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Research Article

On finite groups in which every maximal subgroup of order divisible by p is p-decomposable

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Abstract

We obtain a complete classification of a finite group G in which every maximal subgroup of order divisible by p is p-decomposable for a given prime divisor p of $|G|$ and our results generalize a recent result of Shi and Tian.

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1. Introduction

In this paper, all groups are assumed to be finite. It is known that if a group G can be written as the direct product of a Sylow p-subgroup of G and a Hall p'-subgroup of G, then G is called a p-decomposable group. A group G is not p-decomposable but all of its proper subgroups are p-decomposable, then group G is called inner p-decomposable. A group G is called inner-nilpotent group, if G is non-nilpotent but all of its proper subgroups are nilpotent. A group G is called p-closed, if its Sylow p-subgroup is normal in G. Specially, if $p \nmid |G|$, G is p-closed. Inner-nilpotent group G is a group whose order is $p^\alpha q^\beta$, where p, q are distinct prime number. There is a normal Sylow subgroup and a non-normal Sylow subgroup in G, the non-normal Sylow subgroup is a cyclic group. If the Sylow q-subgroup of G is normal, we call inner-nilpotent group G as a q-fundamental group [2]. In this paper, the symbol $P : Q$ represents the semidirect product of P and Q, where P is normal in G. Shi and Tian [5, Theorem 1.1], characterized the structure of a group in which every maximal subgroup of order divisible by p is nilpotent (or abelian). In this paper, considering any fixed prime divisor p of the order of a group G, we obtain the following result in Theorem 1.1 whose proof is given in Section 3. The symbols appearing in this paper can be found in [3]. In this paper, P_i stands for the Sylow-i subgroup of G.

Theorem 1.1 Suppose that G is a group and p is a given prime divisor of $|G|$. Then every maximal subgroup of G of order divisible by p is p-decomposable if and only if one of the following statements holds:

(1) G is a p-decomposable group.

(2) G is not a p-decomposable group, and the following statements are true.

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(2.1) $G = P : Q$ is an inner p-decomposable group, where P is a Sylow p-subgroup of G and Q is a cyclic Sylow q-subgroup, $p \neq q$, and Q has only one maximal subgroup;
(2.2) $G = Q : P$ is an inner nilpotent group, where P is a Sylow p-subgroup of G and where Q is a Sylow q-subgroup of G.

By Theorem 1.1, the following corollary emerges.
Corollary 1.2 ([5, Theorem 1.1]) Suppose that G is a group and p is any fixed prime divisor of $|G|$, then every maximal subgroup of G of order divisible by p is nilpotent if and only if one of the following statements holds:
(a) G is a nilpotent group;
(b) $G = P : Q$ is an inner-nilpotent group, where $P \in Syl_p(G)$ and $Q \in Syl_q(G)$, P is normal in G, $p \neq q$;
(c) $G = Q : P$ is an inner-nilpotent group, where $Q \in Syl_q(G)$ and $P \in Syl_p(G)$, Q is normal in G, $q \neq p$;
(d) $G = Z_p : K$, where K is an inner-nilpotent group and $(p, |K|)= 1$.

2. A lemma

Lemma 2.1 ([2, Theorem 1]) Inner-p-closed group has the following two forms (1) $G/\Phi(G)$ is a simple group of complex order; (2) G is a q-fundamental group whose order is $p^\alpha q^\beta$.

3. Proof of Theorem 1.1

Proof. The sufficiency part is evident, we only need to prove the necessity part. For a finite group G, it is either p-decomposable or non-p-decomposable. If G is p-decomposable, the conclusion (1) is obviously correct. In the following discussion we suppose G is not p-decomposable.

Now we choose any maximal subgroup H of G. If $p \mid |H|$, then H is p-decomposable; if $p \nmid |H|$, then H is also evidently p-decomposable. Hence, every maximal subgroup of G is p-decomposable. So, G is inner p-decomposable.

In the following we divide our arguments into two cases.

Case 1. G is a p-closed group.

Let P be a Sylow p-subgroup of G, we have $P \trianglelefteq G$. By Schur-Zassenhaus theorem, G has a p-complement Q. Then $G = P : Q$. Choose any maximal subgroup K_1 of Q, we can get PK_1 is a maximal subgroup of G, then PK_1 is p-decomposable. Thus $PK_1 = P \times K_1$. If Q has at least two different maximal subgroups K_1 and K_2. Then $PK_i = P \times K_i$, where i=1, 2. Since $Q = \langle K_1, K_2\rangle$, we have $PQ = P \times Q$, a contradiction. Hence, Q has the unique maximal subgroup K_1, then Q is a cyclic Sylow q-subgroup, where q is a prime number and $p \neq q$. Thus, (2.1) is proved.

