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Properties of Gaussian Generalized Leonardo Numbers

Gaussian Genelleştirilmiş Leonardo Sayılarının Özellikleri

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Abstract

In this research, we introduce and thoroughly examine Gaussian generalized Leonardo numbers, focusing on three distinct cases: Gaussian modified Leonardo numbers, Gaussian Leonardo-Lucas numbers, and Gaussian Leonardo numbers. Our aim is to offer a comprehensive understanding of the behaviour and properties of these sequences.

To this end, we perform a detailed analysis, deriving various identities and matrices associated with these sequences. We also explore key mathematical tools such as recurrence relations, Binet's formulas, generating functions, Simpson's formula, Honsberger's identity, and several summation formulas. This multifaceted approach provides valuable insights into the structure and behaviour of these Gaussian-based sequences. The results we present not only extend existing knowledge but also open the door for future studies that could explore further generalizations and applications of Gaussian generalized Leonardo numbers.

Keywords: Gaussian generalized Leonardo numbers, Gaussian Leonardo-Lucas numbers, Gaussian modified Leonardo numbers, Gaussian Leonardo numbers.

Öz

Bu araştırmada, Gaussian genelleştirilmiş Leonardo sayılarını tanıtıyor ve kapsamlı bir şekilde inceliyoruz ve üç farklı duruma odaklanıyoruz: Gaussian modifiye Leonardo sayıları, Gaussian Leonardo-Lucas sayıları ve Gaussian Leonardo sayıları. Amacımız, bu dizilerin davranışı ve özellikleri hakkında kapsamlı bir anlayış sunmaktır.

Bu amaçla, bu dizilerle ilişkili çeşitli özdeşlikler ve matrisler türeterek derinlemesine bir analiz gerçekleştiriyoruz. Ayrıca, yineleme bağıntıları, Binet formülleri, üreteç fonksiyonlar, Simpson formülü, Honsberger özdeşliği ve çeşitli toplam formülleri gibi temel matematiksel araçları da araştırıyoruz. Bu çok yönlü yaklaşım, bu Gaussian tabanlı dizilerin yapısı ve davranışı hakkında değerli içgörüler sağlar. Sunduğumuz sonuçlar yalnızca mevcut bilgiyi genişletmekle kalmıyor, aynı zamanda Gaussian genelleştirilmiş Leonardo sayılarının daha fazla genelleştirilmesini ve uygulamasını araştırabilecek gelecekteki çalışmalar için de kapı açıyor.

Anahtar Kelimeler: Gaussian genelleştirilmiş Leonardo sayıları, Gaussian Leonardo-Lucas sayıları, Gaussian modifiye Leonardo sayıları, Gaussian Leonardo sayıları.

1. Introduction

Sequences are important in mathematics especially in number theory. Additionally, sequences can be seen everywhere, including in economics, physics, cryptography, biology, engineering, and computer science. One of the most widely studied sequence of numbers is the Fibonacci sequence. There is another sequence which has similar properties to the Fibonacci sequence. It's called Leonardo sequence denoted by Le_n defined by the following recurrence relation for

$n \geq 2$

with the initial conditions $Le_0 = Le_1 = 1$. This sequence corresponds to the sequence A001595 in the on-line encyclopedia of integers sequences in (Sloane 1964). Also, there is a third-order recurrence relation following between Leonardo numbers for $n \geq 2$

$$Le_{n+1} = 2Le_n - Le_{n-2}.$$

Leonardo numbers are introduced and given some properties by Catarino and Borges in (Catarino and Borges

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2020a). Two dimensional recurrences relations of Leonardo numbers from its one-dimensional model is investigated in (Vieira et al. 2019). The matrix representation of Leonardo numbers is given and obtained new identities of Leonardo numbers in (Alp and Kocer 2021). The incomplete Leonardo numbers are defined, and some properties of incomplete Leonardo numbers are given in (Catarino and Borges 2020b). Shannon has defined generalized Leonardo numbers which are considered Asveld's extension and Horadam's generalized sequence in (Shannon 2019). Shannon and Deveci consider some real and complex extensions and generalizations of the Leonardo sequence in (Shannon and Deveci 2022). Tan and Leung introduce Leonardo p-numbers and investigate some basic properties of these numbers. They also define incomplete Leonardo p-numbers which generalize the incomplete Leonardo numbers in (Tan and Leung 2023). Bednarz and Wolowiec-Musial prove some identities for generalized Fibonacci–Leonardo numbers, also define matrix generators for these numbers in (Bednarz and Wolowiec-Musial 2023).

A generalized Leonardo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-3} \tag{1.1}$$

using the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts. To do this, for $n = 1, 2, 3, \ldots$, we define

$$W_{-n} = 2W_{-(n-2)} - W_{-(n-3)}.$$

Recurrence (1.1) is therefore true for all integers n.

Binet's formula for generalized Leonardo numbers can be obtained using (1.1). Binet's formula for generalized Leonardo number is as follows:

$$W_n = \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} = \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{\alpha-\beta} - z_3 \tag{1.2}$$

where

$$z_1 = W_2 - (2-\alpha)W_1 + (1-\alpha)W_0, \tag{1.3}$$

$$z_2 = W_2 - (2-\beta)W_1 + (1-\beta)W_0, \tag{1.4}$$

$$z_3 = W_2 - W_1 - W_0. \tag{1.5}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 2x^2 + 1 = (x^2 - x - 1)(x - 1) = 0.$$

Moreover

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$\beta = \frac{1-\sqrt{5}}{2}$$

$$\gamma = 1.$$

Note that

$\alpha + \beta + \gamma = 2,$

$\alpha\beta + \alpha\gamma + \beta\gamma = 0,$

$\alpha\beta\gamma = -1,$ or

$\alpha + \beta = 1, \alpha\beta = -1.$

Table 1 shows the first several generalized Leonardo numbers, with positive and negative subscripts.

