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# Some remarks regarding the $(p,q)-$Fibonacci and Lucas octonion polynomials

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**Abstract**

We investigate the $(p,q)-$Fibonacci and Lucas octonion polynomials. The main purpose of this paper is using of some properties of the $(p,q)-$Fibonacci and Lucas polynomials. Also for present some results involving these octonion polynomials, we obtain some interesting computational formulas.

## 1. Introduction

Fibonacci, Lucas, Pell and the other special numbers are the special case of the second order linear recurrence $R=\{R_i\}_{i=0}^{\infty}$ if the recurrence relation for $i\geq 2,\ R_i=PR_{i-1}-QR_{i-2}$ holds for its terms, where $P$ and $Q$ are integers such that $D=P^2-4Q\neq 0$ (to exclude a degenerate case) and $R_0,R_1$ are fixed integers. Define the sequences

$$U_n=PU_{n-1}-QU_{n-2} \qquad (1.1)$$
$$V_n=PV_{n-1}-QV_{n-2}$$

for $n\geq 2$. The characteristic equation of them is $x^2-Px+Q=0$ and hence the roots of it are $\alpha=\frac{P+\sqrt{D}}{2}$ and $\beta=\frac{P-\sqrt{D}}{2}$. So by Binet's formula, $U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $V_n=\alpha^n+\beta^n$. Further the generating function for $U_n$ and $V_n$ is

$$\sum_{n=0}^{\infty}U_nx^n=\frac{x}{1-Px+Qx^2}\quad\text{and}\quad\sum_{n=0}^{\infty}V_nx^n=\frac{2-Px}{1-Px+Qx^2}$$

[8, 9].
Polynomials can be defined by Fibonacci-like recursion relations are called Fibonacci polynomials. More mathematicians were involved in the study of Fibonacci polynomials. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The $(p,q)-$Fibonacci polynomials are defined by the recurrence relation

$$F_{p,q,n+1}(x)=p(x)F_{p,q,n}(x)+q(x)F_{p,q,n-1}(x) \qquad (1.2)$$

with the initial conditions $F_{p,q,0}(x)=0,\ F_{p,q,1}(x)=1$. Also for the $p(x)$ and $q(x)$ polynomials with real coefficients the $(p,q)-$Lucas polynomials are defined by the recurrence relation

$$L_{p,q,n+1}(x)=p(x)L_{p,q,n}(x)+q(x)L_{p,q,n-1}(x)$$

with the initial conditions $L_{p,q,0}(x)=2,\ L_{p,q,1}(x)=p(x)$. Let $\alpha_1(x)=\frac{p(x)+\sqrt{p^2(x)+4q(x)}}{2}$ and $\alpha_2(x)=\frac{p(x)-\sqrt{p^2(x)+4q(x)}}{2}$ denote the roots of the characteristic equation

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$$\alpha^2 - p(x)\alpha - q(x) = 0$$

on the recurrence relation of (1.2). Binet formulas for the $(p,q)-$Fibonacci polynomials and $(p,q)-$Lucas polynomials are

$$F_{p,q,n}(x) = \frac{\alpha_1^n(x) - \alpha_2^n(x)}{\alpha_1(x) - \alpha_2(x)} \text{ and } L_{p,q,n}(x) = \alpha_1^n(x) + \alpha_2^n(x).$$

[10]
Note that

$$\begin{aligned} \alpha_1(x) + \alpha_2(x) &= p(x) \\ \alpha_1(x) - \alpha_2(x) &= \sqrt{p^2(x) + 4q(x)} \\ \alpha_1(x).\alpha_2(x) &= -q(x) \\ \frac{\alpha_1(x)}{\alpha_2(x)} &= \frac{-\alpha_1^2(x)}{q(x)}, q(x) \neq 0 \\ \frac{\alpha_2(x)}{\alpha_1(x)} &= \frac{-\alpha_2^2(x)}{q(x)},\ q(x) \neq 0. \end{aligned} \tag{1.3}$$

