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On generalized difference $\mathscr{I}$-convergent sequences in neutrosophic n-normed linear spaces

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Abstract. In this article, we delve into the intricate concepts of $\Delta^m\mathscr{I}$-convergence and $\Delta^m\mathscr{I}$-Cauchy sequences within neutrosophic n-normed linear spaces, unveiling several intriguing properties. Our findings establish that every neutrosophic n-normed linear space is $\Delta^m\mathscr{I}$-complete. We also thoroughly investigate the $\Delta^m\mathscr{I}$-limit and $\Delta^m\mathscr{I}$-cluster points of sequences in relation to the neutrosophic n-norm, proving that the set of all $\Delta^m\mathscr{I}$-cluster points forms a closed set under the topology induced by the neutrosophic n-norm. Additionally, we demonstrate that a linear operator preserves $\Delta^m\mathscr{I}$-convergence if and only if it remains continuous with respect to the neutrosophic n-norm.

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Keywords. Neutrosophic n-normed linear space, $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence, $\Delta^m\mathscr{I}(\mathscr{N}_n)$-completeness, $\Delta^m\mathscr{I}^*(\mathscr{N}_n)$-convergence, $\Delta^m\mathscr{I}(\mathscr{N}_n)$-limit point, $\mathscr{N}_n$-continuous mapping.

1. Introduction

Zadeh [51] stands as the pioneering figure behind the groundbreaking introduction of fuzzy set theory, extending the boundaries of classical set theory. Since its inception, it has been continually refined and integrated across various fields of engineering and science. An intriguing extension of fuzzy sets, introduced by Atanassov [1], is known as intuitionistic fuzzy sets, which enhance the traditional fuzzy sets by incorporating a non-membership function alongside the membership function. Over time, the concept of fuzzy set has been fascinatingly expanded into new and innovative notions and the evolution of fuzzy sets has sparked the growth of numerous concepts in mathematical analysis. As a comprehensive generalization of these concepts, Smarandache [45] defined a new idea named as neutrosophic set by introducing the indeterminacy function to the intuitionistic fuzzy sets, i.e., an element of a neutrosophic set is characterized by a triplet: the truth-membership function, the indeterminacy-membership function, and the falsity-membership function. In a neutrosophic set, each element of the universe is defined by its specific degrees of these notions. The concept of fuzzy normed spaces, introduced by Felbin [6] in 1992, evolved over the years with Saadati and Park's [40] introduction of intuitionistic fuzzy normed spaces in 2006, followed by Karakus et al.'s exploration of statistical convergence [18] within these spaces in 2008 and Kumar et al.'s [26] in 2009 generalization to ideal convergence. Recently, Kirişci and Şimşek [21] introduced neutrosophic normed linear spaces and delved into the concept of statistical convergence, sparking further research into different types of sequence convergence within these spaces. The concept of 2-normed linear spaces, and its significant extension to n-normed linear spaces, was first introduced by Gähler [7, 8], sparking considerable interest among researchers. This idea has since been

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further developed by mathematician such as Kim and Cho [20], Malceski [30], Misiak [32], Gunawan and Mashadi [10], contributing to its rich evolution. Combining these two pivotal notions, in 2023, Murtaza et al. [38] introduced the groundbreaking concept of neutrosophic 2-normed linear space, a significant extension of neutrosophic normed space, and explored its statistical convergence and statistical completeness. Recently, Kumar et al. [27] introduced the innovative concept of neutrosophic n-normed linear spaces, exploring their convergence structures and defining Cauchy sequences within this novel framework.

The concept of statistical convergence, a profound generalization of the traditional convergence of real number sequences, was first independently studied by Fast [5], Steinhaus [46], and Schoenberg [43]. Statistical convergence is a generalized form of convergence that has gained significant attention in various fields of mathematics and applied sciences. It is an extension of the classical notion of convergence and is defined using density-based or probabilistic criteria rather than strict adherence to every term of a sequence. It has wide-ranging applications in several fields: topology [29], approximation theory [49], probability [48], measure theory [28] etc. A particularly fascinating extension, known as $\mathscr{I}$-convergence, was later introduced by Kostyrko et al. [25], where $\mathscr{I}$ represents an ideal-a collection of subsets of natural numbers satisfying specific conditions. Since then, this pivotal concept has been explored in various directions by numerous researchers, including [2, 14, 15, 19, 35–37], contributing to its ongoing development. Also, readers seeking a deeper understanding of summability theory, sequence spaces, and their related concepts are encouraged to explore [11, 12, 22, 39, 41, 42, 47, 50].

1.1. **Motivation.** In 1981, Kizmaz [23] introduced the concept of difference sequence spaces, which was later extended to the form of order second and m by Et and Çolak [3] and Malkowsky et al. [31], respectively. Further developments by Hazarika [13], and Gumus and Nuray [9] explored generalized difference sequence spaces of real numbers in the context of ideals. Since then, this intriguing concept has been nurtured by various researchers in diverse frameworks such as Δ^m-statistical convergence [4], difference $\mathscr{I}$-convergent sequences in IFnNS [19], intuitionistic fuzzy $\mathscr{I}$-convergent difference sequence spaces [17], some classes of ideal convergent sequences and generalized difference matrix operator [33], weighted statistical convergence through difference operator of sequences of fuzzy numbers [34]. Research on sequence convergence in neutrosophic n-normed linear spaces is still in its early stages, with limited progress made thus far. However, the studies conducted to date reveal a compelling similarity in the behavior of sequence convergence within these spaces. From the point of view, the study of generalized difference $\mathscr{I}$-convergence in neutrosophic n-normed linear spaces is very natural. So, keeping these facts in mind, within this specific framwork we define and examine the notion of generalized difference $\mathscr{I}$-convergent sequences, linked with generalized difference $\mathscr{I}$-Cauchy sequences, and present some compelling results in relation to neutrosophic n-norm.

2. Preliminaries

In this section, we present an overview of key definitions and terminology essential for describing our main results. Throughout the study, $\mathbb{N}$ will denote the set of all natural numbers.

Definition 1. *[25] A family $\mathscr{I}$ of subsets of a non empty set $\mathscr{X}$ is said to be an ideal in $\mathscr{X}$ if the following conditions hold:*

(1) $\emptyset \in \mathscr{I}$;
(2) $\mathscr{A}, \mathscr{B} \in \mathscr{I}$ implies $\mathscr{A} \cup \mathscr{B} \in \mathscr{I}$;
(3) $\mathscr{A} \in \mathscr{I}$ and $\mathscr{B} \subset \mathscr{A}$ implies $\mathscr{B} \in \mathscr{I}$.

An ideal $\mathscr{I}$ is called non trivial if $\mathscr{X} \notin \mathscr{I}$ and $\mathscr{I} \neq \emptyset$.

Definition 2. *[25] A non trivial ideal $\mathscr{I} \subset 2^{\mathscr{X}}$ is called admissible if $\{\{x\} : x \in X\} \subset \mathscr{I}$.*

Definition 3. *[25] A non empty family $\mathscr{F}$ of subsets of a non empty set $\mathscr{X}$ is called a filter in $\mathscr{X}$ if the following properties hold:*

(1) $\emptyset \notin \mathscr{F}$;
(2) $\mathscr{A}, \mathscr{B} \in \mathscr{F}$ implies $\mathscr{A} \cap \mathscr{B} \in \mathscr{F}$;
(3) $\mathscr{A} \in \mathscr{F}$ and $\mathscr{A} \subset \mathscr{B}$ implies $\mathscr{B} \in \mathscr{F}$.

If $\mathscr{I} \subset 2^{\mathscr{X}}$ is a non trivial ideal then the class $\mathscr{F}(\mathscr{I}) = \{\mathscr{X} \setminus \mathscr{A} : \mathscr{A} \in \mathscr{I}\}$ is a filter on $\mathscr{X}$ which is called filter associated with the ideal $\mathscr{I}$ [25].

Definition 4. *[25] An admissible ideal $\mathscr{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{\mathscr{A}_1, \mathscr{A}_2, \dots\}$ belonging to $\mathscr{I}$ there exists a countable family of sets $\{\mathscr{B}_1, \mathscr{B}_2, \dots\}$ such that the symmetric difference $\mathscr{A}_i \triangle \mathscr{B}_i$ is finite for each $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathscr{B}_i \in \mathscr{I}$.*

Definition 5. *Let $\mathscr{K} \subset \mathbb{N}$. Then, the natural density of $\mathscr{K}$, denoted by $\delta(\mathscr{K})$, is defined as*

$$\delta(\mathscr{K}) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in \mathscr{K}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

Definition 6. *[9] Let $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal. The sequence $w = \{w_k\}$ of real numbers is named to be $\Delta^m\mathscr{I}$-convergent to $\upsilon \in \mathbb{R}$ if for each $\sigma > 0$ the set*

$$\{k \in \mathbb{N} : |\Delta^m w_k - \upsilon| \geq \sigma\}$$

belongs to $\mathscr{I}$, where $m \in \mathbb{N}$, $\Delta^0 w = (w_k)$, $\Delta w = w_k - w_{k+1}$, $\Delta^m w = (\Delta^m w_k) = (\Delta^{m-1} w_k - \Delta^{m-1} w_{k+1})$ and so that

$$\Delta^m w_k = \sum_{j=0}^{m} (-1)^j \binom{m}{j} w_{k+j}.$$

In this scenario, it is denoted as $\mathscr{I} - \lim \Delta^m w_k = \upsilon$.