Table 1. A few generalized Leonardo numbers.

n	W_n	W_{-n}
0	W_0	
1	W_1	$2W_1$-W_2
2	W_2	$2W_0$-W_1
3	$2W_2$-W_0	$4W_1$-W_0-$2W_2$
4	$4W_2$-W_1-$2W_0$	$4W_0$-$4W_1$+W_2
5	$7W_2$-$2W_1$-$4W_0$	$9W_1$-$4W_0$-$4W_2$
6	$12W_2$-$4W_1$-$7W_0$	$9W_0$-$12W_1$+$4W_2$
7	$20W_2$-$7W_1$-$12W_0$	$22W_1$-$12W_0$-$9W_2$
8	$33W_2$-$12W_1$-$20W_0$	$22W_0$-$33W_1$+$12W_2$
9	$54W_2$-$20W_1$-$33W_0$	$56W_1$-$33W_0$-$22W_2$
10	$88W_2$-$33W_1$-$54W_0$	$56W_0$-$88W_1$+$33W_2$
…	…	…

It is possible to specify three specific cases of the sequence $\{W_n\}$. These are, Modified Leonardo sequence $\{G_n\}_{n\geq 0}$, Leonardo-Lucas sequence $\{H_n\}_{n\geq 0}$ and Leonardo sequence $\{l_n\}_{n\geq 0}$ characterized by the third-order recurrence relations

$$G_n = 2G_{n-1} - G_{n-3}, G_0 = 0, G_1 = 1, G_2 = 2, \tag{1.6}$$

$$H_n = 2H_{n-1} - H_{n-3}, H_0 = 3, H_1 = 2, H_2 = 4, \tag{1.7}$$

$$l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3, \tag{1.8}$$

respectively.

Table 2. The first few values of the special generalized Leonardo numbers

n	0	1	2	3	4	5	6	7	8	9	10	...
G_n	0	1	2	4	7	12	20	33	54	88	143	...
G_{-n}		0	-1	0	-2	1	-4	4	-9	12	-22	...
H_n	3	2	4	5	8	12	19	30	48	77	124	...
H_{-n}		0	4	-3	8	-10	19	-28	48	-75	124	...
l_n	1	1	3	5	9	15	25	41	67	109	177	...
l_{-n}		-1	1	-3	3	-7	9	-17	25	-43	67	...

The sequences $\{G_n\}_{n\geq 0}, \{H_n\}_{n\geq 0}$ and $\{l_n\}_{n\geq 0}$ can be expanded to include negative subscripts by establishing

$$G_{-n} = 2G_{-(n-2)} - G_{-(n-3)}$$
$$H_{-n} = 2H_{-(n-2)} - H_{-(n-3)}$$
$$l_{-n} = 2l_{-(n-2)} - l_{-(n-3)}$$

for $n = 1, 2, 3, ...$, respectively. Therefore, recurrences (1.6)-(1.8) are satisfied for all integers n.

The sequences G_n, H_n and l_n are given by A000071, A001612, A001595 in (Sloane 1964) respectively.

We present the initial values of the modified Leonardo G_n, Leonardo-Lucas H_n, and Leonardo numbers l_n with positive and negative subscripts in Table 2.

Binet's formulas can be used to express modified Leonardo, Leonardo-Lucas, and Leonardo numbers for all integer , employing equations in (1.3) -(1.5)

$$G_n = \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{\alpha - \beta} - z_3 = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - 1,$$

$$H_n = \alpha^n + \beta^n + \gamma^n = \alpha^n + \beta^n + 1,$$

$$l_n = \frac{2(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} - 1,$$

respectively. Since, Binet's formulas of the Fibonacci and Lucas numbers are given by

$$F_n = \frac{\alpha^n}{\alpha - \beta} + \frac{\beta^n}{\beta - \alpha} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n,$$

respectively, we easily see that

$$G_n = F_{n+2} - 1, \quad (1.9)$$

$$H_n = L_n + 1, \quad (1.10)$$

$$l_n = 2F_{n+1} - 1. \quad (1.11)$$

Next, the ordinary generating function of the generalized Leonardo sequence is given.

LEMMA 1.1. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Leonardo sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2}{1 - 2x + x^3}.$$

Proof. Take $r = 2, s = 0, t = -1$ in Lemma 1 in (Soykan 2021).

The above lemma produces the following findings as specific examples.

COROLLARY 1.2. *Generating functions of modified Leonardo, Leonardo-Lucas and Leonardo numbers can be given respectively as*

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1 - 2x + x^3},$$
$$\sum_{n=0}^{\infty} H_n x^n = \frac{3 - 4x}{1 - 2x + x^3}$$
$$\sum_{n=0}^{\infty} l_n x^n = \frac{1 - x + x^2}{1 - 2x + x^3}.$$

Now, we give the definition of Gaussian numbers before giving literature-based information on Gaussian sequences.

Gaussian numbers, or Gaussian integers, are a subset of complex numbers. A complex number is expressed as $z = a + ib$ where a and b are arbitrary real integers and i is the imaginary unit with i^2 = -1. Gaussian integers are a special type of complex number. In other words, z is a Gaussian integer, denoted by $z = a + ib$, where a and b are random integers.

Next, we provide some literature-based information on Gaussian sequences. First, we show some Gaussian numbers with second-order recurrence relations.

- Horadam (1984) introduced Gaussian Fibonacci and Gaussian Lucas numbers which are defined by $GF_{n+2} = GF_{n+1} + GF_n$ and $GL_{n+2} = GL_{n+1} + GL_n$ where $GF_n = F_n + iF_{n+1}$ and $GL_n = L_n + iL_{n+1}$ with $n \geq 0$ (in fact, he defined these numbers as $CF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).
- Pethe and Horadam (1986) introduced Gaussian generalized Fibonacci numbers by $GF_n = F_n + iF_{n-1}$, where $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$.
- Halıcı and Öz (2016) studied Gaussian Pell and Pell Lucas numbers written by
- $GP_n = P_n + iP_{n-1}$, respectively, where $P_n = 2P_{n-1} + P_{n-2}, P_0 = 0, P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}, Q_0 = 2, Q_1 = 2$.
- Aşçı and Gürel (2013) presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by

  $GJ_n = J_n + iJ_{n-1}$,

  $Gj_n = j_n + ij_{n-1,}$

  respectively, where $J_n = J_{n-1} + 2J_{n-2}, J_0 = 0, J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}, j_0 = 2, j_1 = 1$.
- Taşçı (2021) presented and studied Gaussian Mersenne numbers defined by $GM_n = M_n + iM_{n-1}$,

  where $M_n = 3M_{n-1} - 2M_{n-2}, M_0 = 0, M_1 = 1$.
- Taşçı (2018b) introduced and examined the Gaussian Lucas Balancing and Gaussian balancing numbers, denoted by

  $GB_n = B_n + iB_{n-1}$,

  $GC_n = C_n + iC_{n-1}$,

  respectively, where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1$ and $C_n = 6C_{n-1} - C_{n-2}, C_0 = 1, C_1 = 3$.
- Yılmaz and Ertaş (2023) studied Gaussian Oresme numbers and defined them as $GS_n = S_n + iS_{n-1}$

  where Oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}, S_0 = 0, S_1 = \frac{1}{2}$.