Case 2. G is not a p-closed group.

Since G is inner p-decomposable, G is inner p-closed. By Lemma 2.1, we can get (a) $G/\Phi(G)$ is a non-abelian simple group; (b) G is a q-fundamental group.

Assume that the case(a) occurs. Let $\overline{G} = G/\Phi(G)$. By $\overline{G}$ is a non-abelian simple group, we get G is a non-abelian simple group if $\Phi(G) = 1$. By classification of finite simple groups, there are three types of non-abelian simple groups. They are alternating groups, sporadic simple groups and simple groups of Lie type, respectively. In the following we divide our arguments into three cases.

Case 2.1. *G is an alternating group, $G \cong A_n$.* Suppose $p \nmid n$ and $n \geq 6$. A_{n-1} is the maximal subgroup of A_n, evidently, $p \mid |A_{n-1}|$. However, A_{n-1} is a simple group and it's not p-decomposable, a contradiction. So $p \mid n$ and $p > n-1$, thus $p = n$, now we have $A_n = A_p$, and we can select the maximal subgroup $N_G(P)$ of G, where P is a Sylow p-subgroup of G. We can get $N_G(P) = P : C_{\frac{p-1}{2}}$, it's clearly that it is not p-decomposable. Hence it's a contradiction. If $n = 5$, A_5 has three prime factors, they are $2, 3, 5$ respectively. We consider the maximal subgroup S_3 if $p = 3$, since $3 \mid |S_3|$, $N_G(P) = P : C_2$ is non-decomposable, a contradiction. We consider the maximal subgroup D_{10} if p=2 or 5, $p \mid |D_{10}|$, $D_{10} = C_5 : C_2$ is non-decomposable, a contradiction.

Case 2.2. *G is a sporadic simple group.*

Suppose $G \cong M_{11}$, then $\pi(G) = \{2, 3, 5, 11\}$. By [3], We get the maximal subgroup $L_2(11)$ of G whose prime factors are also $p = 2, 3, 5$ or 11. Since $p \mid |L_2(11)|$, $L_2(11) = P_2 \times H$, it contradicts that $L_2(11)$ is a simple group.

Suppose $G \cong Suz$, then $\pi(G) = \{2, 3, 5, 7, 11, 13\}$. By [3], if $p = 2, 3, 5, 11$, we consider the maximal subgroup $M_{12} : 2$ of G. $p \mid |M_{12} : 2|$, but $M_{12} : 2$ is not p-decomposable, a contradiction. If $p = 7$, we consider the maximal subgroup A_7 of G. $7 \mid |A_7|$, then $A_7 = P_7 \times H$, it contradicts that A_7 is a simple group. If $p = 13$, we consider the maximal subgroup $L_3(3) : 2$ of G. $13 \mid |L_3(3) : 2|$, but $L_3(3) : 2$ is not p-decomposable, a contradiction.

Suppose $G \cong Fi_{23}$, then $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 23\}$. By [3], if $p = 2, 3, 5, 7, 17$, we consider the maximal subgroup $S_8(2)$ of G. Since $p \mid |S_8(2)|$, $S_8(2) = P_i \times H$, where $i = 2, 3, 5, 7, 17$, a contradiction. If $p = 13$, we consider the maximal subgroup $O_8^+(3) : S_3$ of G. Since $13 \mid |O_8^+(3) : S_3|$, $O_8^+(3) : S_3$ is not p-decomposable, a contradiction. If $p = 11, 23$, we consider the maximal subgroup $2^{11} \cdot M_{23}$ of G. Since $p \mid |2^{11} \cdot M_{23}|$, $2^{11} \cdot M_{23} = P_i \times H$, where $i = 11, 13$. Then M_{23} is p-decomposable, it contradicts that M_{23} is a non-abelian simple group.