Definition 7. *[9] Let $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal. If $\{k+1 : k \in \mathscr{A}\} \in \mathscr{I}$, for any $\mathscr{A} \in \mathscr{I}$, then $\mathscr{I}$ is named to be a translation invariant ideal.*

If we take a non-trivial ideal $\mathscr{I}$ as the collection of all subsets of $\mathbb{N}$ whose natural density is zero, then $\mathscr{I}$ becomes a translation invariant ideal [9].

Definition 8. *[44] A binary operation $\circledast : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0, 1]$ is named to be a continuous t-norm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathscr{J}$, the below conditions hold:*

(1) $\boxast$ is associative and commutative;
(2) $\boxast$ is continuous;
(3) $\nu_1 \boxast 1 = \nu_1$ for all $\nu_1 \in \mathscr{J}$;
(4) $\nu_1 \boxast \nu_2 \leq \nu_3 \boxast \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Definition 9. *[44] A binary operation $\circledast : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0, 1]$ is named to be a continuous t-conorm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathscr{J}$, the below conditions hold:*

(1) $\circledast$ is associative and commutative;
(2) $\circledast$ is continuous;
(3) $\nu_1 \circledast 0 = \nu_1$ for all $\nu_1 \in \mathscr{J}$;
(4) $\nu_1 \circledast \nu_2 \leq \nu_3 \circledast \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Example 1. *[24] The continuous t-norms are $\nu_1 \boxast \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \boxast \nu_2 = \nu_1.\nu_2$. On the other hand, continuous t-conorms are $\nu_1 \circledast \nu_2 = \max\{\nu_1, \nu_2\}$ and $\nu_1 \circledast \nu_2 = \nu_1 + \nu_2 - \nu_1.\nu_2$.*

Definition 10. *[10] Let $n \in \mathbb{N}$ and $\mathcal{W}$ be a real vector space having dimension $d \geq n$ (d is finite or infinite). A real valued function $\|\cdot, \dots, \cdot\|$ on $\underbrace{\mathcal{W} \times \mathcal{W} \times \ldots \times \mathcal{W}}_{n \text{ times}} = \mathcal{W}^n$, gratifying the below four axioms:*

(1) $\|\kappa_1, \kappa_2, \dots, \kappa_n\| = 0$ if and only if $\kappa_1, \kappa_2, \dots, \kappa_n$ are linearly dependent;
(2) $\|\kappa_1, \kappa_2, \dots, \kappa_n\|$ remains invariant under any permutation of $\kappa_1, \kappa_2, \dots, \kappa_n$;
(3) $\|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \alpha\kappa_n\| = |\alpha| \, \|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \kappa_n\|$ for $\alpha \in \mathbb{R}$ (set of real numbers);
(4) $\|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \tau + \omega\| \leq \|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \tau\| + \|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \omega\|$,

is called an n-norm on $\mathcal{W}$ and the pair $(\mathcal{W}, \|\cdot, \dots, \cdot\|)$ is named to be an n-normed linear space.

As an illustration of n-normed linear space we take $\mathcal{W} = \mathbb{R}^n$ equipped with the Euclidean norm

$$\|w_1, w_2, \dots, w_n\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix}\right)$$

where $w_i = (w_{i1}, w_{i2}, \dots, w_{in}) \in \mathbb{R}^n$. For instance, we get $\|w_1, w_2, \dots, w_n\| \geq 0$ in an n-normed linear space.

Definition 11. *[27] Let $\mathcal{W}$ be a vector space over $\mathscr{H}$ and $\circledast$ and $\circledast$ be continuous t-norm and t-conorm respectively. Let $\Upsilon, \Re, \Psi$ be the functions from $\mathcal{W}^n \times (0,\infty)$ to $[0,1]$. Then, a six tuple $(\mathcal{W}, \Upsilon, \Re, \Psi, \boxast, \circledast)$ is named to be a neutrosophic n-normed linear space (in short Nn-NLS), $(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) \in \mathcal{W}^n \times (0,\infty) \to [0,1]$, if the below conditions hold:*

(1) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) + \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) + \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) \leq 3$;
(2) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) > 0$;
(3) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 1$ *if and only if* w_j *are linearly dependent,* $1 \leq j \leq n$;
(4) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta)$ *is invariant under any permutation of* $\kappa_1, \kappa_2, \ldots, w_n$;
(5) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \kappa w_n; \zeta) = \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ *and* $\kappa \in \mathscr{F}$;
(6) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n + w_n'; \zeta + \tau) \geq \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) \boxast \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n'; \tau)$;
(7) $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta)$ *is non-decreasing continuous in* ζ;
(8) $\lim_{\zeta\to\infty} \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 1$ *and* $\lim_{\zeta\to 0} \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 0$;
(9) $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) > 0$;
(10) $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 0$ *if and only if* w_j *are linearly dependent,* $1 \leq j \leq n$;
(11) $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta)$ *is invariant under any permutation of* $\kappa_1, \kappa_2, \ldots, w_n$;
(12) $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \kappa w_n; \zeta) = \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ *and* $\kappa \in \mathscr{F}$;
(13) $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n + w_n'; \zeta + \tau) \leq \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) \circledast \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n'; \tau)$;
(14) $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta)$ *is non-increasing continuous in* ζ;
(15) $\lim_{\zeta\to\infty} \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 0$ *and* $\lim_{\zeta\to 0} \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 1$;
(16) $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) > 0$;
(17) $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 0$ *if and only if* w_j *are linearly dependent,* $1 \leq j \leq n$;
(18) $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta)$ *is invariant under any permutation of* $\kappa_1, \kappa_2, \ldots, w_n$;
(19) $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \kappa w_n; \zeta) = \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \frac{\zeta}{|\kappa|})$, $\kappa \neq 0$ *and* $\kappa \in \mathscr{F}$;
(20) $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n + w_n'; \zeta + \tau) \leq \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) \circledast \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n'; \tau)$;
(21) $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta)$ *is non-increasing continuous in* ζ;
(22) $\lim_{\zeta\to\infty} \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 0$ *and* $\lim_{\zeta\to 0} \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = 1$.

In the sequal, we shall use the notation $\mathcal{X}$ for neutrosophic n-normed linear space instead of $(\mathcal{W}, \Upsilon, \Re, \Psi, \boxast, \circledast)$ and we denote $\mathscr{N}_n$ to mean neutrosophic n-norm on $\mathcal{X}$.

Example 2. *[27] Let $(\mathcal{W}, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space. Also, let $\nu_1 \boxast \nu_2 = \min(\nu_1, \nu_2)$ and $\nu_1 \circledast \nu_2 = \max(\nu_1, \nu_2)$ for every $\nu_1, \nu_2 \in [0,1]$. If we define $\Upsilon, \Re$ and Ψ as*

$$\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = \frac{\zeta}{\zeta + \|\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n\|}$$
$$\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = \frac{\|\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n\|}{\zeta + \|\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n\|}$$
$$\text{and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n; \zeta) = \frac{\|\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_n\|}{\zeta}.$$

Then, $(\mathcal{W}, \Upsilon, \Re, \Psi, \boxast, \circledast)$ is a neutrosophic n-normed linear space.

Definition 12. *[27] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\{w_k\}$ is named to be convergent to $\upsilon \in \mathcal{W}$ with respect to $\mathscr{N}_n$ if for every $\sigma > 0, \zeta > 0$ and $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$ there exists $k_0 \in \mathbb{N}$ such that $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma$ and $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_k - \upsilon; \zeta) < \sigma$, $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_k - \upsilon; \zeta) < \sigma$ for all $k \geq k_0$. In this scenario, it is denoted as $\mathscr{N}_n - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathscr{N}_n} \upsilon$.*

Definition 13. *[27] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\{w_k\}$ is named to be Cauchy sequence with respect to $\mathscr{N}_n$ if for every $\sigma > 0, \zeta > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_k - w_m; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_k - w_m; \zeta) < \sigma,$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, w_k - w_m; \zeta) < \sigma \text{ for all } k, m \geq k_0.$$

Theorem 1. *[27] Let $\{w_k\}$ be a sequence in $\mathcal{W}$. Then, $\{w_k\}$ is convergent in $(\mathcal{W}, \|\cdot, \ldots, \cdot\|)$ iff $\{w_k\}$ is convergent in $\mathcal{X}$ with respect to neutrosophic n-norm as defined in Example 2.*

3. $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Convergence

In this section we define $\Delta^m\mathscr{I}$-convergence and $\Delta^m\mathscr{I}$-Cauchy sequence with respect to neutrosophic n-norm and develop some of their interesting properties.