  We now introduce a few Gaussian numbers that have recurrence relations of the third order.
- Soykan et al. (2018) presented Gaussian generalized Tribonacci numbers given by $GW_n = W_n + iW_{n-1}$

  where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$ with the initial condition W_0, W_1, W_2.
- Taşçı (2018a) studied Gaussian Padovan and Gaussian Pell-Padovan numbers written by

  $GP_n = P_n + iP_{n-1}$,

  $GR_n = R_n + iR_{n-1}$

  respectively, where $P_n = P_{n-2} + P_{n-3}, P_0 = 1, P_1 = 1, P_2 = 1$ and $R_n = 2R_{n-2} + R_{n-3}, R_0 = 1, R_1 = 1, R_2 = 1$.
- Cerda-Morales (2022) defined Gaussian third-order Jacobsthal numbers as $GJ_n = J_n + iJ_{n-1}$ where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}, J_0 = 0, J_1 = 1, J_2 = 1$.
- Karataş (2022) introduced the complex Leonardo numbers as $CLe_n = Le_n + iLe_{n-1}$, and give some of their properties including Binet's formula, generating function, Cassini, d'Ocagne's identities and calculated some summation formulas.

2. Gaussian Generalized Leonardo Numbers

This section introduces Gaussian generalized Leonardo numbers and examines their features, including Binet's formula and the generating function.

Gaussian generalized Leonardo numbers $\{GW_n\}_{n\geq 0} = GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 2GW_{n-1} - GW_{n-3}, \quad (2.1)$$

with the starting assumptions

$$GW_0 = W_0 + i((2W_1 - W_2)), GW_1 = W_1 + iW_0, GW_2 = W_2 + iW_1$$

not all being zero. The negative subscripts of the sequences $\{GW_n\}$ can be given by

$$GW_{-n} = 2GW_{-(n-2)} - GW_{-(n-3)} \quad (2.2)$$

for $n = 1, 2, 3, \ldots$. Thus, recurrence (2.1) holds for every integer n.

Keep in mind that for every integer n, we get

$$GW_n = W_n + iW_{n-1}. \quad (2.3)$$

The first few generalized Gaussian Leonardo numbers with positive and negative subscripts are presented in Table 3.

There are three specific cases of generalized Gaussian Leonardo numbers. Firstly, Gaussian modified Leonardo numbers, $GW_n(0, 1, 2, +i) = GG_n$, are given by

$$GG_n = 2GG_{n-1} - GG_{n-3} \quad (2.4)$$

with the starting values $GG_0 = 0, GG_1 = 1, GG_2 = 2 + i$.

Secondly, Gaussian Leonardo-Lucas numbers, $GW_n(3, 2, +3i, 4 + 2i) = GH_n$, are given by

$$GH_n = 2GH_{n-1} - GH_{n-3} \quad (2.5)$$

Table 3. First ten generalized Gaussian Leonardo numbers.

n	GW_n	GW_{-n}
0	$W_0+2iW_1-iW_2$	$W_0+2iW_1-iW_2$
1	iW_0+W_1	$2iW_0+(2-i)W_1-W_2$
2	iW_1+W_2	$(2-i)W_0-(1-4i)W_1-2iW_2$
3	$(2+i)W_2-W_0$	$(4-4i)W_1-(1-4i)W_0-(2-i)W_2$
4	$(4+2i)W_2-W_1-(2+i)W_0$	$(4-4i)W_0-(4-9i)W_1-(1-4i)W_2$
5	$(7+4i)W_2-(2+i)W_1-(4+2i)W_0$	$(9-12i)W_1-(4-9i)W_0-(4-4i)W_2$
6	$(12+7i)W_2-(4+2i)W_1-(7+4i)W_0$	$(9-12i)W_0-(12-22i)W_1-(4-9i)W_2$
7	$(20+12i)W_2-(7+4i)W_1-(12+7i)W_0$	$(22-33i)W_1-(12-22iW_0-(9-12i)W_2$
8	$(33+20i)W_2-(12+7i)W_1-(20+12i)W_0$	$(22-33i)W_0-(33-56i)W_1-(12-22i)W_2$
9	$(54+33i)W_2-(20+12i)W_1-(33+20i)W_0$	$(56-88i)W_1-(33-56i)W_0-(22-33i)W_2$
10	$(88+54i)W_2-(33+20i)W_1-(54+33i)W_0$	$(56-88i)W_0-(88-145i)W_1-(33-56i)W_2$
…	…	…

Table 4. Some values of the special cases of Gaussian generalized Leonardo numbers.

n	0	1	2	3	4	5	6	7	8	…
GG_n	0	1	2+i	4+2i	7+4i	12+7i	20+12i	33+20i	54+33i	…
GG_{-n}	0	-i	-1	-2i	-2+i	1-4i	-4+4i	4-9i	-9+12i	…
GH_n	3	2+3i	4+2i	5+4i	8+5i	12+8i	19+12i	30+19i	48+30i	…
GH_{-n}	3	4i	4-3i	-3+8i	8-10i	-10+19i	19-28i	-28+48i	48-75i	…
Gl_n	1-i	1+i	3+i	5+3i	9+5i	15+9i	25+15i	41+25i	67+41i	…
Gl_{-n}	1-i	-1+i	1-3i	-3+3i	3-7i	-7+9i	9-17i	-17+25i	25-43i	…

with the starting values

$GH_0 = 3, GH_1 = 2+3i, GH_2 = 4+2i.$

Thirdly, Gaussian Leonardo numbers, $GW_n(1,1+i,3+i) = Gl_n$, are given by

$$Gl_n = 2Gl_{n-1} - Gl_{n-3} \tag{2.6}$$

with the starting values $Gl_0 = 3, Gl_1 = 1+i, Gl_2 = 3+i$.

The negative subscripts of the sequences $\{GG_n\}_{n\geq 0}, \{GH_n\}_{n\geq 0}$ and $\{Gl_n\}_{n\geq 0}$ is defined by

$$GG_{-n} = 2GG_{-(n-2)} - GG_{-(n-3)},$$
$$GH_{-n} = 2GH_{-(n-2)} - GH_{-(n-3)},,$$
$$Gl_{-n} = 2Gl_{-(n-2)} - Gl_{-(n-3)},$$

for n = 1,2,3,... respectively. As a result, recurrences (2.4)-(2.6) are satisfied for every integer n.