In [5], they introduce $(p,q)-$Fibonacci quaternion polynomials that generalize $h(x)-$Fibonacci quaternion polynomials. Division algebras are defined on real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbf{H}$, and octonions $\mathbb{Q}$. There are different types of sequences of quaternions like Fibonacci Quaternions, Split Fibonacci Quaternions and Complex Fibonacci Quaternions [1].
The octonions in Clifford algebra are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. The field $\mathbb{Q} \cong \mathbb{C}^4$ of octonions

$$\alpha = \sum_{s=0}^{7} \alpha_s e_s, \quad \alpha_i \in \mathbb{R}(i = 0, 1, \cdots, 7)$$

is an eight-dimensional non-commutative and non-associative $\mathbb{R}-$field generated by eight base elements $e_0, e_1, \cdots, e_6$ and $e_7$ which satisfy the non-commutative and non-associative multiplication rules are listed in below Table.

| $\times$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_1$ | $e_1$ | $-e_0$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $e_2$ | $-e_3$ | $-e_0$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_3$ | $e_2$ | $-e_1$ | $-e_0$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $-e_0$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $-e_0$ | $-e_3$ | $e_2$ |
| $e_6$ | $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | $-e_0$ | $-e_1$ |
| $e_7$ | $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ | $-e_0$ |

The multiplication table for the basis of $\mathbb{Q}$

For $n \geq 0$, the Fibonacci octonion numbers that are given for the $n-th$ classic Fibonacci $F_n$ number are defined by the following recurrence relations:

$$\mathbb{Q}_n = \sum_{s=0}^{7} F_{n+s} e_s.$$

Besides $h(x)$ $-$Fibonacci octonion polynomials can be defined by [6] that generalized both Catalan's Fibonacci octonion polynomials $\Psi_n(x)$ and Byrd's Fibonacci octonion polynomials and also $k-$ Fibonacci octonion numbers. Moreover in [2] they derived the Binet formula and generating function of $h(x)$ $-$Fibonacci octonion polynomial sequence.
Let $h(x)$ be a polynomial with real coefficients. The $h(x)-$Fibonacci octonion polynomials $\{O_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$O_{h,n}(x) = \sum_{s=0}^{7} F_{h,n+s}(x) e_s$$

where $F_{h,n}(x)$ is the $n-$th $h(x)-$Fibonacci polynomial in [2].

## 2. Main theorems of the $(p,q)-$Fibonacci and Lucas octonion polynomials

In the main section, we introduce the $(p,q)-$Fibonacci and Lucas octonion polynomials and formulate the Binet-style formula, the generating function and some identities of the $(p,q)-$Fibonacci octonion and Lucas octonion polynomial sequence. In [7], the authors obtained similar results for the $(p,q)-$Fibonacci and Lucas quaternion polynomials.
For $n \geq 0$ the Fibonacci octonion numbers that are given for the $n$-th classic Fibonacci $F_n$ number are defined in [4]. Also $(p,q)-$Fibonacci octonions are investigated by [3].
So $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ are defined by the recurrence relation

$$OF_{p,q,n}(x) = \sum_{k=0}^{7} F_{p,q,n+k}(x)e_k$$

where $F_{p,q,n+k}(x)$ is the $(n+k)-th$ $(p,q)-$Fibonacci polynomial.
The initial conditions of this sequence are given by

$$\begin{aligned} OF_{p,q,0}(x) = \sum_{k=0}^{7} F_{p,q,k}(x)e_k &= e_1 + p(x)e_2 + (p^2(x)+q(x))e_3 + (p^3(x)+2p(x)q(x))e_4 \\ &+ (p^4(x)+3p^2(x)q(x)+q^2(x))e_5 + (p^5(x)+4p^3(x)q(x)+3p(x)q^2(x))e_6 \\ &+ (p^6(x)+5p^4(x)q(x)+6p^2(x)q^2(x)+q^3(x))e_7 \end{aligned}$$

and

$$\begin{aligned} OF_{p,q,1}(x) = \sum_{k=0}^{7} F_{p,q,1+k}(x)e_k &= e_0 + p(x)e_1 + (p^2(x)+q(x))e_2 + (p^3(x)+2p(x)q(x))e_3 \\ &+ (p^4(x)+3p^2(x)q(x)+q^2(x))e_4 + (p^5(x)+4p^3(x)q(x)+3p(x)q^2(x))e_5 \\ &+ (p^6(x)+5p^4(x)q(x)+6p^2(x)q^2(x)+q^3(x))e_6 \\ &+ (p^7(x)+6p^5(x)q(x)+10p^3(x)q^2(x)+4p(x)q^3(x))e_7. \end{aligned}$$

