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Research Article

According to the Frenet Frame Spherical Indicators and Results on $\mathbb{E}^3$

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Abstract: In this study, we showed that the spherical indicator curve frames can correspond to a Bishop frame according to the Serret-Frenet frame of a regular curve.

Keywords: Serret-Frenet frame, Bishop frame, spherical indicator.

1. Introduction and Preliminaries

Curves are one of the critical areas of differential geometry. Space curves were defined as the intersection of two surfaces by Clairaut in the first quarter of the 18th century [9]. Frenet (1847) and Serret, without knowing each other, defined a frame using the derivatives of a regular curve. This frame was called the Serret-Frenet frame, referring to the two. Sometimes it is simply called the Frenet frame. The Frenet frame [7] in Euclidean space $\mathbb{E}^3$ is a frame obtained using the velocity and acceleration vectors of a regular curve. Let the velocity and acceleration vectors of the curve $\pi : I \longrightarrow \mathbb{E}^3$ be π' and π'', respectively. Accordingly, the orthonormal frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ obtained as

$$\mathbf{t} = \frac{\pi'}{\|\pi'\|}, \ \mathbf{b} = \frac{\pi' \wedge \pi''}{\|\pi' \wedge \pi''\|}, \ \mathbf{n} = \mathbf{b} \wedge \mathbf{t}$$

is the Frenet frame. Here, the vector fields $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are called the tangent vector field, the principal normal vector field and the binormal vector field of the curve π, respectively. If the curve π is unit speed ($\|\pi'\| = 1$), then

$$\mathbf{t} = \pi', \ \mathbf{n} = \frac{\pi''}{\|\pi''\|}, \ \mathbf{b} = \mathbf{t} \wedge \mathbf{n}.$$

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Derivative changes of the frame $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b}\}$ are

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n}, \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}' &= -\tau \mathbf{n}. \end{aligned}$$

Here κ and τ are called the first and second curvatures of the curve π, respectively, such that

$$\kappa = \frac{\|\pi' \wedge \pi''\|}{\|\pi''\|^3} \text{ and } \tau = \frac{\det\left(\pi', \pi'', \pi'''\right)}{\|\pi' \wedge \pi''\|^2}. \tag{1}$$

The quintet $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b},\, \kappa,\, \tau\}$ are called Frenet apparatus. Many studies have been done on this frame in geometry, physics and engineering. These studies have also been advanced in non-Euclidean spaces. Some of these studies are spherical indicators of curves. If $X_{\pi(s)} = X(\pi(s)) \in T_\pi \pi(s)$ the unit vector field X is said to be constrained to the curve π. If we take $X = \overrightarrow{PQ}$, while the point P flows on the curve π, the curve drawn by the unit sphere of the point Q is called the spherical indicator on the unite vector field X. Bilici [3] obtained spherical indicators of involute evolute curves with the help of the Frenet frame. Şenyurt and Çalışkan [10] studied the spherical indicators of timelike Bertrand curve pairs. Şenyurt and Demet [11] calculated the geodesic curvatures and natural lifts of the spherical indicators of timelike-spacelike Mannheim curve pairs. Ateş et al. [1] gave tubular surfaces obtained with spherical indicators. Çapın [5] calculated the arc lengths and geodesic curvatures of the spherical indices of curves in the Minkowski space $\mathbb{E}^3_1$. Kula and Yaylı [8] examined slant helices and their spherical indicators. Erkan and Yüce [6] studied the roles of Bézier curves in $\mathbb{E}^2$ and $\mathbb{E}^3$ with the help of Serret-Frenet and curvatures, both using and not using algorithms used in applied mathematics and computer engineering. Frenet frames on Riemannian manifolds have been also investigated, [1, 12].

Many frames can be obtained from one curve. One of them is the Bishop frame. A Bishop frame [4] $\{\mathbf{t},\, \mathbf{n}_1,\, \mathbf{n}_2\}$ on the curve π that rotates about the tangent vector $\mathbf{t}$ by an angle x is

$$\begin{aligned} \mathbf{t} &= \mathbf{t}, \\ \mathbf{n} &= \mathbf{n}_1 \cos x + \mathbf{n}_2 \sin x, \\ \mathbf{b} &= -\mathbf{n}_1 \sin x + \mathbf{n}_2 \cos x. \end{aligned}$$

The derivative change of this frame is

$$\begin{aligned}
\mathbf{t}' &= k_1\mathbf{n}_1 + k_2\mathbf{n}_2, \\
\mathbf{n}_1' &= -k_1\mathbf{t}, \\
\mathbf{n}_2' &= -k_2\mathbf{t}, \\
k_1 &= \kappa\cos x, \\
k_2 &= \kappa\sin x, \\
\tau &= x'.
\end{aligned}$$

Here, the quintet $\{\mathbf{t},\, \mathbf{n}_1,\, \mathbf{n}_2,\, k_1,\, k_2\}$ are called Bishop apparatus.