Keep in mind that for every integer n, we have

$$GG_n = G_n + iG_{n-1},$$
$$GH_n = H_n + iH_{n-1},$$
$$Gl_n = l_n + il_{n-1}.$$

The first eight values of Gaussian modified Leonardo numbers, Gaussian Leonardo-Lucas numbers and Gaussian Leonardo numbers with positive and negative subscripts are given in Table 4.

Next, we present Binet's formula for Gaussian generalized Leonardo numbers.

THEOREM 2.1. *Binet's formula of Gaussian generalized Leonardo numbers can be presented as follows:*

$$GW_n = \left(\frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{\alpha-\beta} - z_3\right) + i\left(\frac{z_1\alpha^n - z_2\beta^n}{\alpha-\beta} - z_3\right)$$

where z_1, z_2, *and* z_3 *are given as*

$z_1 = W_2 - (2-\alpha)W_1 + (1-\alpha)W_0$
$z_2 = W_2 - (2-\beta)W_1 + (1-\beta)W_0,$
$z_3 = W_2 - W_1 - W_0.$

Binet's formulas of the Gaussian generalized Leonardo numbers GW_n have three specific cases which gives the Binet's formulas of the Modified Leonardo sequence $\{GG_n\}_{n\geq 0}$, the Leonardo-Lucas sequence $\{GH_n\}_{n\geq 0}$, and the Gaussian Leonardo sequence $\{Gl_n\}_{n\geq 0}$, respectively as follows:

$$GG_n = \left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta} - 1\right) + i\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - 1\right), \quad (2.7)$$

$$GH_n = (\alpha^{n+1}+\beta^{n+1}+1) + i(\alpha^n+\beta^n+1),$$

$$Gl_n = \left(\frac{2\alpha^{n+1}-2\beta^{n+1}}{\alpha-\beta} - 1\right) + i\left(\frac{2\alpha^{n}-2\beta^{n}}{\alpha-\beta} - 1\right). \quad (2.8)$$

To describe the generating function of Gaussian generalized Leonardo numbers we give the following theorem.

THEOREM 2.2. *Let* $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$ *denote the generating function of Gaussian generalized Leonardo numbers. Then*,

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 2GW_0)x + (GW_2 - 2GW_1)x^2}{1-2x+x^3}. \quad (2.9)$$

Proof. Considering the definition of Gaussian generalized Leonardo numbers, and subtracting $2xf_{GW_n}(x)$ and $-x^3 f_{GW_n}(x)$ from $f_{GW_n}(x)$ we obtain

$$(1-2x-x_3)f_{GW_n}(x) = \sum_{n=0}^{\alpha} GW_n x^n - 2x\sum_{n=0}^{\infty} GW_n x^n - x^3\sum_{n=0}^{\infty} GW_n x^n,$$
$$= \sum_{n=0}^{\alpha} GW_n x^n - 2\sum_{n=0}^{\infty} GW_n x^{n+1} - \sum_{n=0}^{\infty} GW_n x^{n+3},$$
$$= \sum_{n=0}^{\alpha} GW_n x^n - 2\sum_{n=1}^{\infty} GW_{n-1} x^n - \sum_{n=3}^{\infty} GW_{n-3} x^n,$$
$$= GW_0 + GW_1 x + GW_2 x^2 - 2GW_0 x + 2GW_1 x^2 + \sum_{n=3}^{\infty}(GW_n - 2GW_{n-1} - GW_{n-3})x^n,$$
$$= GW_0 + GW_1 x + GW_2 x^2 - 2GW_0 x + 2GW_1 x^2,$$
$$= GW_0 + (GW_1 - 2GW_0)x + (GW_2 - 2GW_1)x^2,$$

and dividing both sides with $1-2x-x^3$ above equation, we get (2.9).

Theorem 2.2 gives following results as special cases,

$$f_{GG_n}(x) = \frac{x+ix^2}{1-2x+x^3}, f_{GH_n}(x) = \frac{3+(-4+3i)x-3ix^2}{1-2x+x^3},$$
$$f_{Gl_n}(x) = \frac{1-i+(-1+3i)x+(1-i)x^2}{1-2x+x^3}.$$

3. Some Identities About Recurrence Relations of Gaussian Generalized Leonardo Numbers

This section introduces the identities for Gaussian modified Leonardo, Gaussian Leonardo-Lucas, and Gaussian Leonardo numbers.

THEOREM 3.1. *For every integer n, the following equations are valid.*

$$GH_n = 4GG_{n+3} - 8GG_{n+2} + 3GG_{n+1}, \quad (3.1)$$
$$5GG_n = 8GH_{n+3} - 7GH_{n+2} - 6GH_{n+1}, \quad (3.2)$$
$$2GG_n = -Gl_{n+3} + Gl_{n+2} + 2Gl_{n+1}, \quad (3.3)$$
$$Gl_n = GG_{n+3} - 3GG_{n+2} + 3GG_{n+1}, \quad (3.4)$$
$$2GH_n = Gl_{n+3} + 2Gl_{n+2} - 5Gl_{n+1}, \quad (3.5)$$
$$5Gl_n = 7GH_{n+3} - 3GH_{n+2} - 9GH_{n+1}. \quad (3.6)$$

Proof. To prove identity (3.1), we can write $GH_n = aGG_{n+3} + bGG_{n+2} + cGG_{n+1}$ and solve the system of equations we get,

$GH_0 = aGG_3 + bGG_2 + cGG_1;$
$GH_1 = aGG_4 + bGG_3 + cGG_2;$
$GH_2 = aGG_5 + bGG_4 + cGG_3.$

Then, we obtain $a=4, b=-8, c=3$. The other identities can be found similarly.

LEMMA 3.2. (*See (Frontczak 2018)) Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be the generating function of the sequence* $\{a_n\}_{n\geq 0}$. *Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *can be given as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n^{x^n}} = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1^{x^n}} = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The following theorem provides the generating functions for even and odd-indexed generalized Leonardo sequences.

THEOREM 3.3. *The sequence* GW_{2n} *and* GW_{2n+1} *have the following generating functions*:

$$f_{GW_{2n}}(x) = \frac{GW_0 + (-4GW_0 + GW_2)x + (2GW_0 - GW_1)x^2}{-x^3+4x^2-4x+1}, \quad (3.7)$$

$$f_{GW_{2n+1}}(x) = \frac{GW_1-(GW_0+4GW_1-2GW_2)x+(2GW_1-GW_2)x^2}{-x^3+4x^2-4x+1}. \quad (3.8)$$

Proof. We only prove (3.7). From Theorem 2.2 we can obtain following identities:

$$f_{GW_n}(\sqrt{x}) = \frac{GW_0+(GW_1-2GW_0)\sqrt{x}}{1-2\sqrt{x}+x^{\frac{3}{2}}},$$

$$f_{GW_n}(-\sqrt{x}) = \frac{GW_0-(GW_1-2GW_0)\sqrt{x}+(GW_2-2GW_1)x}{1+2\sqrt{x}-x^{\frac{3}{2}}}.$$

Thereby, using Lemma 3.2 identity (3.7) can be proved. The other identity can be proved similarly.