Definition 14. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\{w_k\}$ is named to be Δ^m-convergent to $\upsilon \in \mathcal{W}$ with respect to $\mathscr{N}_n$ if for every $\sigma \in (0,1)$, $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$ there exists $k_0 \in \mathbb{N}$ such that*

$$\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma,$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma \text{ for all } k \geq k_0.$$

In this scenario, it is denoted as $\mathscr{N}_n - \lim \Delta^m w_k = \upsilon$ or $w_k \xrightarrow{\Delta^m(\mathscr{N}_n)} \upsilon$.

Definition 15. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\{w_k\}$ is named to be Δ^m-Cauchy sequence with respect to $\mathscr{N}_n$ if for every $\sigma \in (0,1)$, $\zeta > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_i; \zeta) > 1 - \sigma \text{ and}$$
$$\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_i; \zeta) < \sigma,$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_i; \zeta) < \sigma,$$

for all $k, i \geq k_0$.

Definition 16. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$ and $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then, $\{w_k\}$ is named to be generalized difference $\mathscr{I}$-convergent to $\upsilon \in \mathcal{W}$ with respect to $\mathscr{N}_n$ (named as $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence) if for every $\sigma \in (0,1), \zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$ such that*

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma \text{ and}$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I}.$$

In this scenario, it is denoted as $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$ or $w_k \xrightarrow{\Delta^m\mathscr{I}(\mathscr{N}_n)} \upsilon$. And, υ is called $\Delta^m\mathscr{I}(\mathscr{N}_n)$-limit of $\{w_k\}$.

Example 3. *Let $\mathcal{W} = \mathbb{R}^n$ with*

$$\|w_1, w_2, \ldots, w_n\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix}\right)$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \circledast \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \circledast \nu_2 = \max\{\nu_1, \nu_2\}$ for every $\nu_1, \nu_2 \in [0,1]$. We take Nn-NLS as defined in Example 2. Let $\mathscr{I}$ be a class of subsets of $\mathbb{N}$ such that natural density of each subset is zero. Then, $\mathscr{I}$ becomes a nontrivial admissible ideal. Now, we define a sequence $\{w_k\} \in \mathcal{W}$ by

$$\Delta^m w_k = \begin{cases} (1, 0, \ldots, 0) = \mathbf{1}, & \text{if } k = i^2,\ i \in \mathbb{N} \\ (0, 0, \ldots, 0) = \mathbf{0}, & \text{otherwise} \end{cases}.$$

Then for any $\sigma \in (0,1)$ and $\zeta > 0$, we have

$$\begin{aligned} &\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k; \zeta) \\ &\geq \sigma \text{and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k; \zeta) \geq \sigma\} \\ &= \{k \in \mathbb{N} : \|\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k\| \geq \frac{\zeta\sigma}{1-\sigma} > 0 \text{ or } \|\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k\| \\ &\geq \zeta\sigma > 0\} \subseteq \{k \in \mathbb{N} : k = i^2,\ i \in \mathbb{N}\}. \end{aligned}$$

Since $\delta(\{k \in \mathbb{N} : k = i^2,\ i \in \mathbb{N}\}) = 0$, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \mathbf{0}$.

From Definition 16, we can easily prove the following lemma.

Lemma 1. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, for every $\sigma \in (0,1), \zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$, the below properties are gratified:*

(1) $\mathscr{I}(\mathscr{N}_n)-\lim \Delta^m w_k = \upsilon$;
(2) $\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)\le 1-\sigma\}\in\mathscr{I}$, $\{k\in\mathbb{N}:\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)\ge\sigma\}\in\mathscr{I}$ and $\{k\in\mathbb{N}:\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)\ge\sigma\}\in\mathscr{I}$;
(3) $\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)>1-\sigma$ and $\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)<\sigma,\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)<\sigma\}\in\mathscr{F}(\mathscr{I})$;
(4) $\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)>1-\sigma\}\in\mathscr{F}(\mathscr{I})$, $\{k\in\mathbb{N}:\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)<\sigma\}\in\mathscr{F}(\mathscr{I})$ and $\{k\in\mathbb{N}:\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)<\sigma\}\in\mathscr{F}(\mathscr{I})$;
(5) $\mathscr{I}-\lim\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)=1$, $\mathscr{I}-\lim\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)=0$ and $\mathscr{I}-\lim\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta)=0$.

Theorem 2. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\{w_k\}$ is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergent, $\Delta^m\mathscr{I}(\mathscr{N}_n)$-limit of $\{w_k\}$ is unique.*

Proof. If possible, let $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon_1$ and $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon_2$ where $\upsilon_1\ne\upsilon_2$. Let $\sigma\in(0,1)$ be given. Choose $\varpi\in(0,1)$ such that $(1-\varpi)\circledast(1-\varpi)>1-\sigma$ and $\varpi\circledast\varpi<\sigma$. For any $\zeta>0$ and nonzero $\kappa_1,\kappa_2,\ldots,\kappa_{n-1}\in\mathcal{W}$, we define

$$\mathscr{M}_{\Upsilon,1}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Upsilon\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_1;\frac{\zeta}{2}\right)\le 1-\varpi\right\};$$
$$\mathscr{B}_{\Upsilon,2}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Upsilon\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_2;\frac{\zeta}{2}\right)\le 1-\varpi\right\};$$
$$\mathscr{M}_{\Re,1}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Re\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_1;\frac{\zeta}{2}\right)\ge\varpi\right\};$$
$$\mathscr{B}_{\Re,2}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Re\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_2;\frac{\zeta}{2}\right)\ge\varpi\right\};$$
$$\mathscr{M}_{\Psi,1}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Psi\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_1;\frac{\zeta}{2}\right)\ge\varpi\right\};$$
$$\mathscr{B}_{\Psi,2}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Psi\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_2;\frac{\zeta}{2}\right)\ge\varpi\right\}.$$

Since $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon_1$, by Lemma 1,

$$\mathscr{M}_{\Upsilon,1}(\varpi,\zeta),\mathscr{M}_{\Re,1}(\varpi,\zeta)\text{ and }\mathscr{M}_{\Psi,1}(\varpi,\zeta)\in\mathscr{I}.$$

Again, as $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon_2$,

$$\mathscr{B}_{\Upsilon,2}(\varpi,\zeta),\mathscr{B}_{\Re,2}(\varpi,\zeta)\text{ and }\mathscr{B}_{\Psi,2}(\varpi,\zeta)\in\mathscr{I}.$$

Let $\mathscr{A}_{(\Upsilon,\Re,\Psi)}(\sigma,\zeta)=[\mathscr{M}_{\Upsilon,1}(\varpi,\zeta)\cup\mathscr{B}_{\Upsilon,2}(\varpi,\zeta)]\cap[\mathscr{M}_{\Re,1}(\varpi,\zeta)\cup\mathscr{B}_{\Re,2}(\varpi,\zeta)]$
$\cap[\mathscr{M}_{\Psi,1}(\varpi,\zeta)\cup\mathscr{B}_{\Psi,2}(\varpi,\zeta)]$. Then $\mathscr{A}_{(\Upsilon,\Re,\Psi)}(\sigma,\zeta)\in\mathscr{I}$. Obviously $\mathbb{N}\setminus\mathscr{A}_{(\Upsilon,\Re,\Psi)}(\sigma,\zeta)\in\mathscr{F}(\mathscr{I})$. So, let $k\in\mathbb{N}\setminus\mathscr{A}_{(\Upsilon,\Re,\Psi)}(\sigma,\zeta)$. So, there are three possible cases to be considered.

Case-1 : If $k\in\mathbb{N}\setminus(\mathscr{M}_{\Upsilon,1}(\varpi,\zeta)\cup\mathscr{B}_{\Upsilon,2}(\varpi,\zeta))$, then we have

$$\begin{aligned}&\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\upsilon_1-\upsilon_2;\zeta)\\&\ge\Upsilon\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_1;\frac{\zeta}{2}\right)\circledast\Upsilon\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_2;\frac{\zeta}{2}\right)\\&>(1-\varpi)\circledast(1-\varpi)>1-\sigma.\end{aligned}$$

Since $\sigma>0$ is arbitrary, $\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\upsilon_1-\upsilon_2;\zeta)=1$ for all $\zeta>0$. Hence $\upsilon_1=\upsilon_2$.

Case-2 : If $k\in\mathbb{N}\setminus(\mathscr{M}_{\Re,1}(\varpi,\zeta)\cup\mathscr{B}_{\Re,2}(\varpi,\zeta))$, then we can have

$$\begin{aligned}&\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\upsilon_1-\upsilon_2;\zeta)\\&\le\Re\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_1;\frac{\zeta}{2}\right)\circledast\Re\left(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon_2;\frac{\zeta}{2}\right)\\&<\varpi\circledast\varpi<\sigma.\end{aligned}$$

Since $\sigma>0$ is arbitrary, $\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\upsilon_1-\upsilon_2;\zeta)=0$ for all $\zeta>0$. Hence $\upsilon_1=\upsilon_2$.

Case-3 : If $k\in\mathbb{N}\setminus(\mathscr{M}_{\Psi,1}(\varpi,\zeta)\cup\mathscr{B}_{\Psi,2}(\varpi,\zeta))$, then by similar technique as above we can arrive at $\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\upsilon_1-\upsilon_2;\zeta)=0$ i.e., $\upsilon_1=\upsilon_2$.