In this study, we examined the spherical indicator curve frames using angles according to the Serret-Frenet frame of a regular curve. We showed that these frames can correspond to a Bishop frame. We expressed and proved the results. We reinforced the study with an example.

2. According to the Frenet Frame Spherical Indicators and Results

Let $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b},\, \kappa \neq 0,\, \tau \neq 0\}$ be the Frenet apparatus of a unit speed curve

$$\begin{array}{rccc}
\pi: & J & \longmapsto & \mathbb{E}^3 \\
 & s & \longmapsto & \pi(s).
\end{array}$$

The Darboux vector and the pol vector of this curve are

$$\begin{aligned}
\mathbf{w} &= \tau\mathbf{t} + \kappa\mathbf{b}, \\
\mathbf{c} &= \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi,
\end{aligned}$$

respectively. Here

$$\cos\phi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}, \tag{2}$$

$$\sin\phi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} \tag{3}$$

and ϕ are the angles between the pole vector c and the binormal vector $\mathbf{b}$.

From now on, unless we state otherwise, we will consider a curve π as a curve with a unit speed and curvatures $\kappa \neq 0, \tau \neq 0$.

Theorem 2.1 *Let the Frenet apparatuses of a curve* $\pi : J \longrightarrow \mathbb{E}^3$ *be* $\{\mathbf{t},\, \mathbf{n},\, \mathbf{b},\, \kappa,\, \tau\}$ *and the* *tangents indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$ *be the Frenet apparatuses* $\{\mathbf{t}_{\mathbf{t}}, \mathbf{n}_{\mathbf{t}}, \mathbf{b}_{\mathbf{t}}, \kappa_{\mathbf{t}}, \tau_{\mathbf{t}}\}$. *Therefore*

$$\begin{aligned}
\mathbf{t}_{\mathbf{t}} &= \mathbf{n}, \\
\mathbf{n}_{\mathbf{t}} &= -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi, \\
\mathbf{b}_{\mathbf{t}} &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi, \\
\kappa_{\mathbf{t}} &= \sec\phi, \\
\tau_{\mathbf{t}} &= \frac{\phi'}{\kappa}.
\end{aligned}$$

Here, $\phi' = \dfrac{\mathrm{d}\phi}{\mathrm{d}s}$.

Proof On condition that $\dfrac{\mathrm{d}\pi_{\mathbf{t}}}{\mathrm{d}s} = \dfrac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \pi_{\mathbf{t}}'$,

$$\begin{aligned}
\pi_{\mathbf{t}}' &= \kappa\mathbf{n}, \\
\pi_{\mathbf{t}}'' &= -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}, \\
\pi_{\mathbf{t}}''' &= -3\kappa\kappa'\mathbf{t} + \left(\kappa'' - \kappa^3 - \kappa\tau^2\right)\mathbf{n} + 2\left(\kappa'\tau + \kappa\tau'\right)\kappa\tau\mathbf{b}.
\end{aligned}$$

Using Equation (1), we obtain the first and second curvatures of the curve $\pi_{\mathbf{t}} = \mathbf{t}$ is

$$\kappa_{\mathbf{t}} = \frac{\|\pi_{\mathbf{t}}' \wedge \pi_{\mathbf{t}}''\|}{\|\pi_{\mathbf{t}}''\|^3} = \sec\phi$$

and

$$\tau_{\mathbf{t}} = \frac{\det\left(\pi_{\mathbf{t}}', \pi_{\mathbf{t}}'', \pi_{\mathbf{t}}'''\right)}{\|\pi_{\mathbf{t}}' \wedge \pi_{\mathbf{t}}''\|^2} = \frac{\phi'}{\kappa},$$

respectively. If we take the derivative of the curve $\pi_{\mathbf{t}} = \mathbf{t}$ with respect to its arc parameter $s_{\mathbf{t}}$,

$$\frac{\mathrm{d}\pi_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}}\kappa\mathbf{n}.$$

If so,

$$\frac{\mathrm{d}\pi_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}} = \mathbf{t}_{\mathbf{t}} = \mathbf{n}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = \frac{1}{\kappa}. \tag{4}$$

On the other hand, if we use (2), (3) and (4), we have

$$\mathbf{n}_{\mathbf{t}} = \frac{\dfrac{\mathrm{d}\mathbf{t}_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}}}{\left\|\dfrac{\mathrm{d}\mathbf{t}_{\mathbf{t}}}{\mathrm{d}s_{\mathbf{t}}}\right\|} = -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi,$$

and

$$\mathbf{b_t} = \mathbf{t_t} \wedge \mathbf{n_t} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi.$$