The following is a consequence of Theorem 3.3.

COROLLARY 3.4.

(a) $f_{GG_{2n}}(x) = \frac{(2+i)x-x^2}{-x^3+4x^2-4x+1}$ *and*

$f_{GG_{2n+1}}(x) = \frac{-ix^2+2ix+1}{-x^3+4x^2-4x+1}$

(b) $f_{GH_{2n}}(x) = \frac{(4-3i)x^2-(8-2i)x+3}{-x^3+4x^2-4x+1}$ *and*

$f_{GH_{2n+1}}(x) = \frac{4ix^2-(3+8i)x+2+3i}{-x^3+4x^2-4x+1}$

(c) $f_{Gl_{2n}}(x) = \frac{(1-3i)x^2-(1-5i)x+1-i}{-x^3+4x^2-4x+1}$ *and*

$f_{Gl_{2n+1}}(x) = \frac{-(1-i)x^2+(1-i)x+1+i}{-x^3+4x^2-4x+1}$

From Corollary 3.4 we can obtain the following corollary which describes the identities on Gaussian generalized Leonardo sequences.

COROLLARY 3.5.

(a): $(2+i)Gl_{2n-2}-Gl_{2n-4} = ((1-i))GG_{2n} -(1-5i)GG_{2n-2}+(1-3i)GG_{2n-4}$

(b): $(2+i)GH_{2n-2}-GH_{2n-4} = 3GG_{2n} +(-8+2i)GG_{2n-2}+(4-3i)GG_{2n-4}$

(c): $(4-3i)Gl_{2n-4}+(-8+2i)Gl_{2n-2}+3Gl_{2n} = (1-i)GH_{2n}-(1-5i)GH_{2n-2}+(1-3i)GH_{2n-4}$

(d): $-iGH_{2n-3}+2iGH_{2n-1}+GH_{2n+1} = (2+3i)GG_{2n+1} -(3+8i)GG_{2n-1}+4iGG_{2n-3}$

(e): $(-1+i)GG_{2n-3}+(1-i)GG_{2n-1}+(1+i)GG_{2n+1} = -iGl_{2n-3}+2iGl_{2n-1}+Gl_{2n+1}$

(f): $4iGl_{2n-3}-(3+8i)Gl_{2n-1}+(2+3i)Gl_{2n+1} = (-1+i)GH_{2n-3}+(1-i)GH_{2n-1}+(1+i)GH_{2n+1}.$

Proof. From Corollary 3.4 we obtain

$$(-x^2+(2+i)x)f_{Gl_{2n}}(x) = ((1-3i)x^2 -(1-5i)x+1-i)f_{GG_{2n}}(x).$$

The first part is equal to

$$\begin{aligned}(-x^2+(2+i)x)\sum_{n=0}^{\infty}Gl_{2n}x^n &= -\sum_{n=0}^{\infty}Gl_{2n}x^{n+2} +(2+i)\sum_{n=0}^{\infty}Gl_{2n}x^{n+1}\\ &= (2+i)\sum_{n=1}^{\infty}Gl_{2n-2}x^n-\sum_{n=2}^{\infty}Gl_{2n-4}x^n\\ &= (2+i)Gl_0x+\sum_{n=2}^{\infty}((2+i)Gl_{2n-2}-Gl_{2n-4})x^n\\ &= (3-i)x+\sum_{n=2}^{\infty}((2+i)Gl_{2n-2}-Gl_{2n-4})x^n\end{aligned}$$

whereas the second part is equal to

$$\begin{aligned}&((1-3i)x^2-(1-5i)x+1-i)\sum_{n=0}^{\infty}GG_{2n}x^n\\ &= (1-i)\sum_{n=0}^{\infty}GG_{2n}x^n-(1-5i)\sum_{n=0}^{\infty}GG_{2n}x^{n+1} +(1-3i)\sum_{n=0}^{\infty}GG_{2n}x^{n+2}\\ &= (1-i)\sum_{n=0}^{\infty}GG_{2n}x^n-(1-5i)\sum_{n=1}^{\infty}GG_{2n-2}x^n +(1-3i)\sum_{n=2}^{\infty}GG_{2n-4}x^n\\ &= (1-i)GG_2x+\sum_{n=2}^{\infty}(((1-i))GG_{2n}-(1-5i)GG_{2n-2} +(1-3i)GG_{2n-4})x^n\\ &= (3-i)x+\sum_{n=2}^{\infty}(((1-i))GG_{2n}-(1-5i)GG_{2n-2} +(1-3i)GG_{2n-4})x^n\end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can present other identities similarly.

We can get Honsberger's identity related to generalised Leonardo numbers and modified Leonardo numbers given below.

THEOREM 3.6. *For all integers m, n the following identity holds*:

$$GW_{m+n} = G_{m+1}GW_n-GW_{n-1}G_{m-1}-GW_{n-2}G_m.$$

Proof. First, we assume that $m,n\geq 0$. We prove Theorem 3.6 by mathematical induction on m. If $m=0$ we get

$$GW_n = G_1GW_n-GW_{n-1}G_{-1}-GW_{n-2}G_0 = GW_n$$

since $G_1 = 1, G_0 = 0, G_{-1} = 0$. If $m = 1$ we get

$$GW_{n+1} = G_2 GW_n - GW_{n-1} G_0 - GW_{n-2} G_1 = 2GW_n - GW_{n-2}.$$

since $G_2 = 2, G_1 = 1, G_0 = 0$. We assume that the identity given holds for $m \leq k$. For $m = k+1$, we get

$$\begin{aligned}
&GW_{(k+1)+n} = 2GW_{n+k} - GW_{n+k-2}\\
&= 2(G_{k+1}GW_n - GW_{n-1}G_{k-1} - GW_{n-2}G_k)\\
&-(G_{k-1}GW_n - GW_{n-1}G_{k-3} - GW_{n-2}G_{k-2})\\
&= GW_n(2G_{k+1} - G_{k-1}) - GW_{n-1}(2G_{k-1} - G_{k-3})\\
&-GW_{n-2}(2G_k - G_{k-2})\\
&= GW_n G_{k+2} - GW_{n-1}G_k - GW_{n-2}G_{k+1}\\
&= GW_n G_{(k+1)+1} - GW_{n-1}G_{(k+1)-1} - GW_{n-2}G_{(k+1)}
\end{aligned}$$

Consequently, by mathematical induction on , this proves Theorem 3.6. The case $m, n < 0$, can be proved similarly.