In all the above three cases, we get $\upsilon_1 = \upsilon_2$, which is a contradiction. Hence $\Delta^m \mathscr{I}(\mathscr{N}_n)$-limit of $\{w_k\}$ is unique. □

Now, we give two interesting results based on the translation invariant ideal.

Proposition 1. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon \implies \mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_{k+1} = \upsilon$ under the condition $\mathscr{I}$ is a translation invariant ideal.*

Proof. Let $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. Then

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma\}.$$

Since $\mathscr{I}$ is a translation invariant ideal,

$$\{k+1 \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_{k+1} - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_{k+1} - \upsilon; \zeta) \geq \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_{k+1} - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I},$$

i.e., $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_{k+1} = \upsilon$. □

Proposition 2. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^{m-1} w_k = \upsilon \implies \mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$ under the condition $\mathscr{I}$ is an admissible translation invariant ideal.*

Proof. Let $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^{m-1} w_k = \upsilon$. Since $\mathscr{I}$ is a translation invariant ideal, by Proposition 1 we have $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^{m-1} w_{k+1} = \upsilon$. Since $(\Delta^m w_k) = (\Delta^{m-1} w_k - \Delta^{m-1} w_{k+1})$, therefore $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. □

Theorem 3. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\mathscr{N}_n - \lim \Delta^m w_k = \upsilon$ then $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$.*

Proof. Let $\mathscr{N}_n - \lim \Delta^m w_k = \upsilon$. Then, for every $\sigma \in (0, 1)$, $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \dots, \kappa_{n-1} \in \mathcal{W}$ there exists $k_0 \in \mathbb{N}$ such that

$$\Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma, \quad \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma$$

for all $k \geq k_0$. Therefore, it is immediate that the set

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma\}$$

is finite. Since $\mathscr{I}$ is an admissible ideal, hence

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I}.$$

So, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. □

But, in general, the converse of the above theorem need not be true which can be illustrated as given below.

Example 4. *Let $\mathcal{W} = \mathbb{R}^n$ with*

$$\|w_1, w_2, \dots, w_n\| = abs \left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix} \right)$$

where $w_i = (w_{i1}, w_{i2}, \dots, w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \circledast \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \circledast \nu_2 = \max\{\nu_1, \nu_2\}$ for every $\nu_1, \nu_2 \in [0, 1]$. We take Nn-NLS as defined in Example 2. Let $\mathscr{I}$ be a class of subsets of $\mathbb{N}$ such that natural density of each subset is zero. Then, $\mathscr{I}$ becomes a nontrivial admissible ideal. Now, we define a sequence $\{w_k\} \in \mathcal{W}$ by

$$\Delta^m w_k = \begin{cases} (k, 0, \dots, 0), & \text{if } k = i^2, \ i \in \mathbb{N} \\ (0, 0, \dots, 0) = \boldsymbol{0}, & \text{otherwise} \end{cases}.$$

Then we obtain $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \mathbf{0}$, *but* $w_k \xrightarrow{\Delta^m(\mathscr{N}_n)}\!\!\!\!\!\!\!/\;\; \mathbf{0}$ *as it is not convergent in* $\mathbb{R}^n$, *so from Theorem 1, we have it is not convergent in* $\mathcal{W}$ *with respect to* $\mathscr{N}_n$.

Theorem 4. *Let* $\mathcal{W}$ *be a real vector space,* $\{w_k\}$ *and* $\{l_k\}$ *be two sequences in a Nn-NLS* $\mathcal{X}$*. Then, the below statements hold good:*

(1) *If* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon_1$ *and* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m l_k = \upsilon_2$, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m (w_k + l_k) = \upsilon_1 + \upsilon_2$;
(2) *If* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m (\kappa w_k) = \kappa\upsilon$, $\kappa \neq 0$;
(3) *If* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon_1$ *and* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m l_k = \upsilon_2$, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m (w_k - l_k) = \upsilon_1 - \upsilon_2$.

Proof. It is straightforward. So, we omit details. □

Theorem 5. *Let* $\{w_k\}$ *be a sequence in a Nn-NLS* $\mathcal{X}$*. Then,* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$ *if and only if there exists a* $\mathscr{C} \subseteq \mathbb{N}$ *such that* $\mathscr{C} \in \mathscr{F}$ *and*

$$\mathscr{N}_n - \lim_{\substack{k\to\infty \\ k\in\mathscr{C}}} \Delta^m w_k = \upsilon.$$

Proof. First, suppose that $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. Then, for any $\zeta > 0, q \in \mathbb{N}$ and for all nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$,

$$\mathscr{Y}_{\mathscr{N}_n}(q,\zeta) = \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) > 1 - \frac{1}{q} \text{ and } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \frac{1}{q},$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \frac{1}{q}\} \in \mathscr{F}(\mathscr{I})$$

and

$$\mathscr{T}_{\mathscr{N}_n}(q,\zeta) = \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \frac{1}{q} \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \frac{1}{q},$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \frac{1}{q}\} \in \mathscr{I}.$$

We observe $\mathscr{Y}_{\mathscr{N}_n}(q+1,\zeta) \subseteq \mathscr{Y}_{\mathscr{N}_n}(q,\zeta)$. Now, we shall show that

$$\text{for } k \in \mathscr{Y}_{\mathscr{N}_n}(q,\zeta),\ \mathscr{N}_n - \lim_{k\to\infty} \Delta^m w_k = \upsilon.$$

Suppose that $w_k \xrightarrow{\Delta^m(\mathscr{N}_n)}\!\!\!\!\!\!\!/\;\; \upsilon$. Then for some $\sigma \in (0,1)$, $\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma$, $\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma$ and $\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma$ holds good except at most finite number of terms $k \in \mathscr{Y}_{\mathscr{N}_n}(q,\zeta)$. Let

$$\mathscr{D}_{\mathscr{N}_n}(\sigma,\zeta) = \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma,$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma\},$$

where $\sigma > \frac{1}{q}$. Then $\mathscr{D}_{\mathscr{N}_n}(\sigma,\zeta) \in \mathscr{I}$ as $\mathscr{I}$ is an admissible. As, $\sigma > \frac{1}{q}$, $\mathscr{Y}_{\mathscr{N}_n}(q,\zeta) \subseteq \mathscr{D}_{\mathscr{N}_n}(\sigma,\zeta)$ and hence $\mathscr{Y}_{\mathscr{N}_n}(q,\zeta) \in \mathscr{I}$ which contradicts $\mathscr{Y}_{\mathscr{N}_n}(q,\zeta) \in \mathscr{F}(\mathscr{I})$. Therefore, for $k \in \mathscr{Y}_{\mathscr{N}_n}(q,\zeta)$, $\mathscr{N}_n - \lim_{k\to\infty} \Delta^m w_k = \upsilon$.

Conversely suppose that there exists a $\mathscr{C} \subseteq \mathbb{N}$ such that $\mathscr{C} \in \mathscr{F}(\mathscr{I})$ and

$$\mathscr{N}_n - \lim_{\substack{k\to\infty \\ k\in\mathscr{C}}} \Delta^m w_k = \upsilon.$$

Then for every $\sigma \in (0,1), \zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$ there exists $k_0 \in \mathbb{N}$ such that

$$\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma,$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \sigma \text{ for all } k \geq k_0,\ k \in \mathscr{C}.$$

Thus

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma \text{ and}$$
$$\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I},$$

since $\mathscr{I}$ is an admissible. Hence $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. This completes the proof. □

Theorem 6. *Let* $\{w_k\}$ *and* $\{l_k\}$ *be two sequences in a Nn-NLS* $\mathcal{X}$ *such that* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m l_k = \upsilon$ *and* $\{k \in \mathbb{N} : \Delta^m l_k \neq \Delta^m w_k\} \in \mathscr{I}$*. Then,* $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$.

Proof. Suppose $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m l_k = \upsilon$. Then, for every $\sigma \in (0,1), \zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$, the set $\mathscr{A} \in \mathscr{I}$ where

$$\mathscr{A} = \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m l_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m l_k - \upsilon; \zeta) \ge \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m l_k - \upsilon; \zeta) \ge \sigma\}.$$

It is clear that

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma\} \subseteq \mathscr{A} \cup \{k \in \mathbb{N} : \Delta^m l_k \ne \Delta^m w_k\}.$$

By the condition,

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma\} \in \mathscr{I},$$

i.e., $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. □

4. $\Delta^m \mathscr{I}^*(\mathscr{N}_n)$-Convergence

Now, we proceed with the notion of $\Delta^m \mathscr{I}^*$-convergence in a neutrosophic n-normed linear space $\mathcal{X}$.

Definition 17. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\{w_k\}$ is named to be generalized difference $\mathscr{I}^*$-convergent to $\upsilon \in \mathcal{W}$ with regards to $\mathscr{N}_n$ (in short $\Delta^m \mathscr{I}^*(\mathscr{N}_n)$-convergence) if there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p \to \infty} \Delta^m w_{k_p} = \upsilon$. In this case, we write $\mathscr{I}^*(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$ or $w_k \xrightarrow{\Delta^m \mathscr{I}^*(\mathscr{N}_n)} \upsilon$ and υ is called $\Delta^m \mathscr{I}^*(\mathscr{N}_n)$-limit of $\{w_k\}$.*

We establish the connection between $\Delta^m \mathscr{I}^*(\mathscr{N}_n)$ and $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergence.