According to these, the proof ends. □

Corollary 2.2 *On the tangent indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$, *there is a Bishop frame* $\{\mathbf{n}, -\mathbf{t}, \mathbf{b}\}$ *that rotates about the tangent vector* $\mathbf{t_t} = \mathbf{n}$ *by an angle* ϕ *and the following equations exist*

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s_{\mathbf{t}}} &= a_1(-\mathbf{t}) + a_2\mathbf{b}, \\
\frac{\mathrm{d}(-\mathbf{t})}{\mathrm{d}s_{\mathbf{t}}} &= -a_1\mathbf{n}, \\
\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{t}}} &= -a_2\mathbf{n}, \\
a_1 &= 1, \\
a_2 &= \tan\phi,
\end{aligned}$$

where a_1 *and* a_2 *are the first and second curvatures of the Bishop frame* $\{\mathbf{n}, -\mathbf{t}, \mathbf{b}\}$, *respectively.*

Proof It is seen from Theorem 2.1 that the frame $\{\mathbf{n}, -\mathbf{t}, \mathbf{b}\}$ is a Bishop frame. We have

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s_{\mathbf{t}}} &= \frac{d\mathbf{n}}{ds}\frac{ds}{ds_{\mathbf{t}}} = (-\kappa\mathbf{t} + \tau\mathbf{b})\frac{1}{\kappa} \\
&= -\mathbf{t} + \left(\frac{\tau}{\kappa}\right)\mathbf{b},
\end{aligned}$$

$$\frac{\mathrm{d}(-\mathbf{t})}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}(-\mathbf{t})}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = -\kappa\mathbf{n}\frac{1}{\kappa} = -\mathbf{n},$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{t}}} = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{t}}} = -\frac{\tau}{\kappa}\mathbf{n}.$$

Therefore

$$\begin{aligned}
a_1 &= -1, \\
a_2 &= \frac{\tau}{\kappa} = \tan\phi.
\end{aligned}$$

If so, the proof ends. □

Theorem 2.3 *For a curve* $\pi : J \longrightarrow \mathbb{E}^3$, *let apparatuses of the tangents indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$ *be* $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}, \kappa_{\mathbf{t}}, \tau_{\mathbf{t}}\}$ *and let apparatuses of the principal normal indicator curve* $\pi_{\mathbf{t}} = \mathbf{t}$ *be*

$\{\mathbf{t_n}, \mathbf{n_n}, \mathbf{b_n}, \kappa_\mathbf{n}, \tau_\mathbf{n}\}$. *There are the following equations*

$$\begin{aligned}
\mathbf{t_n} &= \mathbf{n_t}, \\
\mathbf{n_n} &= \mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega, \\
\mathbf{b_n} &= \mathbf{b_t}\sin\omega + \mathbf{t_t}\cos\omega, \\
\kappa_\mathbf{n} &= \sqrt{1+\left(\frac{\phi'}{\|\mathbf{w}\|}\right)^2}, \\
\tau_\mathbf{n} &= -\frac{\omega'}{\|\mathbf{w}\|},
\end{aligned}$$

where $\cos\omega = \frac{\sqrt{\kappa_n^2-1}}{\kappa_\mathbf{n}}$, $\sin\omega = \frac{1}{\kappa_\mathbf{n}}$ *and the angle* ω *is the angle between vectors* $\mathbf{b_t}$ *and* $\mathbf{n_n}$.

Proof On condition that $\frac{\mathrm{d}\pi_\mathbf{n}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \pi'_\mathbf{n}$,

$$\begin{aligned}
\pi'_\mathbf{n} &= -\kappa\mathbf{t}+\tau\mathbf{b}, \\
\pi''_\mathbf{n} &= -\kappa'\mathbf{t} - \left(\kappa^2+\tau^2\right)\mathbf{n}+\tau'\mathbf{b}, \\
\pi'''_\mathbf{n} &= \left[-\kappa'' + \left(\kappa^2+\tau^2\right)\kappa\right]\mathbf{t} - 3\left(\kappa'\tau+\kappa\tau'\right)\mathbf{n} + \left[\tau''+\left(\kappa^2+\tau^2\right)\kappa\right]\mathbf{b}.
\end{aligned}$$