Suppose $G \cong J_4$, then $\pi(G) = \{2, 3, 5, 7, 11, 23, 29, 31, 37, 43\}$. By [3], if $p = 2, 3, 5, 11, 37$, we consider the maximal subgroup $U_3(11) : 2$ of G. Since $p \mid |U_3(11) : 2|$, $U_3(11) : 2$ is not p-decomposable, a contradiction. If $p = 23$, we consider the maximal subgroup $L_2(23) : 5$ of G. Since $23 \mid |L_2(23) : 5|$, $L_2(23) : 5$ is not p-decomposable, a contradiction. If $p = 7$, we consider the maximal subgroup $2_+^{1+12} \cdot M_{22} : 2$ of G. Since $7 \mid |2_+^{1+12} \cdot M_{22} : 2|$, $2_+^{1+12} \cdot M_{22} : 2$ is not p-decomposable, a contradiction. If $p = 29$, we consider the maximal subgroup $29 : 28$ of G. Since $29 \mid |29 : 28|$, $29 : 28$ is not p-decomposable, a contradiction. If $p = 43$, we consider the maximal subgroup $43 : 14$. Since $43 \mid |43 : 14|$, $43 : 14$ is not p-decomposable, a contradiction.

We can also get a contradiction when G is an other sporadic group according to the Atlas form [3].

Case 2.3. *G is a simple group of Lie type.*

Let G be a classical group.

Suppose $G \cong L_n(q), q = r^t$, where r is a prime. If $p = r$ or $p \nmid (q^n - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PGL_{n-1}(q)$, $p \mid |PGL_{n-1}(q)|$, so $PGL_{n-1}(q)$ is p-decomposable, a contradiction. If $p \mid (q^n - 1)$ but $p \nmid (q^i - 1)$, where $i < n$, G has a subgroup $N_G(P) = \frac{q^n-1}{q-1} : C_n$ [4]. It is not p-decomposable, so it is a contradiction.

Suppose $G \cong PSU_{2n}(q)$. If $p \mid q\prod_{i=1}^{2n-1}(q^i - (-1)^i)$ but $p \nmid (q^{2n} - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PGU_{2n-1}(q)$, $p \mid |PGU_{2n-1}(q)|$, so $PGU_{2n-1}(q)$ is p-decomposable, a contradiction. If $p \mid (q^{2n} - 1)$, then by [1, Tables] and [6, Proposition

4.1.17], G has a subgroup $PSL_n(q^2).(q-1).2$. So $p \mid |PSL_n(q^2).(q-1).2|$, and thus $SL_n(q^2).(q-1).2$ is p-decomposable, a contradiction.

Suppose $G \cong PSU_{2n+1}(q)$. If $p \mid q\prod_{i=1}^{2n}(q^i-(-1)^i)$ but $p \nmid (q^{2n+1}+1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PGU_{2n}(q)$, $p \mid |PGU_{2n}(q)|$, so $PGU_{2n}(q)$ is p-decomposable, a contradiction. If $p \mid (q^{2n+1}+1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $\frac{q^{2n+1}+1}{q+1}:(2n+1)$, $p \mid |(\frac{q^{2n+1}+1}{q+1}:(2n+1))|$, so $\frac{q^{2n+1}+1}{q+1}:(2n+1)$ is p-decomposable, a contradiction.

Suppose $G \cong PSp_{2n}(q)$. If $p \mid q\prod_{i=1}^{n-1}(q^{2i}-1)$ but $p \nmid (q^{2n}-1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $E_q^{1+(2n-2)}:((q-1)\times PSP_{2n-2}(q))$, $p \mid |E_q^{1+(2n-2)}:((q-1)\times PSP_{2n-2}(q))|$, so $E_q^{1+(2n-2)}:((q-1)\times PSP_{2n-2}(q))$ is p-decomposable, a contradiction. If $p \mid (q^{2n}-1)$ but $p \nmid \prod_{i=1}^{n-1}(q^{2i}-1)$, then by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PSO^-_{2n}(q)$, $p \mid |PSO^-_{2n}(q)|$, so $PSO^-_{2n}(q)$ is p-decomposable, a contradiction.

Suppose $G \cong P\Omega_{2n+1}(q)$. If $p \mid q(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $P\Omega^+_{2n}(q).2$, $p \mid |P\Omega^+_{2n}(q).2|$, so $P\Omega^+_{2n}(q).2$ is p-decomposable, a contradiction. If $p \mid (q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $P\Omega^-_{2n}(q).2$, $p \mid |P\Omega^-_{2n}(q).2|$, so $P\Omega^-_{2n}(q).2$ is p-decomposable, a contradiction.