Theorem 7. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\mathscr{I}^*(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$, $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$.*

Proof. Since $\mathscr{I}^*(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$, there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p \to \infty} \Delta^m w_{k_p} = \upsilon$ i.e., for every $\sigma \in (0,1)$, $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$ there exists $p_0 \in \mathbb{N}$ such that

$$\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_p} - \upsilon; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_p} - \upsilon; \zeta) < \sigma, \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_p} - \upsilon; \zeta) < \sigma \text{ for all } p \ge p_0.$$

So,

$$\{k_p \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_p} - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_p} - \upsilon; \zeta) \ge \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_p} - \upsilon; \zeta) \ge \sigma\} \subseteq \{k_1, k_2, \ldots, k_{p_0 - 1}\}.$$

Let $\mathscr{G} = \mathbb{N} \setminus \mathscr{K}$. Then,

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma\} \subseteq \mathscr{G} \cup \{k_1, k_2, \ldots, k_{p_0 - 1}\}.$$

Since $\mathscr{I}$ is an admissible ideal,

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \le 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \ge \sigma\} \in \mathscr{I}.$$

This shows that $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. □

In general, the converse of the above Theorem need not be true which can illustrated by the following example.

Example 5. *Let $\mathcal{W} = \mathbb{R}^n$ with*

$$\|w_1, w_2, \ldots, w_n\| = abs \left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix} \right)$$

where $w_i = (w_{i1}, w_{i2}, \dots, w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \circledast \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \circledast \nu_2 = \max\{\nu_1, \nu_2\}$ for every $\nu_1, \nu_2 \in [0,1]$. We consider the neutrosophic n-normed linear space defined as in Example 2. Let $\mathbb{N} = \bigcup_i \mathscr{D}_i$ be a decomposition of $\mathbb{N}$ such that for any $r \in \mathbb{N}$ each $\mathscr{D}_i$ contains infinitely many $i's$ where $i \geq r$ and $\mathscr{D}_i \cap \mathscr{D}_r = \emptyset$ whenever $i \neq r$. Let $\mathscr{I}$ be the class of all subsets of $\mathbb{N}$ which intersects only a finite number of $\mathscr{D}_i's$. Then, $\mathscr{I}$ becomes a non trivial admissible ideal of $\mathbb{N}$. Now we define a sequence $\{w_k\} \in \mathcal{W}$ by $\Delta^m w_k = (\frac{1}{k}, 0, \dots, 0) \in \mathbb{R}^n$ if $k \in \mathscr{D}_k$. Let $\boldsymbol{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$. Then for $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \dots, \kappa_{n-1} \in \mathcal{W}$, we have

$$\Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k; \zeta) = \frac{\zeta}{\zeta + \|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k\|} \to 1,$$
$$\Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k; \zeta) = \frac{\|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k\|}{\zeta + \|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k\|} \to 0,$$
$$\text{and } \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k; \zeta) = \frac{\|\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k\|}{\zeta} \to 0$$

as $k \to \infty$. Since $\mathscr{I}$ is an admissible ideal, therefore $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \boldsymbol{0}$.

Now, if possible, let $\mathscr{I}^*(\mathscr{N}_n) - \lim \Delta^m w_k = \boldsymbol{0}$. Then, there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p\to\infty} \Delta^m w_{k_p} = \boldsymbol{0}$. Since $\mathscr{K} \in \mathscr{F}(\mathscr{I})$, there is $\mathscr{G} \in \mathscr{I}$ such that $\mathbb{N} \setminus \mathscr{K} = \mathscr{G}$. Now by the construction of $\mathscr{I}$, there is $j \in \mathbb{N}$ such that $\mathscr{G} \subset \bigcup_{i=1}^{j} \mathscr{D}_i$. But then $\mathscr{D}_{j+1} \subset \mathscr{K}$ and therefore $\Delta^m w_{k_p} = (\frac{1}{j+1}, 0, \dots, 0)$ for infinitely many $k_p \in \mathscr{K}$ which contradicts $\mathscr{N}_n - \lim_{p\to\infty} \Delta^m w_{k_p} = \boldsymbol{0}$. Therefore $\{w_k\}$ is not $\Delta^m \mathscr{I}^*(\mathscr{N}_n)$-convergent to $\boldsymbol{0} \in \mathcal{W}$.

Then, question normally arises that under what condition the converse of the above Theorem is true. We investigate it in the following theorem.

Theorem 8. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$ and $\mathscr{I}$ satisfies the condition (AP) then $\mathscr{I}^*(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$.*

Proof. Suppose that $\mathscr{I}$ satisfies the condition (AP) and $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. Then, for every $\sigma \in (0,1), \zeta > 0$ and nonzero $\kappa_1, \kappa_2, \dots, \kappa_{n-1} \in \mathcal{W}$ such that

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma \text{ and } \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I}.$$

Define

$$\begin{aligned} \mathscr{A}_j = \{k \in \mathbb{N} : & 1 - \frac{1}{j} \leq \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < 1 - \frac{1}{j+1} \text{ or} \\ & \frac{1}{j+1} < \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq \frac{1}{j} \text{ and} \\ & \frac{1}{j+1} < \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq \frac{1}{j}\}. \end{aligned}$$

Clearly, $\{\mathscr{A}_1, \mathscr{A}_2, \dots\}$ is countable and pairwise disjoint and each $\mathscr{A}_j \in \mathscr{I}$. Since $\mathscr{I}$ satisfies the condition (AP), there exists a countable family $\{\mathscr{B}_1, \mathscr{B}_2, \dots\}$ of subsets of $\mathbb{N}$ belonging to $\mathscr{I}$ and $\mathscr{A}_i \triangle \mathscr{B}_i$ is finite for each i and $\mathscr{G} = \cup_i \mathscr{B}_i \in \mathscr{I}$. Now from the associated filter of $\mathscr{I}$ there is $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ such that $\mathscr{K} = \mathbb{N} \setminus \mathscr{G}$. It is sufficient to prove the theorem that the subsequence $\{w_k\}_{k \in \mathscr{K}}$ is Δ^m-convergent to υ with regard to $\mathscr{N}_n$. Let $\varpi \in (0,1)$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varpi$. Then, it is immediate that

$$\begin{aligned} &\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \varpi \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \varpi \text{ and } \\ &\qquad \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \varpi\} \\ &\subseteq \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \leq 1 - \frac{1}{k_0} \text{ or } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \frac{1}{k_0} \text{ and } \\ &\qquad \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) \geq \frac{1}{k_0}\} \subseteq \bigcup_{i=1}^{k_0+1} \mathscr{A}_i. \end{aligned}$$

Since $\mathscr{A}_i \triangle \mathscr{B}_i$, $i = 1, 2, \ldots, k_0 + 1$, are finite, there is $p_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{k_0+1} \mathscr{B}_i\right) \cap \{k \in \mathbb{N} : k \geq p_0\} = \left(\bigcup_{i=1}^{k_0+1} \mathscr{A}_i\right) \cap \{k \in \mathbb{N} : k \geq p_0\}. \tag{1}$$

If $k \geq p_0$ and $k \in \mathscr{K}$, $k \notin \bigcup_{i=1}^{k_0+1} \mathscr{B}_i$. So, by (1), $k \notin \bigcup_{i=1}^{k_0+1} \mathscr{A}_i$. Therefore, for every $k \geq p_0$ and $k \in \mathscr{K}$ we get

$$\begin{aligned} &\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) > 1 - \varpi, \\ &\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \varpi, \\ \text{and } &\Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \zeta) < \varpi. \end{aligned}$$

Since $\varpi \in (0, 1)$ is arbitrary, we have $\mathscr{I}^*(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$. Hence proved. □

5. $\Delta^m \mathscr{I}(\mathscr{N}_n)$-Completeness

Now, we proceed with the notion of generalized difference $\mathscr{I}$-Cauchy sequence in neutrosophic n-normed linear spaces.

Definition 18. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\{w_k\}$ is named to be generalized difference $\mathscr{I}$-Cauchy sequence with regard to $\mathscr{N}_n$ (in short $\Delta^m \mathscr{I}(\mathscr{N}_n)$-Cauchy) if for every $\sigma \in (0, 1), \zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$ there exists a natural number $k_0 = k_0(\sigma)$ such that*

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_{k_0}; \zeta) \leq 1 - \sigma \ \text{ or } \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_{k_0}; \zeta) \geq \sigma$$
$$\text{and } \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_{k_0}; \zeta) \geq \sigma\} \in \mathscr{I}.$$

Now, we proceed with the investigations of relation ship between $\Delta^m \mathscr{I}(\mathscr{N}_n)$-Cauchy sequence and $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergence of a sequence.