Using Equation (1), the first and second curvatures of the curve $\pi_\mathbf{n} = \mathbf{n}$ are obtained as

$$\kappa_\mathbf{n} = \frac{\|\pi'_\mathbf{n} \wedge \pi''_\mathbf{n}\|}{\|\pi''_\mathbf{n}\|^3} = \sqrt{1+\left(\frac{x'}{\|\mathbf{w}\|}\right)^2} \tag{5}$$

and

$$\tau_\mathbf{n} = \frac{\det\left(\pi'_\mathbf{n}, \pi''_\mathbf{n}, \pi'''_\mathbf{n}\right)}{\|\pi'_\mathbf{n} \wedge \pi''_\mathbf{n}\|^2} = -\frac{\omega'}{\|\mathbf{w}\|}, \tag{6}$$

respectively. If we take the derivative of the curve $\pi_\mathbf{n} = \mathbf{n}$ with respect to its arc parameter $s_\mathbf{n}$,

$$\frac{\mathrm{d}\pi_\mathbf{n}}{\mathrm{d}s_\mathbf{n}} = \frac{d\mathbf{n}}{ds_\mathbf{n}} = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_\mathbf{n}} = \frac{\mathrm{d}s}{\mathrm{d}s_\mathbf{n}}\left(-\kappa\mathbf{t}+\tau\mathbf{b}\right)$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}s_\mathbf{n}} = \frac{1}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{\|\mathbf{w}\|}. \tag{7}$$

If so,

$$\frac{\mathrm{d}\pi_\mathbf{n}}{\mathrm{d}s_\mathbf{n}} = \mathbf{t_n} = -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi = \mathbf{n_t}.$$

On the other hand, if we use (5), (6) and (7), we have

$$\mathbf{n_n} = \frac{\dfrac{d\mathbf{t_n}}{ds_\mathbf{n}}}{\left\|\dfrac{d\mathbf{t_n}}{ds_\mathbf{n}}\right\|} = \frac{\sqrt{\kappa_n^2 - 1}}{\kappa_\mathbf{n}}\left(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi\right) - \frac{1}{\kappa_\mathbf{n}}\mathbf{n}.$$

If we say $\cos\omega = \dfrac{\sqrt{\kappa_n^2 - 1}}{\kappa_\mathbf{n}}$, $\sin\omega = \dfrac{1}{\kappa_\mathbf{n}}$, we get

$$\mathbf{n_n} = \mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega,$$

and

$$\mathbf{b_n} = \mathbf{t_n} \wedge \mathbf{n_n} = \mathbf{b_t}\sin\omega + \mathbf{t_t}\cos\omega.$$

□

Corollary 2.4 *The frame* $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}\}$ *is a Bishop frame rotating about the tangent vector* $\mathbf{t_n} = \mathbf{n_t}$ *by an angle* $-\omega$ *on the principal normals indicator curve* $\pi_\mathbf{n} = \mathbf{n}$. *We have the following equations*

$$\begin{aligned}
\frac{d\mathbf{n_t}}{ds_\mathbf{n}} &= b_1\mathbf{b_t} - b_2\mathbf{t_t},\\
\frac{d\mathbf{b_t}}{ds_\mathbf{n}} &= -b_1\mathbf{n_t},\\
\frac{d\mathbf{t_t}}{ds_\mathbf{n}} &= b_2\mathbf{n_t},\\
b_1 &= \frac{\phi'}{\|\mathbf{w}\|},\\
b_2 &= -1,
\end{aligned}$$

where b_1 *and* b_2, $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}\}$ *are the first and second Bishop curvatures of the Bishop frame, respectively.*

Proof It is seen from Theorem 2.3 that the frame $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}\}$ is a Bishop frame. We have

$$\begin{aligned}
\frac{d\mathbf{n_t}}{ds_\mathbf{n}} &= \frac{d\mathbf{t_n}}{ds_\mathbf{n}}\\
&= \kappa_\mathbf{n}\mathbf{n_n}\\
&= \kappa_\mathbf{n}\left[\mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega\right]\\
&= \kappa_\mathbf{n}\mathbf{b_t}\cos\omega - \kappa_\mathbf{n}\mathbf{t_t}\sin\omega,
\end{aligned}$$

$$\frac{d\mathbf{b_t}}{ds_\mathbf{n}} = \frac{d\left(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi\right)}{ds}\frac{ds}{ds_\mathbf{n}} = -\frac{\phi'}{\|\mathbf{w}\|}\mathbf{n_t},$$

$$\frac{d\mathbf{t_t}}{ds_\mathbf{n}} = \frac{d\mathbf{n}}{ds}\frac{ds}{ds_\mathbf{n}} = \mathbf{n_t}.$$

Therefore

$$\begin{aligned} b_1 &= \kappa_{\mathbf{n}} \cos(\omega) = \kappa_{\mathbf{n}} \frac{\sqrt{\kappa_n^2 - 1}}{\kappa_{\mathbf{n}}} = \frac{\phi'}{\|w\|}, \\ b_2 &= -\kappa_{\mathbf{n}} \sin(\omega) = -\kappa_{\mathbf{n}} \frac{1}{\kappa_{\mathbf{n}}} = -1. \end{aligned}$$

