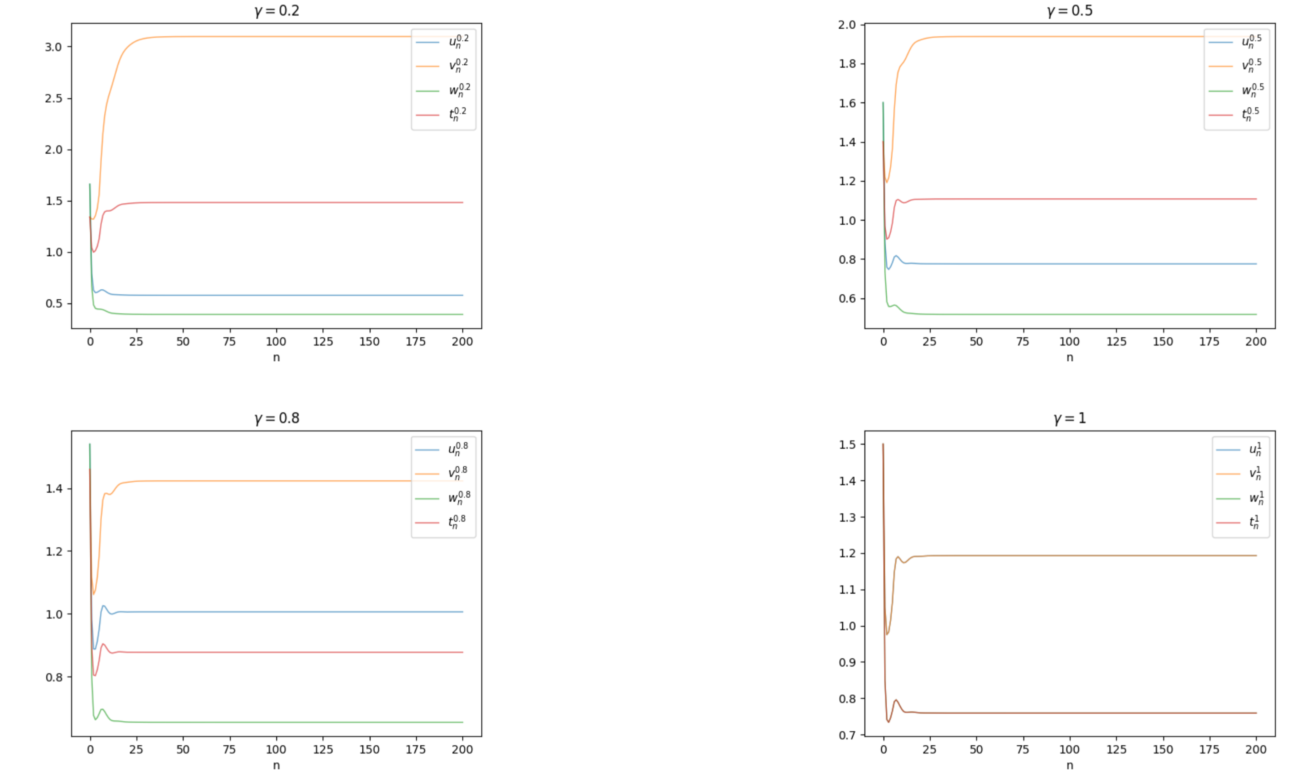

Figure 1: The solution of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

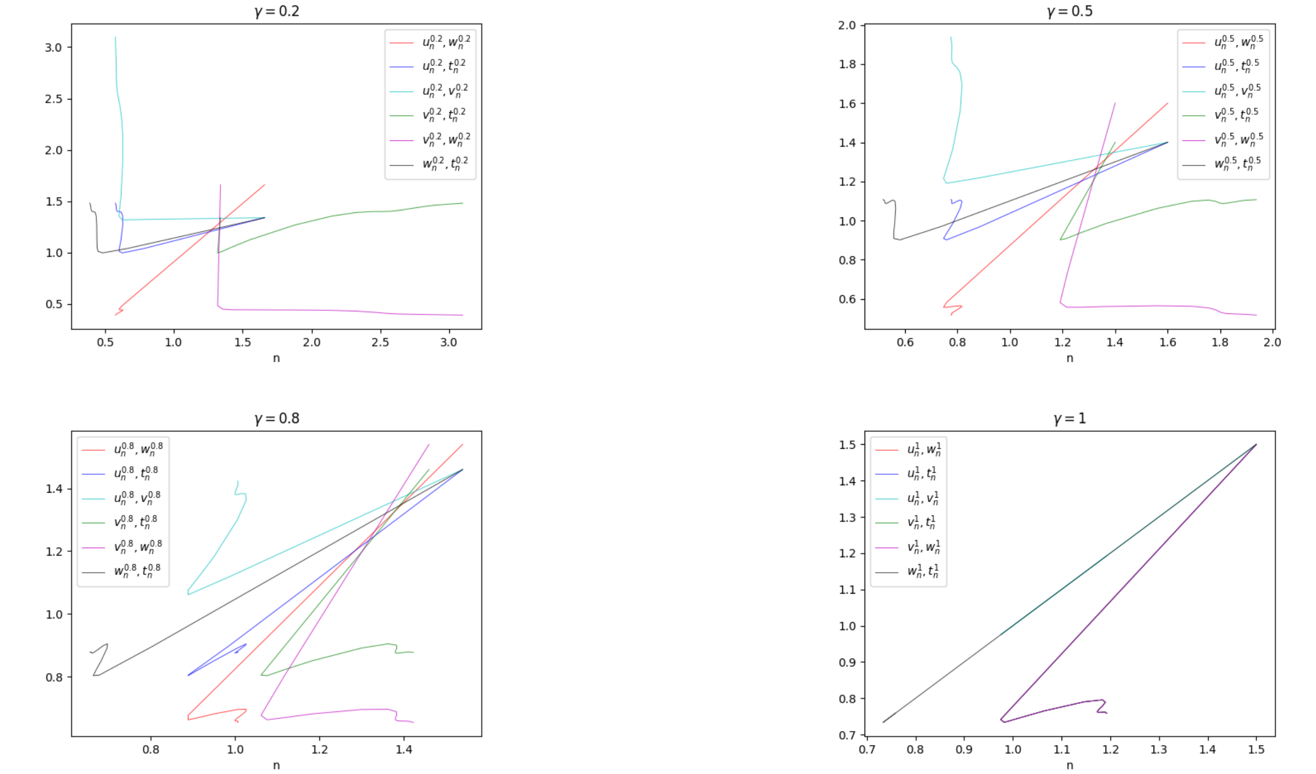

Figure 2: The attractors of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

Example 4.2. *Consider following system when $m = 3$ for* (1.2).

$$\alpha_{n+1} = \tau_1 + \frac{\alpha_n}{\beta_{n-1} + \beta_{n-2} + \beta_{n-3}}, \beta_{n+1} = \tau_2 + \frac{\beta_n}{\alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3}}. \tag{4.4}$$

Also the parameters τ_1, τ_2 *and the initial conditions* α_{-i}, β_{-i}, *for* $i = \{0,1,2,3\}$, *are triangular fuzzy numbers, respectively,*

$$\tau_1(x) = \begin{cases} 5x-3, & 0.6 \le x \le 0.8 \\ -5x+5, & 0.8 \le x \le 1 \end{cases}, \qquad \tau_2(x) = \begin{cases} 5x-2, & 0.4 \le x \le 0.6 \\ -5x+4, & 0.6 \le x \le 0.8 \end{cases}, \tag{4.5}$$

$$\alpha_{-i}(x) = \begin{cases} 5x-\frac{3}{2}, & 0.3 \le x \le 0.5 \\ -5x+\frac{7}{2}, & 0.5 \le x \le 0.7 \end{cases}, \quad \beta_{-i}(x) = \begin{cases} 5x-\frac{3}{2}, & 0.3 \le x \le 0.5 \\ -5x+\frac{7}{2}, & 0.5 \le x \le 0.7 \end{cases}. \tag{4.6}$$