Suppose $G \cong P\Omega^+_{2n}(q)$. If $p \mid q\prod_{i=1}^{n-1}(q^{2i}-1)$, but $p \nmid (q^n-1)$, by [1, Tables] and [6, Proposition 4.1.17]], G has a subgroup $PSp_{2n-2}(q)$, $p \mid |PSp_{2n-2}(q)|$, so $PSp_{2n-2}(q)$ is p-decomposable, a contradiction. If $p \mid q^n-1$, let the prime factor q of $|\Omega^+_{2n}(q)|$ be of power t, then by [1, Tables] and [6, Proposition 4.1.17], we can find a subgroup $E_q^{t-\frac{n(n-1)}{2}}:GL_n(q)$, $p \mid |E_q^{t-\frac{n(n-1)}{2}}:GL_n(q)|$, so $E_q^{t-\frac{n(n-1)}{2}}:GL_n(q)$ is p-decomposable, a contradiction.

Suppose $G \cong P\Omega^-_{2n}(q)$. If $p \mid q\prod_{i=1}^{n-1}(q^{2i}-1)$, but $p \nmid (q^n+1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PSp_{2n-2}(q)$. Since $p \mid |PSp_{2n-2}(q)|$, $PSp_{2n-2}(q)$ is p-decomposable, a contradiction. Suppose $p \mid (q^n+1)$ and n is an odd number. By [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $GU_n(q)$. Since $p \mid |GU_n(q)|$, $GU_n(q)$ is p-decomposable, a contradiction. Suppose $p \mid (q^n+1)$ and n is an even number. G has a subgroup $P\Omega^-_n(q^2).2$. Since $p \mid |P\Omega^-_n(q^2).2|$, $P\Omega^-_n(q^2).2$ is p-decomposable, a contradiction.

Let G be an exceptional group.

Suppose $G \cong G_2(q)$, $|G_2(q)| = q^6(q^6-1)(q^2-1)$. If $p \mid q^6(q^3-1)(q^2-1)$, but $p \nmid (q^3+1)$, by [7, Table 4.1], G has a maximal subgroup $SL_3(q):2$. Since $p \mid |SL_3(q):2|$, $SL_3(q):2$ is p-decomposable, a contradiction. If $p \mid q^3+1$, G has a maximal subgroup $SU_3(q):2$. Since $p \mid |SU_3(q):2|$, $SU_3(q):2$ is p-decomposable, a contradiction.

Suppose $G \cong^3 D_4(q)$, $|^3D_4(q)| = q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$. If $p \mid q^{12}(q^6-1)(q^2-1)$, by [7, Theorem 4.3.], G has a maximal subgroup $G_2(q)$. Since $p \mid |G_2(q)|$, then $G_2(q)$ is p-decomposable, a contradiction. If $p \mid (q^2+q+1)$, by [7, Theorem 4.3.], G has a maximal subgroup $(C_{q^2+q+1}\times C_{q^2+q+1}):SL_2(3)$. Since $p \mid |(C_{q^2+q+1}\times C_{q^2+q+1}):SL_2(3)|$, $(C_{q^2+q+1}\times C_{q^2+q+1}):SL_2(3)$ is p-decomposable, a contradiction. If $p \mid (q^2-q+1)$, by [7, Theorem 4.3.], G has a maximal subgroup $(C_{q^2-q+1}\times C_{q^2-q+1}):SL_2(3)$. Since $p \mid |(C_{q^2-q+1}\times C_{q^2-q+1}):SL_2(3)|$, $(C_{q^2-q+1}\times C_{q^2-q+1}):SL_2(3)$ is p-decomposable, a contradiction. If $p \mid (q^4-q^2+1)$, by [7, Theorem 4.3.], G has a maximal subgroup $C_{q^4-q^2+1}:4$. Since $p \mid |C_{q^4-q^2+1}:4|$, $C_{q^4-q^2+1}:4$ is p-decomposable, a contradiction.

We can also get a contradiction when G is an other exceptional group.

From (b) of [2], we have $G = Q:P$, where G is an inner nilpotent group, P is a Sylow p-subgroup of G and where Q is a Sylow q-subgroup of G. (2.2) is proved.

□

4. Proof of Corollary 1.2

Proof. It is obviously that (b) and (c) are true from (2.1) and (2.2) of Theorem 1.1. From 1 of Theorem 1.1, if G is p-decomposable, then $G = P \times K$, where P is the Sylow p-subgroup of G, and K is the Hall p'-subgroup of G. Suppose K is nilpotent, then G is nilpotent, (a) is proved. Suppose K is not nilpotent. Let $N < K$. Then $P \times N < G$, so $p \mid |P \times N|$. So, N is nilpotent, and we can get K is inner-nilpotent. If P is not a group of order p. Let $H < P$ and $|H| = p$. Then $p \mid |H \times K|$. So $H \times K$ is nilpotent, and we get K is nilpotent, a contradiction. Hence, P is a group of order p, (d) is proved. □

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