Thus, the proof is completed. □

Theorem 2.5 *Let the Frenet apparatuses of a curve* $\pi : J \longrightarrow \mathbb{E}^3$ *be* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$ *and the Frenet apparatuses of the binormal indicator curve* $\pi_{\mathbf{b}} = \mathbf{b}$ *be* $\{\mathbf{t_b}, \mathbf{n_b}, \mathbf{b_b}, \kappa_{\mathbf{b}}, \tau_{\mathbf{b}}\}$. *We have*

$$\begin{aligned} \mathbf{t_b} &= -\mathbf{n}, \\ \mathbf{n_b} &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi, \\ \mathbf{b_b} &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi, \\ \kappa_{\mathbf{b}} &= \csc\phi, \\ \tau_{\mathbf{b}} &= -\frac{\phi'}{\tau}\cdot \end{aligned}$$

Here, the angle ϕ *is the angle between vectors* $\mathbf{t}$ *and* $\mathbf{n_b}$.

Proof If we take the derivative of the curve $\pi_{\mathbf{b}} = \mathbf{b}$ with respect to its arc parameter $s_{\mathbf{b}}$,

$$\frac{\mathrm{d}\pi_{\mathbf{b}}}{\mathrm{d}s_{\mathbf{b}}} = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = -\tau\mathbf{n}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}}\cdot$$

For this reason

$$\mathbf{t_b} = -\mathbf{n} \text{ ve } \frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = \frac{1}{\tau}\cdot \tag{8}$$

Accordingly

$$\kappa_{\mathbf{b}} = \csc\phi,$$

and

$$\mathbf{n_b} = \mathbf{t}\cos\phi - \mathbf{b}\sin\phi.$$

On the other hand, we obtain

$$\mathbf{b_b} = \mathbf{t_b} \wedge \mathbf{n_b} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi.$$

Also, if we consider Equation (8),

$$\tau_{\mathbf{b}} = \frac{\mathrm{d}(-\phi)}{\mathrm{d}s_{\mathbf{c}}} = \frac{\mathrm{d}(-\phi)}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} = -\frac{\phi'}{\tau}\cdot$$

□

Corollary 2.6 *The frame* $\{\mathbf{t}, -\mathbf{n}, \mathbf{b}\}$ *is a Bishop frame rotating about the tangent vector* $\pi_{\mathbf{b}} = \mathbf{b}$ *by an angle* $-\phi$ *on the binormals indicator curve* $\pi_{\mathbf{n}} = \mathbf{n}$. *We have the following equations*

$$\begin{aligned}
\frac{\mathrm{d}(-\mathbf{n})}{\mathrm{d}s_{\mathbf{b}}} &= c_1\mathbf{t} - c_2\mathbf{b}, \\
\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s_{\mathbf{b}}} &= -c_1(-\mathbf{n}), \\
\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{b}}} &= c_2(-\mathbf{n}), \\
c_1 &= \cot\phi, \\
c_2 &= -1,
\end{aligned}$$

where c_1 *and* c_2, $\{\mathbf{t}, -\mathbf{n}, \mathbf{b}\}$ *are the first and second Bishop curvatures of the Bishop frame, respectively.*

Proof It is seen from Theorem 2.5 that the frame $\{\mathbf{t}, -\mathbf{n}, \mathbf{b}\}$ is a Bishop frame. We have

$$\begin{aligned}
\frac{\mathrm{d}(-\mathbf{n})}{\mathrm{d}s_{\mathbf{b}}} &= \frac{\mathrm{d}(-\mathbf{n})}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = (\kappa\mathbf{t} - \tau\mathbf{b})\frac{1}{\tau}, \\
&= \left(\frac{\kappa}{\tau}\right)\mathbf{t} - \mathbf{b},
\end{aligned}$$

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s_{\mathbf{b}}} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = -(-\mathbf{n})\cot\phi,$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s_{\mathbf{b}}} = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = -\mathbf{n}.$$

If so,

$$\begin{aligned}
c_1 &= \frac{\kappa}{\tau} = \cot\phi, \\
c_2 &= -1.
\end{aligned}$$

On the other hand, if we consider (8),

$$\tau_{\mathbf{b}} = \frac{\mathrm{d}\phi}{\mathrm{d}s_{\mathbf{b}}} = \frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{b}}} = \phi'\frac{1}{\tau}.$$

□

Theorem 2.7 *Let the Frenet apparatuses of a curve* $\pi : J \longrightarrow \mathbb{E}^3$ *be* $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau\}$ *and let the Frenet apparatuses of the spherical indicator curve of the pol vector* $\pi_{\mathbf{c}} = \mathbf{c}$ *be* $\{\mathbf{t_c}, \mathbf{n_c}, \mathbf{b_c}, \kappa_{\mathbf{c}}, \tau_{\mathbf{c}}\}$.