For all integers m, n taking $GW_n = GG_n$ or $GW_n = GH_n$ or $GW_n = Gl_n$, respectively, we get,

$$\begin{aligned}
&GG_{m+n} = G_{m+1}GG_n - GG_{n-l}G_{m-l} - GG_{n-2}G_m,\\
&GH_{m+n} = G_{m+1}GH_n - G_{m-1}GH_{n-1} - G_m GH_{n-2},\\
&Gl_{m+n} = G_{m+1}Gl_n - G_{m-1}Gl_{n-1} + G_m Gl_{n-2}.
\end{aligned}$$

4. Simpson's Formula

In this section, we present Simpson's formula of generalized Gaussian Leonardo numbers. This is a special case of (Soykan 2020, Theorem 4.1). We give the proof by calculating determinant and using Binet's formula of Gaussian generalized Leonardo numbers.

Theorem 4.1 (Simpson's formula of generalized Gaussian Leonardo numbers). *For all integer n, we can write following equality*

$$\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} = (-1)^n \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix}$$
$$= (-1)^n(3+i)[W_0^2 - W_1^2 + W_2^2 - 2W_0^2 W_2 - 2W_0 W_1^2$$
$$+3W_0 W_1 W_2 + 4W_1^2 W_2 - 4W_1 W_2^2].$$

Proof. Putting $t = -1$ in Theorem 3 in (Soykan 2021) we can obtain

$$\begin{aligned}
&\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix}\\
&= \begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} + i \begin{vmatrix} W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \\ W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix}\\
&= (-1)^n \begin{vmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{vmatrix}\\
&= (-1)^n[-GW_0^3 + 2GW_0 GW_1 GW_{-1} + GW_2 W_{-2} GW_0\\
&-GW_1^2 GW_{-2} - GW_2 GW_{-1}^2]\\
&= (-1)^n(3+i)[W_0^2 - W_1^2 + W_2^2 - 2W_0^2 W_2 - 2W_0 W_1^2\\
&+3W_0 W_1 W_2 + 4W_1^2 W_2 - 4W_1 W_2^2].
\end{aligned}$$

From Theorem 4.1 we get the following corollary.

Corollary 4.2. *For all integer n, we get the following identities:*

(a): $\begin{vmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{vmatrix} = (-1)^n(3+i),$

(b): $\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{vmatrix} = 5(-1)^n(3+i),$

(c): $\begin{vmatrix} Gl_{n+2} & Gl_{n+1} & Gl_n \\ Gl_{n+1} & Gl_n & Gl_{n-1} \\ Gl_n & Gl_{n-1} & Gl_{n-2} \end{vmatrix} = 4(-1)^{n+1}(3+i).$

5. Sum Formulas

In this section, we identify some sum formulas of generalized Gaussian Leonardo numbers.

THEOREM 5.1. *For all integers $n \geq 0$, we have sum formulas given below*

(a): $\sum_{k=0}^{n} GW_k = (n+3)GW_n - (n+2)GW_{n+2} + (n+3)GW_{n+1}$
$+2GW_2 - 3GW_1 - 2GW_0,$

(b): $\sum_{k=0}^{n} GW_{2k} = (n+1)GW_{2n} + (n+2)GW_{2n+1}$
$-(n+1)GW_{2n+2} + GW_2 - 2GW_1,$

(c): $\sum_{k=0}^{n} GW_{2k+1} = (n+1)GW_{2n+1} - nGW_{2n+2} + (n+1)GW_{2n}$
$-GW_0.$

Proof. From (2.3) we can write the following sum formulas.

$$\sum_{k=0}^{n} GW_k = \sum_{k=0}^{n} W_k + i\sum_{k=0}^{n} W_{k-1},$$
$$\sum_{k=0}^{n} GW_{2k} = \sum_{k=0}^{n} W_{2k} + i\sum_{k=0}^{n} W_{2k-1},$$
$$\sum_{k=0}^{n} GW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + i\sum_{k=0}^{n} W_{2k}.$$

Using sum formulas in (Soykan 2023 Theorem 48 (*b*)) we can write

(a): $\sum_{k=0}^{n} GW_k = (n+3)W_n - (n+2)W_{n+2} + (n+3)W_{n+1} + 2W_2 - 3W_1 - 2W_0,$

(b): $\sum_{k=0}^{n} GW_{2k} = (n+1)W_{2n} + (n+2)W_{2n+1} - (n+1)W_{2n+2} + W_2 - 2W_1,$

(c): $\sum_{k=0}^{n} GW_{2k+1} = (n+1)W_{2n+1} + (n+1)W_{2n} - nW_{2n+2} - W_0.$

So that, the proof is done easily.

The previous theorem gives the following corollaries.

COROLLARY 5.2.

(a): $\sum_{k=0}^{n} GG_k = (n+3)GG_n - (n+2)GG_{n+2} + (n+3)GG_{n+1} + 1 + 2i,$

(b): $\sum_{k=0}^{n} GH_k = (n+3)GH_n - (n+2)GH_{n+2} + (n+3)GH_{n+1} - 4 - 5i,$

(c): $\sum_{k=0}^{n} Gl_k = (n+3)Gl_n - (n+2)Gl_{n+2} + (n+3)Gl_{n+1} + 1 + i.$

Next, we give sum formulas which are given by even subscripts.

COROLLARY 5.3.

(a): $\sum_{k=0}^{n} GG_{2k} = (n+1)GG_{2n} + (n+2)GG_{2n+1} - (n+1)GG_{2n+2} + i,$

(b): $\sum_{k=0}^{n} GH_{2k} = (n+1)GH_{2n} + (n+2)GH_{2n+1} - (n+1)GH_{2n+2} - 4i,$

(c): $\sum_{k=0}^{n} Gl_{2k} = (n+1)Gl_{2n} + (n+2)Gl_{2n+1} - (n+1)Gl_{2n+2} + 1 - i.$

Next, we give sum formulas which are given by odd subscripts.

COROLLARY 5.4.