Also $OF_{p,q,n}(x)$ is written by a recurrence relation of order two;

$$\begin{aligned} OF_{p,q,n+1}(x) &= \sum_{k=0}^{7} F_{p,q,n+1+k}(x)e_k \\ &= \sum_{k=0}^{7} (p(x)F_{p,q,n+k}(x) + q(x)F_{p,q,n-1+k}(x))e_k \\ &= p(x)\sum_{k=0}^{7} F_{p,q,n+k}(x)e_k + q(x)\sum_{k=0}^{7} F_{p,q,n-1+k}(x)e_k \end{aligned}$$

and thus,

$$OF_{p,q,n+1}(x) = p(x)OF_{p,q,n}(x) + q(x)OF_{p,q,n-1}(x).$$

For the $n-th$ $(p,q)-$Lucas octonion polynomials $OL_{p,q,n}(x) = \sum_{k=0}^{7} L_{p,q,n+k}(x)e_k$ ,where $L_{p,q,n+k}$ is the $(n+k)-th$ $(p,q)-$Lucas polynomial. For $n \geq 1$

$$OL_{p,q,n+1}(x) = p(x)OL_{p,q,n}(x) + q(x)OL_{p,q,n-1}(x)$$

with the initial conditions.

**Theorem 2.1.** *The generating functions for the $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ and the $(p,q)-$Lucas octonion polynomials $OL_{p,q,n}(x)$ are*

$$g_{OF}(t) = \frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{1-p(x)t-q(x)t^2}$$

*and*

$$g_{OL}(t) = \frac{OL_{p,q,0}(x) + [OL_{p,q,1}(x) - p(x)OL_{p,q,0}(x)]t}{1-p(x)t-q(x)t^2}.$$

*respectively.*

*Proof.* The generating function $g_{OF}(t)$ for $OF_{p,q,n}(x)$ is to be of the form

$$\sum_{n=0}^{\infty} OF_{p,q,n}(x)t^n = OF_{p,q,0}(x) + OF_{p,q,1}(x)t + OF_{p,q,2}(x)t^2 + \cdots + OF_{p,q,n}(x)t^n + \cdots. \tag{2.1}$$

The formal power series expansions of $g_{OF}(t), -p(x)tg_{OF}(t)$ and $-q(x)t^2g_{OF}(t)$ are

$$\begin{aligned}
g_{OF}(t) &= \sum_{n=0}^{\infty} OF_{p,q,n}(x)t^n = OF_{p,q,0}(x) + OF_{p,q,1}(x)t + OF_{p,q,2}(x)t^2 \\
&\quad + \cdots + OF_{p,q,n}(x)t^n + \cdots \\
-p(x)tg_{OF}(t) &= -p(x)OF_{p,q,0}(x)t - p(x)OF_{p,q,1}(x)t^2 - p(x)OF_{p,q,2}(x)t^3 \\
&\quad - \cdots - p(x)OF_{p,q,n}(x)t^{n+1} - \cdots \\
-q(x)t^2g_{OF}(t) &= -q(x)OF_{p,q,0}(x)t^2 - q(x)OF_{p,q,1}(x)t^3 - q(x)OF_{p,q,2}(x)t^4 \\
&\quad - \cdots - q(x)OF_{p,q,n}(x)t^{n+2} - \cdots
\end{aligned}$$

respectively. So the expansion for $g_{OF}(t) - g_{OF}(t)p(x)t - g_{OF}(t)q(x)t^2$ is

$$\begin{aligned}
g_{OF}(t)[1-p(x)t-q(x)t^2] &= OF_{p,q,0}(x) + OF_{p,q,1}(x)t - p(x)OF_{p,q,0}(x)t \\
&\quad + [OF_{p,q,2}(x) - p(x)OF_{p,q,1}(x) - q(x)OF_{p,q,0}(x)]t^2 \\
&\quad + [OF_{p,q,3}(x) - p(x)OF_{p,q,2}(x) - q(x)OF_{p,q,1}(x)]t^3 \\
&\quad + \ldots + [OF_{p,q,n}(x) - p(x)OF_{p,q,n-1}(x) - q(x)OF_{p,q,n-2}(x)]t^n \\
&\quad + \ldots \\
&= OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t.
\end{aligned}$$