We have

$$\begin{aligned}
\mathbf{t_c} &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi, \\
\mathbf{n_c} &= \mathbf{n}\cos\theta - [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\sin\theta, \\
\mathbf{b_c} &= \mathbf{n}\sin\theta + [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\cos\theta, \\
\kappa_{\mathbf{c}} &= \sqrt{1+\left(\frac{\|\mathbf{w}\|}{\phi'}\right)^2}, \\
\tau_{\mathbf{c}} &= -\frac{\theta'}{\phi'},\ \ \phi \neq 0.
\end{aligned}$$

Here, the angle θ is the angle between vectors $\mathbf{n}$ and $\mathbf{n_c}$, and $\cos\theta = \dfrac{\|w\|}{\sqrt{(\phi')^2 + \|\mathbf{w}\|^2}}$, $\sin\theta = \dfrac{\phi'}{\sqrt{(\phi')^2 + \|\mathbf{w}\|^2}}$.

Proof If we take the derivative of the curve $\pi_{\mathbf{c}} = \mathbf{c} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi$ with respect to its arc parameter $s_{\mathbf{c}}$,

$$\frac{d\pi_{\mathbf{c}}}{ds_{\mathbf{c}}} = \frac{d\mathbf{c}}{ds}\frac{ds}{ds_{\mathbf{c}}} = \phi'(\mathbf{t}\cos\phi - \mathbf{b}\sin\phi)\frac{ds}{ds_{\mathbf{c}}}$$

and provided that $\phi' \neq 0$,

$$\mathbf{t_c} = \mathbf{t}\cos\phi - \mathbf{b}\sin\phi \text{ ve } \frac{ds}{ds_{\mathbf{c}}} = \frac{1}{\phi'}. \tag{9}$$

Since

$$\frac{d\mathbf{t_c}}{ds_{\mathbf{c}}} = \kappa_{\mathbf{c}}\mathbf{n_c} = -\frac{d(\mathbf{t}\cos\phi - \mathbf{b}\sin\phi)}{ds}\frac{ds}{ds_{\mathbf{b}}} = -(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi) + \frac{\|\mathbf{w}\|}{\phi'}\mathbf{n},$$

$$\kappa_{\mathbf{c}} = \sqrt{1+\left(\frac{\|\mathbf{w}\|}{\phi'}\right)^2},$$

$$\mathbf{n_c} = \mathbf{n}\cos\theta - [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\sin\theta, \tag{10}$$

and

$$\mathbf{b_c} = \mathbf{t_c} \wedge \mathbf{n_c} = \mathbf{n}\sin\theta + [\mathbf{t}\sin\phi + \mathbf{b}\cos\phi]\cos\theta. \tag{11}$$

On the other hand, if we consider (9),

$$\tau_{\mathbf{c}} = \frac{d(-\theta)}{ds_{\mathbf{c}}} = \frac{d(-\theta)}{ds}\frac{ds}{ds_{\mathbf{c}}} = -\frac{\theta'}{\phi'}.$$

□

Corollary 2.8 *The frame*

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi,\\ \mathbf{v}_2 &= \mathbf{n},\\ \mathbf{v}_3 &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi\end{aligned}$$

is the pol vector field indicator curve $\pi_{\mathbf{c}} = \mathbf{c}$*, a Bishop frame rotating about the tangent vector* $\mathbf{v}_1 = \mathbf{t}\cos\phi - \mathbf{b}\sin\phi$ *by an angle* $-\theta$*. We have*

$$\begin{aligned}\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}s_{\mathbf{c}}} &= d_1\mathbf{v}_2 - d_2\mathbf{v}_3,\\ \frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}s_{\mathbf{c}}} &= -d_1\mathbf{v}_1,\\ \frac{\mathrm{d}\mathbf{v}_3}{\mathrm{d}s_{\mathbf{c}}} &= d_2\mathbf{v}_1,\\ d_1 &= \frac{\|\mathbf{w}\|}{\phi'},\\ d_2 &= -1,\end{aligned}$$

where d_1 *and* d_2 *are the first and second Bishop curvatures of the Bishop frame* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*, respectively.*

Proof If we use the following equations

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi,\\ \mathbf{v}_2 &= \mathbf{n},\\ \mathbf{v}_3 &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi,\end{aligned}$$

with (9), (10) and (11), we obtain

$$\begin{aligned}\mathbf{t}_{\mathbf{c}} &= \mathbf{v}_1,\\ \mathbf{n}_{\mathbf{c}} &= \mathbf{v}_2\cos\theta - \mathbf{v}_3\sin\theta,\\ \mathbf{b}_{\mathbf{c}} &= \mathbf{v}_2\sin\theta + \mathbf{v}_3\cos\theta.\end{aligned}$$

This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a Bishop frame. Accordingly,