Theorem 9. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\{w_k\}$ is $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergent, it is $\Delta^m \mathscr{I}(\mathscr{N}_n)$-Cauchy sequence.*

Proof. Let $\{w_k\}$ is $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergent to υ. For a given $\sigma \in (0, 1)$, choose $\varpi \in (0, 1)$ such that $(1 - \varpi) \circledast (1 - \varpi) > 1 - \sigma$ and $\varpi \circledast \varpi < \sigma$. Then for any $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$, each of the following sets

$$\begin{aligned} &\mathscr{A}_1 = \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \frac{\zeta}{2}) > 1 - \varpi\}, \\ &\mathscr{A}_2 = \{k \in \mathbb{N} : \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \frac{\zeta}{2}) < \varpi\} \\ \text{and } &\mathscr{A}_3 = \{k \in \mathbb{N} : \Psi(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \frac{\zeta}{2}) < \varpi\} \end{aligned}$$

belongs to $\mathscr{F}(\mathscr{I})$. Let $\mathscr{B} = \mathscr{A}_1 \cap \mathscr{A}_2 \cap \mathscr{A}_3$. Then $\mathscr{B} \in \mathscr{F}(\mathscr{I})$. So, let $k \in \mathscr{B}$. Choose a fixed $k_0 \in \mathscr{B}$. Then,

$$\begin{aligned} &\Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_{k_0}; \zeta) \\ &\geq \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \frac{\zeta}{2}) \circledast \Upsilon(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_0} - \upsilon; \frac{\zeta}{2}) \\ &> (1 - \varpi) \circledast (1 - \varpi) \\ &> (1 - \sigma) \end{aligned}$$

and

$$\begin{aligned} &\Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \Delta^m w_{k_0}; \zeta) \\ &\leq \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_k - \upsilon; \frac{\zeta}{2}) \circledast \Re(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, \Delta^m w_{k_0} - \upsilon; \frac{\zeta}{2}) \\ &< \varpi \circledast \varpi \\ &< \sigma. \end{aligned}$$

Similarly we have $\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m\Delta^m w_{k_0};\zeta)<\sigma$. Therefore,

$$\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)>1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)<\sigma, \\ \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)<\sigma\}\in\mathscr{F}(\mathscr{I}).$$

Hence, $\{w_k\}$ is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence. □

Theorem 10. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\{w_k\}$ is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence, it is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergent.*

Proof. Let $\{w_k\}$ be $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence but not $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergent. Then for $\sigma\in(0,1),\zeta>0$ and nonzero $\kappa_1,\kappa_2,\ldots,\kappa_{n-1}\in\mathcal{W}$ there exists $k_0=k_0(\sigma)\in\mathbb{N}$ such that $\mathscr{K}\in\mathscr{I}$ where

$$\mathscr{K}=\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)\le 1-\sigma \text{ or } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)\ge\sigma \\ \text{and } \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)\ge\sigma\}.$$

And, $\mathscr{M}\in\mathscr{I}$ where

$$\mathscr{M}=\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})>1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})<\sigma, \\ \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})<\sigma\}.$$

Consequently

$$\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)\ge 2\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})>1-\sigma$$

and

$$\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)\le 2\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})<\sigma,$$
$$\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)\le 2\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})<\sigma,$$

if

$$\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})>\frac{1-\sigma}{2};$$
$$\text{and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})<\frac{\sigma}{2};\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\upsilon;\frac{\zeta}{2})<\frac{\sigma}{2}$$

respectively. This yields

$$\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)>1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)<\sigma, \\ \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k-\Delta^m w_{k_0};\zeta)<\sigma\}\in\mathscr{I},$$

which means $\mathscr{K}^c\in\mathscr{I}$ that implies $\mathscr{K}\in\mathscr{F}(\mathscr{I})$ by which we arrive at a contradiction. Hence, $\{w_k\}$ is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergent. □

Definition 19. *A Nn-NLS is named to be $\Delta^m\mathscr{I}$-complete with regard to $\mathscr{N}_n$ (in short $\Delta^m\mathscr{I}(\mathscr{N}_n)$-complete) if every $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergent.*

Remark 1. *In the light of Theorem 10, we see every Nn-NLS is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-complete.*

Now, from Theorems 5, 9 and 10 we get the following result.

Theorem 11. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, the below properties are gratified:*

(1) $\{w_k\}$ is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergent;
(2) $\{w_k\}$ is $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Cauchy;
(3) There exists a $\mathscr{C}\subseteq\mathbb{N}$ such that $\mathscr{C}\in\mathscr{F}(\mathscr{I})$ and the subsequence $\{w_k\}_{k\in\mathscr{C}}$ is an Δ^m-Cauchy sequence with regard to $\mathscr{N}_n$.

6. $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Limit Point

First we recall some basic topological terminology from [16].

Definition 20. *[16] For $\sigma \in (0,1), \zeta > 0, \alpha \in \mathcal{W}$ and every $\kappa_1, \kappa_2, \ldots, \kappa_{n-1} \in \mathcal{W}$, the open ball (also named as $\mathscr{N}_n$-open ball) centered at α and of radius σ with respect to ζ, denoted by $\mathscr{B}(\alpha, \sigma; \zeta)$, is defined by*

$$\mathscr{B}(\alpha,\sigma;\zeta) = \{w \in \mathcal{W} : \Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\alpha - w;\zeta) > 1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\alpha - w;\zeta) < \sigma, \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\alpha - w;\zeta) < \sigma\}.$$

Definition 21. *A subset $\mathscr{O}$ of $\mathcal{W}$ is called open with respect to $\mathscr{N}_n$ (named as $\mathscr{N}_n$-open set) if for each $\alpha \in \mathscr{O}$ there exists an $\mathscr{N}_n$-open ball of some radius which is contained in $\mathscr{O}$.*

Let $\mathcal{X}$ be an Nn-NLS. If we take a collection $\mathscr{T}_{\mathscr{N}_n}$ as $\mathscr{T}_{\mathscr{N}_n} = \{\mathscr{O} \subset \mathcal{W} : \mathscr{O}$ is an $\mathscr{N}_n -$ open set$\}$. Then $\mathscr{T}_{\mathscr{N}_n}$ becomes a topology on $\mathcal{X}$. A subset $\mathscr{U}$ of $\mathcal{W}$ is named to be bounded with respect to $\mathscr{N}_n$ (denoted as $\mathscr{N}_n$-bounded) if there exist $\zeta > 0$ and $\sigma \in (0,1)$ such that for each $\alpha \in \mathscr{U}$, $\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\alpha;\zeta) > 1-\sigma$ and $\Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\alpha;\zeta) < \sigma$, $\Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\alpha;\zeta) < \sigma$ holds for every $\kappa_1,\kappa_2,\ldots,\kappa_{n-1} \in \mathcal{W}$ [16].

Here, we define some topological notions with regard to $\mathscr{N}_n$ which will be needed to develop our results.

Definition 22. *Let $\mathscr{Y} \subseteq \mathcal{W}$ and $\upsilon \in \mathcal{W}$. Then, υ is named to be a limit point of $\mathscr{Y}$ with respect to $\mathscr{N}_n$ ($\mathscr{N}_n$-limit point) if for every $\mathscr{N}_n$-open ball centered at υ contains at least one point of $\mathscr{Y}$ different from υ.*

$\mathscr{Y}$ is named to be closed in $\mathcal{W}$ with regard to $\mathscr{N}_n$ ($\mathscr{N}_n$-closed set) if it contains all of its $\mathscr{N}_n$-limit point. Throughout our discussion $\overline{\mathscr{A}}$ denotes closure of $\mathscr{A}$ with respect to $\mathscr{N}_n$.

Now, we define $\Delta^m(\mathscr{N}_n)$-limit point, $\Delta^m\mathscr{I}(\mathscr{N}_n)$-limit point and $\Delta^m\mathscr{I}(\mathscr{N}_n)$-cluster point of a sequence $\{w_k\}$.

Definition 23. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then a point $\alpha \in \mathcal{W}$ is named to be a Δ^m-limit point of $\{w_k\}$ with respect to $\mathscr{N}_n$ (in short $\Delta^m(\mathscr{N}_n)$-limit point) if there is a subsequence of $\{w_k\}$ which is Δ^m-convergent to α with respect to $\mathscr{N}_n$.*

We denote $\Delta^m\mathscr{L}^{\mathscr{N}_n}(w_k)$ to mean the set of all $\Delta^m(\mathscr{N}_n)$-limit points of $\{w_k\}$.

Definition 24. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, a point $\alpha \in \mathcal{W}$ is named to be a $\Delta^m\mathscr{I}$-limit point of $\{w_k\}$ with respect to $\mathscr{N}_n$ (in short $\Delta^m\mathscr{I}(\mathscr{N}_n)$-limit point) if there exists a $\mathscr{M} \subseteq \mathbb{N}$ such that $\mathscr{M} \notin \mathscr{I}$ and the subsequence $\{w_k\}_{k\in\mathscr{M}}$ is Δ^m-convergent to υ with respect to $\mathscr{N}_n$.*

We denote $\Delta^m\Lambda^{\mathscr{N}_n}(w_k)$ to mean the set of all $\Delta^m\mathscr{I}(\mathscr{N}_n)$-limit points of $\{w_k\}$.