By using (4.5) *and* (4.6), *the bounded support sets for* $\gamma \in (0,1]$ *are as follows*

$$\begin{cases} supp\tau_1 \subseteq [0.6,1], & supp\tau_2 \subseteq [0.4,0.8], \\ supp\alpha_{-i} \subseteq [0.3,0.7], & supp\beta_{-i} \subseteq [0.3,0.7]. \end{cases} \tag{4.7}$$

This example shows persistence and boundedness of FDEs system (1.2) *if condition* $\frac{1}{m} < \min\{\tau_{1,l}, \tau_{1,r}, \tau_{2,l}, \tau_{2,r}\}$ *is satisfied. Moreover, note that as* $n \to \infty$, *every positive solution of FDEs system* (1.2) *converges to a unique equilibrium point* $(\tilde{\alpha}, \tilde{\beta})$ *in given* (3.32) *as it can be seen in Figure* (3). *Figure* (3) *shows the attractors of system* (1.2) *for* $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$ *and* $\gamma = 1$.

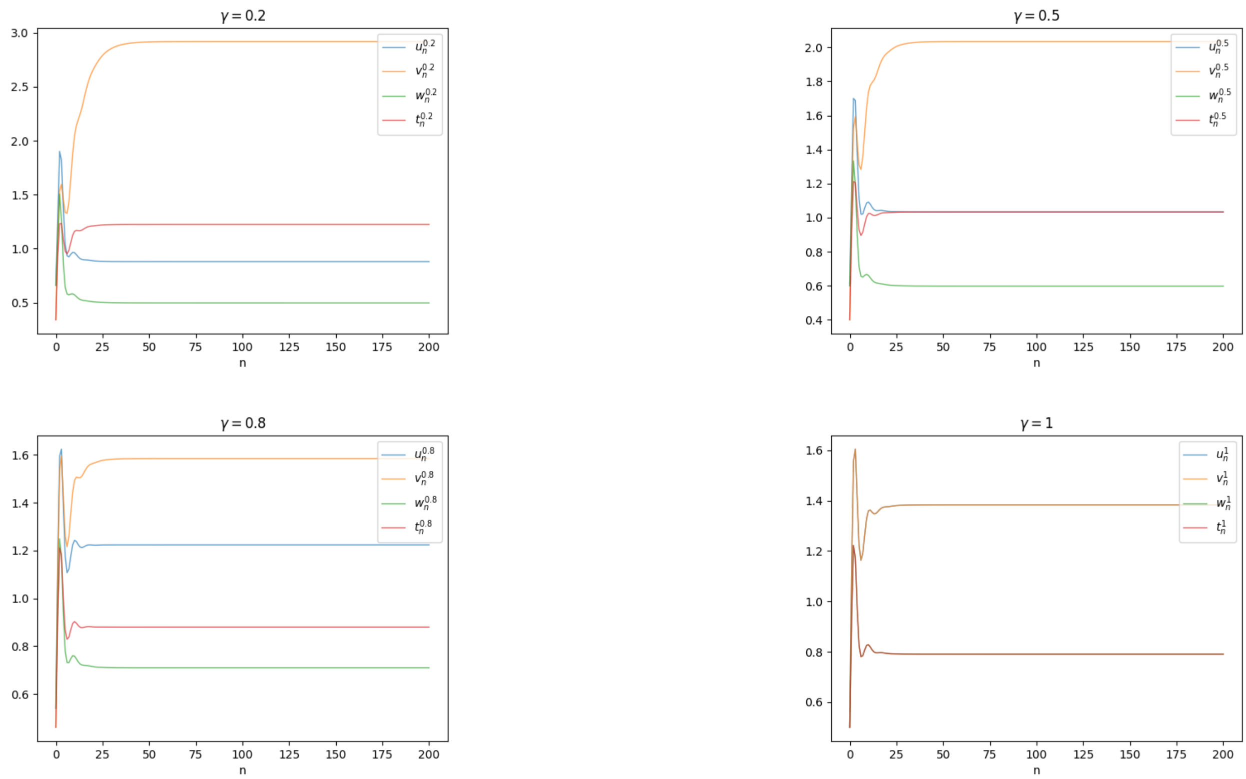

Figure 3: The solution of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

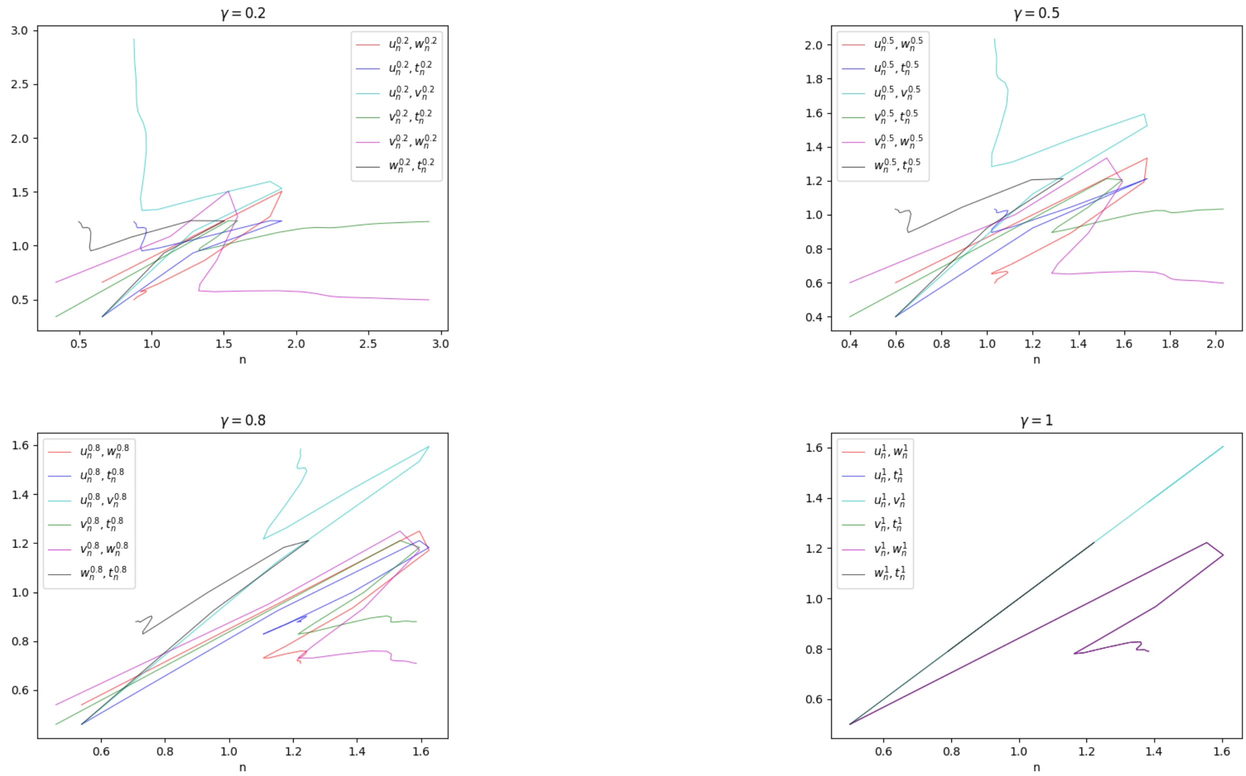

Figure 4: The attractors of FDEs system (1.2) at $\gamma = 0.2$, $\gamma = 0.5$, $\gamma = 0.8$, $\gamma = 1$.

Declarations

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