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}s_{\mathbf{c}}} &= \frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} \\
&= \frac{\mathrm{d}\left(\mathbf{t}\cos\phi - \mathbf{b}\sin\phi\right)}{\mathrm{d}s}\frac{1}{\phi'} \\
&= \frac{\|\mathbf{w}\|}{\phi'}\mathbf{n} - \left(\mathbf{t}\sin\phi + \mathbf{b}\cos\phi\right) \\
&= \frac{\|\mathbf{w}\|}{\phi'}\mathbf{v}_2 - \mathbf{v}_3,
\end{aligned}$$

$$\frac{\mathrm{d}\mathbf{v}_2}{\mathrm{d}s_{\mathbf{c}}} = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} = -\frac{\|\mathbf{w}\|}{\phi'}\mathbf{v}_1,$$

$$\frac{\mathrm{d}\mathbf{v}_3}{\mathrm{d}s_{\mathbf{c}}} = \frac{\mathrm{d}\mathbf{v}_3}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}s_{\mathbf{c}}} = \mathbf{v}_1.$$

Therefore

$$\begin{aligned}
d_1 &= \frac{\|\mathbf{w}\|}{\phi'}, \\
d_2 &= -1.
\end{aligned}$$

□

Example 2.9 *Let a curve π be defined as*

$$\begin{aligned}
\pi : \; J &\longmapsto \mathbb{E}^3 \\
t &\longmapsto \pi(t) = \left(\frac{2t^3}{3}, t^2, t\right)
\end{aligned}$$

in $\mathbb{E}^3$. The Frenet apparatuses of the curve π are

$$\begin{aligned}
\mathbf{t} &= \frac{1}{2t^2+1}\left(2t^2, 2t, 1\right), \\
\mathbf{n} &= \frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t, -8t^4+2, -8t^3-4t\right), \\
\mathbf{b} &= \frac{1}{2\left(2t^2+1\right)}\left(-2, 4t, -4t^2\right), \\
\kappa &= \frac{2}{\left(2t^2+1\right)^2}, \\
\tau &= \frac{-2}{\left(2t^2+1\right)^2}.
\end{aligned}$$

From (2) and (3), it is obtained that

$$\cos\phi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} = \frac{1}{\sqrt{2}},$$
$$\sin\phi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}} = -\frac{1}{\sqrt{2}}.$$

Accordingly, we can easily calculate the following apparatuses:

If the Frenet apparatuses of the tangent indicator curve $\pi_{\mathbf{t}} = \mathbf{t}$ *are* $\{\mathbf{t_t}, \mathbf{n_t}, \mathbf{b_t}, \kappa_{\mathbf{t}}, \tau_{\mathbf{t}}\}$*, then from Theorem 2.1*

$$\mathbf{t_t} = \mathbf{n} = \frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t, -8t^4+2, -8t^3-4t\right),$$
$$\mathbf{n_t} = -\mathbf{t}\cos\phi + \mathbf{b}\sin\phi = -\frac{1}{2\sqrt{2}\left(2t^2+1\right)}\left(4t^2-2, 8t, -4t^2+2\right),$$
$$\mathbf{b_t} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right),$$
$$\kappa_{\mathbf{t}} = \sec\phi = \sqrt{2},$$
$$\tau_{\mathbf{t}} = \frac{\phi'}{\kappa} = 0.$$

If the Frenet apparatuses of the tangent indicator curve $\pi_{\mathbf{n}} = \mathbf{n}$ *are* $\{\mathbf{t_n}, \mathbf{n_n}, \mathbf{b_n}, \kappa_{\mathbf{n}}, \tau_{\mathbf{n}}\}$*, then from Theorem 2.3*

$$\mathbf{t_n} = \mathbf{n_t} = -\frac{1}{2\sqrt{2}\left(2t^2+1\right)}\left(4t^2-2, 8t, -4t^2+2\right),$$
$$\mathbf{n_n} = \mathbf{b_t}\cos\omega - \mathbf{t_t}\sin\omega = -\frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t, -8t^4+2, -8t^3-4t\right),$$
$$\mathbf{b_n} = \mathbf{b_t}\sin\omega + \mathbf{t_t}\cos\omega = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right),$$
$$\kappa_{\mathbf{n}} = \sqrt{1+\left(\frac{\phi'}{\|\mathbf{w}\|}\right)^2} = 1,$$
$$\tau_{\mathbf{n}} = -\frac{\omega'}{\|\mathbf{w}\|} = 0.$$