(a): $\sum_{k=0}^{n} GG_{2k+1} = (n+1)GG_{2n+1} + (n+1)GG_{2n} - nGG_{2n+2},$

(b): $\sum_{k=0}^{n} GH_{2k+1} = (n+1)GH_{2n+1} + (n+1)GH_{2n} - nGH_{2n+2} - 3,$

(c): $\sum_{k=0}^{n} Gl_{2k+1} = (n+1)Gl_{2n+1} + (n+1)Gl_{2n} - nGl_{2n+2} - 1 + i.$

5.1. Sums of Squares and other sum formulas.

THEOREM 5.5. *For all integers* $n \geq 0$, *we have sum formulas given below*

(a):
$$\begin{aligned}\sum_{k=0}^{n} GW_k^2 &= \left(n+\frac{7}{2}\right)GW_{n+3}^2 + \left(n+\frac{9}{2}\right)GW_{n+2}^2 + \left(n+\frac{7}{2}\right)GW_{n+1}^2 - 2(n+4)GW_{n+2}GW_{n+3} \\ &+ 2(n+5)GW_{n+2} - 2(n+4)GW_{n+1}GW_{n+3} \\ &+ 6GW_1GW_2 - 8GW_0GW_1 + 6GW_0GW_2 - \frac{5}{2}GW_2^2 - \frac{7}{2}GW_1^2 - \frac{5}{2}GW_0^2. \end{aligned} \tag{5.1}$$

(b):
$$\begin{aligned}\sum_{k=0}^{n} GW_kGW_{k+1} &= (n+4)GW_{n+1}^2 + (n+5)GW_{n+2}^2 + (n+3)GW_{n+3}^2 + \left(2n+\frac{15}{2}\right)GW_{n+1}GW_{n+2} \\ &- (2n+\frac{13}{2})GW_{n+1}GW_{n+3} - (2n+\frac{15}{2})GW_{n+2}GW_{n+3} \\ &- 3GW_0^2 - 4GW_1^2 - 2GW_2^2 - \frac{11}{2}GW_0GW_1 + \frac{9}{2}GW_0GW_2 + \frac{11}{2}GW_1GW_2;\end{aligned}$$

(c):
$$\begin{aligned}\sum_{k=0}^{n} GW_kGW_{k+2} &= (n+3)GW_{n+1}^2 + (n+4)GW_{n+2}^2 + (n+2)GW_{n+3}^2 + \left(2n+\frac{15}{2}\right)GW_{n+1}GW_{n+2} \\ &- \left(2n+\frac{11}{2}\right)GW_{n+1}GW_{n+3} - \left(2n+\frac{11}{2}\right)GW_{n+2}GW_{n+3} \\ &- 2GW_0^2 - 3GW_1^2 - \frac{11}{2}GW_1GW_0 + \frac{7}{2}GW_0GW_2 + \frac{7}{2}GW_1GW_2;\end{aligned}$$

As a result of Theorem 5.5, we can give the following corollary.

COROLLARY 5.6.

(a): (*i*)
$$\begin{aligned}\sum_{k=0}^{n} GG_k^2 &= \left(n+\frac{7}{2}\right)GG_{n+3}^2 + \left(n+\frac{9}{2}\right)GG_{n+2}^2 + \left(n+\frac{7}{2}\right)GG_{n+1}^2 - 2(n+4)GG_{n+2}GG_{n+3} \\ &+ 2(n+5)GG_{n+l}GG_{n+2} - 2(n+4)GG_{n+l}GG_{n+3} + 1 - 4i,\end{aligned}$$

$$(ii)\sum_{k=0}^{n} GH_k^2 = \left(n+\frac{7}{2}\right)GH_{n+3}^2 + \left(n+\frac{9}{2}\right)GH_{n+2}^2$$
$$+\left(n+\frac{7}{2}\right)GH_{n+1}^2 - 2(n+4)GH_{n+2}GH_{n+3}$$
$$+2(n+5)GH_{n+1}GH_{n+2} - 2(n+4)GH_{n+1}GH_{n+3}$$
$$+1-22i;$$
$$(iii)\sum_{k=0}^{n} Gl_k^2 = \left(n+\frac{7}{2}\right)Gl_{n+3}^2 + \left(n+\frac{9}{2}\right)Gl_{n+2}^2$$
$$+\left(n+\frac{7}{2}\right)Gl_{n+1}^2 - 2(n+4)Gl_{n+2}Gl_{n+3}$$
$$+2(n+5)Gl_{n+1}Gl_{n+2} - 2(n+4)Gl_{n+1}Gl_{n+3} - 5i,$$

(b): $$(i)\sum_{k=0}^{n} GG_kGG_{k+1} = (n+4)GG_{n+1}^2 + (n+5)GG_{n+2}^2$$
$$+(n+3)GG_{n+3}^2 + \left(2n+\frac{15}{2}\right)GG_{n+1}GG_{n+2}$$
$$-\left(2n+\frac{13}{2}\right)GG_{n+1}GG_{n+3} - \left(2n+\frac{15}{2}\right)GG_{n+2}GG_{n+3}$$
$$+1-\frac{5}{2}i,$$
$$(ii)\sum_{k=0}^{n} GH_kGH_{k+1} = (n+4)GH_{n+1}^2 + (n+5)GH_{n+2}^2$$
$$+(n+3)GH_{n+3}^2 + \left(2n+\frac{15}{2}\right)GH_{n+l}GH_{n+2}$$
$$-\left(2n+\frac{13}{2}\right)GH_{l+1}GH_{n+3} - \left(2n+\frac{15}{2}\right)GH_{n+2}GH_{n+3}$$
$$+1-\frac{29}{2}i,$$
$$(iii)\sum_{k=0}^{n} Gl_kGl_{k+1} = (n+4)Gl_{n+1}^2 + (n+5)Gl_{n+2}^2$$
$$+(n+3)Gl_{n+3}^2 + \left(2n+\frac{15}{2}\right)Gl_{n+1}Gl_{n+2}$$
$$-\left(2n+\frac{13}{2}\right)Gl_{n+1}Gl_{n+3} - \left(2n+\frac{15}{2}\right)Gl_{n+2}Gl_{n+3}$$
$$+2-i.$$