Definition 25. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. A point $\alpha \in \mathcal{W}$ is named to be $\Delta^m\mathscr{I}$-cluster point with respect to $\mathscr{N}_n$ of $\{w_k\}$ (in short $\Delta^m\mathscr{I}(\mathscr{N}_n)$-cluster point) if for every $\sigma \in (0,1), \zeta > 0$ and nonzero $\kappa_1,\kappa_2,\ldots,\kappa_{n-1} \in \mathcal{W}$,*

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k - \alpha;\zeta) > 1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k - \alpha;\zeta) < \sigma, \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k - \alpha;\zeta) < \sigma\} \notin \mathscr{I}.$$

We denote $\Delta^m\Gamma^{\mathscr{N}_n}(w_k)$ to mean the set of all $\Delta^m\mathscr{I}(\mathscr{N}_n)$-cluster points of $\{w_k\}$.

Theorem 12. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\Delta^m\Lambda^{\mathscr{N}_n}(w_k) \subseteq \Delta^m\Gamma^{\mathscr{N}_n}(w_k)$ holds good.*

Proof. Let $\upsilon \in \Delta^m\Lambda^{\mathscr{N}_n}(w_k)$. Then, there exists a $\mathscr{M} = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\mathscr{M} \notin \mathscr{I}$ and the subsequence $\{w_k\}_{k\in\mathscr{M}}$ is Δ^m-convergent to υ with respect to $\mathscr{N}_n$, i.e., for every $\sigma \in (0,1)$, $\zeta > 0$ and nonzero $\kappa_1,\kappa_2,\ldots,\kappa_{n-1} \in \mathcal{W}$ there exists $n_0 \in \mathbb{N}$ such that

$$\Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_{k_n} - \upsilon;\zeta) > 1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_{k_n} - \upsilon;\zeta) < \sigma, \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_{k_n} - \upsilon;\zeta) < \sigma$$

holds for all $n \geq n_0$. Let

$$\mathscr{A} = \{k \in \mathbb{N} : \Upsilon(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k - \alpha;\zeta) > 1-\sigma \text{ and } \Re(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k - \alpha;\zeta) < \sigma, \Psi(\kappa_1,\kappa_2,\ldots,\kappa_{n-1},\Delta^m w_k - \alpha;\zeta) < \sigma\}.$$

Then, it is obvious that $\mathscr{A} \supseteq \mathscr{M} \setminus \{k_1,k_2,\ldots,k_{n_0-1}\}$. Since $\mathscr{I}$ is an admissible ideal, $\mathscr{A} \notin \mathscr{I}$. Therefore, $\upsilon \in \Delta^m\Gamma^{\mathscr{N}_n}(w_k)$. Hence proved. □

Theorem 13. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\Delta^m\Gamma^{\mathscr{N}_n}(w_k) \subseteq \Delta^m\mathscr{L}^{\mathscr{N}_n}(w_k)$ holds good.*

Proof. Let $\upsilon \in \Delta^m\Gamma^{\mathscr{N}_n}(w_k)$. Then, for every $\sigma \in (0,1)$, $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \dots, \kappa_{n-1} \in \mathcal{W}$ the set

$$\mathscr{M} = \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) < \sigma, \\ \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) < \sigma\} \notin \mathscr{I}.$$

Since $\mathscr{I}$ is an admissible ideal, $\mathscr{M}$ must be infinite. So, we can write $\mathscr{M}$ as $\{k_1 < k_2 < \cdots < k_n < \cdots\}$. Thus we have a subsequence $\{w_k\}_{k\in\mathscr{M}}$ of $\{w_k\}$ such that $w_{k_n} \xrightarrow{\Delta^m(\mathscr{N}_n)} \upsilon$, i.e., $\upsilon \in \Delta^m\mathscr{L}^{\mathscr{N}_n}(w_k)$. Hence proved. □

Theorem 14. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. Then, $\Delta^m\Gamma^{\mathscr{N}_n}(w_k)$ is a closed set in $\mathcal{W}$ with respect to the topology induced by $\mathscr{N}_n$-norm.*

Proof. Let $\alpha \in \overline{\Delta^m\Gamma^{\mathscr{N}_n}(w_k)}$. Choose $\sigma \in (0,1)$ and $\zeta > 0$. Then, we have an element $\mu \in \Delta^m\Gamma^{\mathscr{N}_n}(w_k) \cap \mathscr{B}(\alpha, \sigma; \zeta)$. We select $\vartheta \in (0,1)$ such that $\mathscr{B}(\mu, \vartheta; \zeta) \subset \mathscr{B}(\alpha, \sigma; \zeta)$. Then for every nonzero $\kappa_1, \kappa_2, \dots, \kappa_{n-1} \in \mathcal{W}$, it follows that

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \mu; \zeta) > 1 - \vartheta \text{ and } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \mu; \zeta) < \vartheta, \\ \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \mu; \zeta) < \vartheta\} \\ \subseteq \{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) < \sigma, \\ \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) < \sigma\}.$$

Since $\mu \in \Delta^m\Gamma^{\mathscr{N}_n}(w_k)$,

$$\{k \in \mathbb{N} : \Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) > 1 - \sigma \text{ and } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) < \sigma, \\ \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, \Delta^m w_k - \alpha; \zeta) < \sigma\} \notin \mathscr{I}.$$

Therefore $\alpha \in \Delta^m\Gamma^{\mathscr{N}_n}(w_k)$. Hence the result is proved. □

Theorem 15. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathcal{X}$. If $\mathscr{I}(\mathscr{N}_n) - \lim \Delta^m w_k = \upsilon$, $\Delta^m\Lambda^{\mathscr{N}_n}(w_k) = \Delta^m\Gamma^{\mathscr{N}_n}(w_k) = \{\upsilon\}$.*

Proof. It is omitted. □

7. Continuous Linear Operators Preserving $\Delta^m\mathscr{I}(\mathscr{N}_n)$-Convergence

In this section, we explore the notion of continuous mapping and prove that a linear operator preserves $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence iff it is continuous on $\mathcal{W}$. Throughout this section we denote $\mathscr{G}(\kappa_1) = z_1$, $\mathscr{G}(\kappa_2) = z_2, \dots, \mathscr{G}(\kappa_{n-1}) = z_{n-1}$.

Definition 26. *A mapping $\mathscr{G} : \mathcal{W} \to \mathcal{W}$ is named to be continuous at $\upsilon \in \mathcal{W}$ with regard to $\mathscr{N}_n$ (in short $\mathscr{N}_n$-continuous) if for $\sigma \in (0,1)$ and $\zeta > 0$ and nonzero $\kappa_1, \kappa_2, \dots, \kappa_{n-1} \in \mathcal{W}$ and $z_1, z_2, \dots, z_{n-1} \in \mathcal{W}$ there exist $\sigma_1 = \sigma_1(\sigma, \zeta, \upsilon) \in (0,1)$ and $\zeta_1 = (\sigma, \zeta, \upsilon) > 0$ such that for all $w \in \mathcal{W}$ such that*

$$\Upsilon(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, w - \upsilon; \zeta_1) > 1 - \sigma_1 \text{ and } \Re(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, w - \upsilon; \zeta_1) < \sigma_1, \\ \Psi(\kappa_1, \kappa_2, \dots, \kappa_{n-1}, w - \upsilon; \zeta_1) < \sigma_1$$

implies

$$\Upsilon(z_1, z_2, \dots, z_{n-1}, \mathscr{G}(w) - \mathscr{G}(\upsilon); \zeta) > 1 - \sigma \text{ and } \Re(z_1, z_2, \dots, z_{n-1}, \mathscr{G}(w) - \mathscr{G}(\upsilon); \zeta) < \sigma, \\ \Psi(z_1, z_2, \dots, z_{n-1}, \mathscr{G}(w) - \mathscr{G}(\upsilon); \zeta) < \sigma.$$

If $\mathscr{G}$ is $\mathscr{N}_n$-continuous at each point of $\mathcal{W}$, then $\mathscr{G}$ is called $\mathscr{N}_n$-continuous on $\mathcal{W}$.

Now, we define sequential continuity of a mapping for the generalized difference sequence with regard to $\mathscr{N}_n$.