If the Frenet apparatuses of the tangent indicator curve $\pi_{\mathbf{b}} = \mathbf{b}$ *are* $\{\mathbf{t_b}, \mathbf{n_b}, \mathbf{b_b}, \kappa_{\mathbf{b}}, \tau_{\mathbf{b}}\}$, *then from Theorem 2.5*

$$\begin{aligned}
\mathbf{t_b} &= -\mathbf{n} = -\frac{1}{2\left(2t^2+1\right)^2}\left(8t^3+4t,-8t^4+2,-8t^3-4t\right), \\
\mathbf{n_b} &= \mathbf{t}\cos\phi - \mathbf{b}\sin\phi = \frac{1}{2\sqrt{2}\left(2t^2+1\right)}\left(4t^2-2,8t,-4t^2+2\right), \\
\mathbf{b_b} &= \mathbf{t}\sin\phi + \mathbf{b}\cos\phi = \left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right), \\
\kappa_{\mathbf{b}} &= \csc\phi = -\sqrt{2}, \\
\tau_{\mathbf{b}} &= 0.
\end{aligned}$$

Since $\pi_{\mathbf{c}} = \mathbf{c} = \mathbf{t}\sin\phi + \mathbf{b}\cos\phi = \left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$*, the spherical indicator of the pole vector* $\pi_{\mathbf{c}} = \mathbf{c}$ *is a point.*

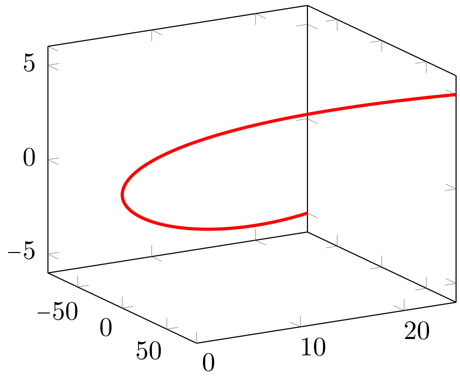

Figure 1: The curve π

Conclusion

Curves are a subject that is used in many fields such as science, engineering, computer design, astronomy studies, and geography. Examining curves means examining the changes in curves. These changes are called the differential geometry of curves. The characterization of curves can be examined with the differential of curves. A lot of work has been done on this subject so far. We have given the sources related to these in the previous sections. Sometimes it is easier to give an idea about a curve with the help of spherical indicators. In this way, spherical indicators of curves are also important. In the studies so far, spherical indicators have been examined with the help of the curvatures of their curves. In this study, we examined spherical indicators depending on the angle between the tangent vector field of a curve and the Darboux vector field. We saw that with this technique, operations and calculations become simpler. In addition, in this study, we showed that spherical indicators (tangent spherical indicators, primary normal spherical indicators, binormal spherical indicators) correspond to a Bishop frame according to the Frenet frame of a

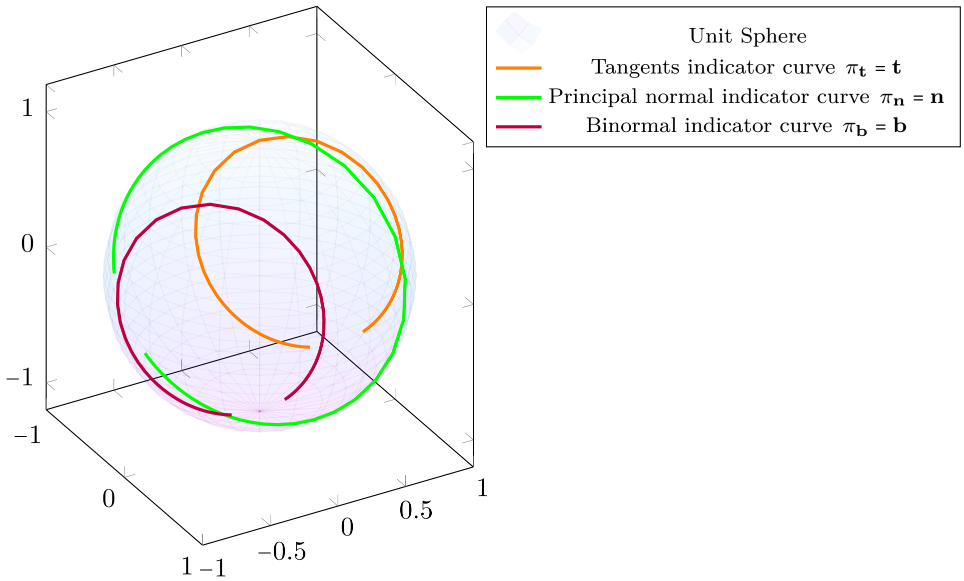

Figure 2: Spherical indicators

regular curve. We could not fully achieve our goals with this study due to lack of time. We could not examine the indicators of a regular curve according to the Darboux frame and the Sabban conflict. These will be addressed in other studies later.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Abdullah Yıldırım]: Collected the data, contributed to completing the research and solving the problem, wrote the manuscript (%75).

Author [Ali Toktimur]: Contributed to research method or evaluation of data, contributed to completing the research and solving the problem (%25).

Conflicts of Interest

The authors declare no conflict of interest.

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