(c): $$(i)\sum_{k=0}^{n} GG_kGG_{k+2} = (n+3)GG_{n+1}^2 + (n+4)GG_{n+2}^2$$
$$+(n+2)GG_{n+3}^2 + \left(2n+\frac{15}{2}\right)GG_{n+1}GG_{n+2}$$
$$-\left(2n+\frac{11}{2}\right)GG_{n+1}GG_{n+3} - \left(2n+\frac{11}{2}\right)GG_{n+2}GG_{n+3}$$
$$+1-\frac{1}{2}i,$$
$$(ii)\sum_{k=0}^{n} GH_kGH_{k+2} = (n+3)GH_{n+1}^2 + (n+4)GH_{n+2}^2$$
$$+(n+2)GH_{n+3}^2 + \left(2n+\frac{15}{2}\right)GH_{n+1}GH_{n+2}$$
$$-\left(2n+\frac{11}{2}\right)GH_{n+1}GH_{n+3} - \left(2n+\frac{11}{2}\right)GH_{n+2}GH_{n+3}$$
$$+1-\frac{49}{2}i,$$
$$(iii)\sum_{k=0}^{n} Gl_kGl_{k+2} = (n+3)Gl_{n+1}^2 + (n+4)Gl_{n+2}^2$$
$$+(n+2)Gl_{n+3}^2 + \left(2n+\frac{15}{2}\right)Gl_{n+1}Gl_{n+2}$$
$$-\left(2n+\frac{11}{2}\right)Gl_{n+1}Gl_{n+3} - \left(2n+\frac{11}{2}\right)Gl_{n+2}Gl_{n+3}$$
$$+2-i.$$

6. Matrix Formulation of GW_n

Consider the sequence $\{W_n\}$ defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-3}$$

with the starting values

$$W_0 = 0, W_1 = 1, W_2 = 2.$$

We define the square matrix A of order 3 as

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $A =-1$. Then we give the following lemma.

LEMMA 6.1. *For* $n \geq 0$ *the following identity is true*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}. \qquad (6.1)$$

Proof. The identity (6.1) can be proved by mathematical induction on n. If $n=0$ we obtain

$$\begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n=k$. Thus, the following identity is true:

$$\begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n=k+1$, we get

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 2GW_{k+2} - GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix} = \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof is completed.

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

For the proof see (Soykan 2022).

We define

$$N_{GW}=\begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \quad (6.2)$$

$$E_{GW}=\begin{pmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{pmatrix}. \quad (6.3)$$

Now, we have the following theorem with N_{GW} and E_{GW}

THEOREM 6.2. *Using* N_{GW} *and* E_{GW}, *we get*

$A^n N_{GW} = E_{GW}$.

Proof. Note that we get

$$A^n N_{GW}=\begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}\begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$a_{11} = GW_2 G_{n+1} - GW_1 G_{n-1} - GW_0 G_n$;

$a_{12} = GW_1 G_{n+1} - GW_0 G_{n-1} - GW_{-l} G_n$,

$a_{13} = GW_0 G_{n+1} - GW_{-1} G_{n-1} - GW_{-2} G_n$,

$a_{21} = GW_2 G_n - GW_1 G_{n-2} - GW_0 G_{n-1}$,

$a_{22} = GW_1 G_n - GW_0 G_{n-2} - GW_{-1} G_{n-1}$,

$a_{23} = GW_0 G_n - GW_{-1} G_{n-2} - GW_{-2} G_{n-1}$,

$a_{31} = GW_2 G_{n-1} - GW_1 G_{n-3} - GW_0 G_{n-2}$,

$a_{32} = GW_1 G_{n-1} - GW_0 G_{n-3} - GW_{-1} G_{n-2}$,

$a_{33} = GW_0 G_{n-1} - GW_{-1} G_{n-3} - GW_{-2} G_{n-2}$,

Using the Theorem 3.6 we see that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{pmatrix}.$$

Hence, the proof is done.

By taking, $GW_n = GG_n$ with GG_0, GG_1, GG_2 in (6.2) and (6.3), $GW_n = GH_n$ with GH_0, GH_1, GH_2 in (6.2) and (6.3), $GW_n = Gl_n$ with Gl_0, Gl_1, Gl_2 in (6.2) and (6.3) respectively, we get:

$$N_{GG}=\begin{pmatrix} 2+i & 1 & 0 \\ 1 & 0 & -i \\ 0 & -i & -1 \end{pmatrix}, E_{GG}=\begin{pmatrix} GG_{n+2} & GG_{n+1} & GG_n \\ GG_{n+1} & GG_n & GG_{n-1} \\ GG_n & GG_{n-1} & GG_{n-2} \end{pmatrix}$$

$$N_{GH}=\begin{pmatrix} 4+2i & 2+3i & 3 \\ 2+3i & 3 & 4i \\ 3 & 4i & 4-3i \end{pmatrix}, E_{GH}=\begin{pmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{pmatrix}$$

$$N_{Gl}=\begin{pmatrix} 3+i & 1+i & 1-i \\ +i1 & 1-i & -1+i \\ 1-i & -1+i & 1-3i \end{pmatrix}, E_{Gl}=\begin{pmatrix} Gl_{n+2} & Gl_{n+1} & Gl_n \\ Gl_{n+1} & Gl_n & Gl_{n-1} \\ Gl_n & Gl_{n-1} & Gl_{n-2} \end{pmatrix}$$

From Theorem 6.2, we can write the following corollary.

COROLLARY 6.3. *The following identities are hold.*

(a): $A^n N_{GG} = E_{GG}$.

(b): $A^n N_{GH} = E_{GH}$.

(c): $A^n N_{Gl} = E_{Gl}$.

7. Conclusion

There have been several studies on numerical sequences in the literature, and they have been widely employed in various scientific disciplines, including physics, engineering, architecture, nature, and art. In this study, we describe the Gaussian generalized Leonardo sequence and concentrate on three distinct cases: Gaussian modified Leonardo numbers, Gaussian Leonardo-Lucas numbers, and Gaussian Leonardo numbers.

- In the first section, we provide some background information about generalized Leonardo numbers and give some information about Gaussian sequences from literature.
- In the second section, we define Gaussian generalized Leonardo numbers and give some properties such as Binet's formula and generating function.
- In the third section, we present some identities, using recurrence relation and generating function, on Gaussian modified Leonardo, Gaussian Leonardo-Lucas, and Gaussian Leonardo numbers.
- In the fourth section, we find Simpson's formula of the Gaussian generalized Leonardo numbers.
- In the fifth section, we identify some sum formulas of Gaussian generalized Leonardo numbers.
- In sixth section, we give the square matrix using triangular sequence and introduce some identities about Gaussian generalized Leonardo numbers.

Finally, we want to continue this study for the dual, hyperbolic and dual hyperbolic generalized Leonardo numbers.

People who is willing to study on this subject, the results we present would be a great help to them.

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