Definition 27. *A mapping $\mathscr{G} : \mathcal{W} \to \mathcal{W}$ is named to be sequentially continuous at $\upsilon \in \mathcal{W}$ with regard to $\mathscr{N}_n$ if for any sequence $\{w_k\} \in \mathcal{W}$, $\mathscr{N}_n - \lim \mathscr{G}(\Delta^m w_k) = \mathscr{G}(\upsilon)$ whenever $\mathscr{N}_n - \lim \Delta^m w_k = \upsilon$.*

Theorem 16. *A mapping $\mathscr{G} : \mathcal{W} \to \mathcal{W}$ is $\mathscr{N}_n$-continuous if and only if it is sequentially continuous with regard to $\mathscr{N}_n$.*

Proof. It is straightforward. So, we omit details. □

Definition 28. *A mapping $\mathscr{G}:\mathcal{W}\to\mathcal{W}$ is named to preserve $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence in $\mathcal{W}$ if $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon$ implies $\mathscr{I}(\mathscr{N}_n)-\lim\mathscr{G}(\Delta^m w_k)=\mathscr{G}(\upsilon)$ for any sequence $\{w_k\}\in\mathcal{W}$.*

Theorem 17. *A linear operator $\mathscr{G}:\mathcal{W}\to\mathcal{W}$ preserves $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence in $\mathcal{W}$ if and only if $\mathscr{G}$ is $\mathscr{N}_n$-continuous on $\mathcal{W}$.*

Proof. Let $\{w_k\}$ be a sequence in $\mathcal{W}$ such that $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon$. Let $\mathscr{G}$ is $\mathscr{N}_n$-continuous on $\mathcal{W}$. So, for $\sigma\in(0,1)$ and $\zeta>0$ and nonzero $\kappa_1,\kappa_2,\dots,\kappa_{n-1}\in\mathcal{W}$ and $z_1,z_2,\dots,z_{n-1}\in\mathcal{W}$ there exist $\sigma_1=\sigma_1(\sigma,\zeta,\upsilon)\in(0,1)$ and $\zeta_1=(\sigma,\zeta,\upsilon)>0$ such that for all $w\in\mathcal{W}$ such that

$$\Upsilon(\kappa_1,\kappa_2,\dots,\kappa_{n-1},w-\upsilon;\zeta_1)>1-\sigma_1 \text{ and } \Re(\kappa_1,\kappa_2,\dots,\kappa_{n-1},w-\upsilon;\zeta_1)<\sigma_1,\\ \Psi(\kappa_1,\kappa_2,\dots,\kappa_{n-1},w-\upsilon;\zeta_1)<\sigma_1$$

implies

$$\Upsilon(z_1,z_2,\dots,z_{n-1},\mathscr{G}(w)-\mathscr{G}(\upsilon);\zeta)>1-\sigma \text{ and } \Re(z_1,z_2,\dots,z_{n-1},\mathscr{G}(w)-\mathscr{G}(\upsilon);\zeta)<\sigma,\\ \Psi(z_1,z_2,\dots,z_{n-1},\mathscr{G}(w)-\mathscr{G}(\upsilon);\zeta)<\sigma.$$

Since $\mathscr{I}(\mathscr{N}_n)-\lim\Delta^m w_k=\upsilon$,

$$\mathscr{M}=\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)>1-\sigma_1 \text{ and } \Re(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)<\sigma_1,\\ \Psi(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)<\sigma_1\}\in\mathscr{F}(\mathscr{I}).$$

Now, consider the open balls

$$\mathscr{B}(\upsilon,\sigma_1,\zeta_1)=\{w\in\mathcal{W}:\Upsilon(\kappa_1,\kappa_2,\dots,\kappa_{n-1},w-\upsilon;\zeta_1)>1-\sigma_1 \text{ and } \Re(\kappa_1,\kappa_2,\dots,\kappa_{n-1},w-\upsilon;\zeta_1)<\sigma_1,\\ \Psi(\kappa_1,\kappa_2,\dots,\kappa_{n-1},w-\upsilon;\zeta_1)<\sigma_1\}$$

and

$$\mathscr{B}(\mathscr{G}(\upsilon),\sigma,\zeta)=\{\mathscr{G}(w)\in\mathcal{W}:\Upsilon(z_1,z_2,\dots,z_{n-1},\mathscr{G}(w)-\mathscr{G}(\upsilon);\zeta)>1-\sigma \text{ and}\\ \Re(z_1,z_2,\dots,z_{n-1},\mathscr{G}(w)-\mathscr{G}(\upsilon);\zeta)<\sigma,\ \Psi(z_1,z_2,\dots,z_{n-1},\mathscr{G}(w)-\mathscr{G}(\upsilon);\zeta)<\sigma\}$$

centered at υ and $\mathscr{G}(\upsilon)$ respectively. From the above fact, it follows that if $w\in\mathscr{B}(\upsilon,\sigma_1,\zeta_1)$ then $\mathscr{G}(w)\in\mathscr{B}(\mathscr{G}(\upsilon),\sigma,\zeta)$. Hence, we can have

$$\mathscr{M}\subseteq\{k\in\mathbb{N}:\Upsilon(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)>1-\sigma \text{ and}\\ \Re(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)<\sigma,\Psi(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)<\sigma\}\in\mathscr{F}(\mathscr{I}),$$

as $\mathscr{M}\in\mathscr{F}(\mathscr{I})$. Therefore $\mathscr{I}(\mathscr{N}_n)-\lim\mathscr{G}(\Delta^m w_k)=\mathscr{G}(\upsilon)$, i.e., $\mathscr{G}$ preserves $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence.

Conversely, suppose that $\mathscr{G}$ preserves $\Delta^m\mathscr{I}(\mathscr{N}_n)$-convergence. We shall show that $\mathscr{G}$ is $\mathscr{N}_n$-continuous on $\mathcal{W}$. If possible, let $\mathscr{G}$ is not $\mathscr{N}_n$-continuous at some point υ of $\mathcal{W}$. Then there are some $\sigma\in(0,1)$ and $\zeta>0$ such that $w\in\mathscr{B}(\upsilon,\sigma_1,\zeta_1)$ but $\mathscr{G}(w)\notin\mathscr{B}(\mathscr{G}(\upsilon),\sigma,\zeta)$ for all $\sigma_1=\sigma_1(\sigma,\zeta,\upsilon)\in(0,1)$. Since $\mathscr{G}$ is not sequentially continuous, there is a sequence $\{w_k\}\in\mathcal{W}$ such that $\mathscr{N}_n-\lim\Delta^m w_k=\upsilon$ but $\mathscr{N}_n-\lim\mathscr{G}(\Delta^m w_k)\neq\mathscr{G}(\upsilon)$. Since $\mathscr{N}_n-\lim\Delta^m w_k=\upsilon$,

$$\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)>1-\sigma_1 \text{ and } \Re(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)<\sigma_1,\\ \Psi(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)<\sigma_1\}\in\mathscr{F}(\mathscr{I}).$$

In fact

$$\{k\in\mathbb{N}:\Upsilon(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)>1-\sigma_1 \text{ and } \Re(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)<\sigma_1,\\ \Psi(\kappa_1,\kappa_2,\dots,\kappa_{n-1},\Delta^m w_k-\upsilon;\zeta_1)<\sigma_1\}\\ \subseteq\{k\in\mathbb{N}:\Upsilon(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)\le1-\sigma \text{ or}\\ \Re(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)\ge\sigma \text{ and } \Psi(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)\ge\sigma\},$$

therefore

$$\{k\in\mathbb{N}:\Upsilon(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)\le1-\sigma \text{ or } \Re(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)\ge\sigma\\ \text{and } \Psi(z_1,z_2,\dots,z_{n-1},\mathscr{G}(\Delta^m w_k)-\mathscr{G}(\upsilon);\zeta)\ge\sigma\}\in\mathscr{F}(\mathscr{I}).$$

This gives $\mathscr{I}(\mathscr{N}_n) - \lim \mathscr{G}(\Delta^m w_k) \neq \mathscr{G}(v)$, which contradicts the fact that $\mathscr{G}$ preserves $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergence. So, our assumption $\mathscr{G}$ is not $\mathscr{N}_n$-continuous at some point v of $\mathcal{W}$ is wrong. Therefore $\mathscr{G}$ is $\mathscr{N}_n$-continuous on $\mathcal{W}$. This completes the proof. □

8. Conclusion and Future Developments

Incorporating the latest advancements, this paper has unveiled the innovative concepts of $\Delta^m \mathscr{I}(\mathscr{N}_n)$ and $\Delta^m \mathscr{I}^*(\mathscr{N}_n)$-convergence, presenting a groundbreaking generalization of generalized difference statistical convergence within summability theory. We have delved into the key properties and revealed the intricate relationship between these two pioneering concepts using the condition (AP). In this work, we have explored the concept of $\Delta^m \mathscr{I}(\mathscr{N}_n)$-Cauchy sequences, highlighting the interrelationship with $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergent sequences. Furthermore, we have established the completeness of every Nn-NLS within this generalized difference ideal-driven framework. We have conducted an in-depth exploration of $\Delta^m \mathscr{I}(\mathscr{N}_n)$-limit and $\Delta^m \mathscr{I}(\mathscr{N}_n)$-cluster points of sequences within the framework of the neutrosophic n-norm. Our results have shown that the set of all $\Delta^m \mathscr{I}(\mathscr{N}_n)$-cluster points constitutes a closed set under the topology induced by this norm. Furthermore, we have established that a linear operator preserves $\Delta^m \mathscr{I}(\mathscr{N}_n)$-convergence if, and only if, it remains continuous in relation to the neutrosophic n-norm. Research on sequence convergence in neutrosophic n-normed linear spaces is still in its early stages, with limited progress made thus far. Building upon the insights gained from this research, future studies may extend this notion to encompass compact operator, further exploring its connections to double sequences within the framework of $\mathscr{N}_n$. This concept can also be applied to convergence-related challenges across various branches of science and engineering, offering valuable insights and solutions.

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