Hence $OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t$ is a finite series, so we can rewrite $[1-p(x)t-q(x)t^2]g_{OF}(t) = OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t$ and hence

$$g_{OF}(t) = \frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{1-p(x)t-q(x)t^2} \tag{2.2}$$

as we claimed.
Similarly, it can be also proved that $g_{OL}(t) = \frac{OL_{p,q,0}(x)+[OL_{p,q,1}(x)-p(x)OL_{p,q,0}(x)]t}{1-p(x)t-q(x)t^2}$. □

**Lemma 2.2.** *For the generating function given in Theorem 2.1, we have*

$$\begin{aligned}
g_{OF}(t) &= \frac{1}{\alpha_1(x)-\alpha_2(x)}\left(\frac{OF_{p,q,1}(x)-\alpha_2(x)OF_{p,q,0}(x)}{1-\alpha_1(x)t} - \frac{OF_{p,q,1}(x)-\alpha_1(x)OF_{p,q,0}(x)}{1-\alpha_2(x)t}\right) \\
g_{OL}(t) &= \frac{1}{\alpha_1(x)-\alpha_2(x)}\left(\frac{OL_{p,q,1}(x)-\alpha_2(x)OL_{p,q,0}(x)}{1-\alpha_1(x)t} - \frac{OL_{p,q,1}(x)-\alpha_1(x)OL_{p,q,0}(x)}{1-\alpha_2(x)t}\right).
\end{aligned}$$

*Proof.* Using the expression of $g_{OF}(t)$ in Teorem 2.1 and (1.3) ,we found

$$\frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{1-p(x)t-q(x)t^2} = \frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - p(x)OF_{p,q,0}(x)]t}{(1-\alpha_1(x)t)(1-\alpha_2(x)t)}$$

$$= \left(\frac{OF_{p,q,0}(x) + [OF_{p,q,1}(x) - (\alpha_1(x)+\alpha_2(x))OF_{p,q,0}(x)]t}{(1-\alpha_1(x)t)(1-\alpha_2(x)t)}\right) \times \left(\frac{\alpha_1(x)-\alpha_2(x)}{\alpha_1(x)-\alpha_2(x)}\right)$$

$$= \frac{\left\{\begin{array}{c} \alpha_1(x)OF_{p,q,0}(x) + \alpha_1(x)OF_{p,q,1}(x)t - \alpha_1^2(x)OF_{p,q,0}(x)t \\ -\alpha_1(x)\alpha_2(x)OF_{p,q,0}(x)t - \alpha_2(x)OF_{p,q,0}(x) - \alpha_2(x)OF_{p,q,1}(x)t \\ +\alpha_1(x)\alpha_2(x)OF_{p,q,0}(x)t + \alpha_2^2(x)OF_{p,q,0}(x)t + OF_{p,q,1}(x) - OF_{p,q,1}(x) \end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))(1-\alpha_1(x)t)(1-\alpha_2(x)t)}$$

$$= \frac{\left\{\begin{array}{c} OF_{p,q,1}(x)(1-\alpha_2(x)t) + \alpha_2(x)OF_{p,q,0}(x)(-1+\alpha_2(x)t) \\ +OF_{p,q,1}(x)(-1+\alpha_1(x)t) + \alpha_1(x)OF_{p,q,0}(x)(1-\alpha_1(x)t) \end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))(1-\alpha_1(x)t)(1-\alpha_2(x)t)}$$

$$= \frac{\left\{\begin{array}{c} (1-\alpha_2(x)t)(OF_{p,q,1}(x) - \alpha_2(x)OF_{p,q,0}(x)) \\ -(1-\alpha_1(x)t)(OF_{p,q,1}(x) - \alpha_1(x)OF_{p,q,0}(x)) \end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))(1-\alpha_1(x)t)(1-\alpha_2(x)t)}$$

$$= \frac{1}{\alpha_1(x)-\alpha_2(x)}\left[\frac{OF_{p,q,1}(x)-\alpha_2(x)OF_{p,q,0}(x)}{1-\alpha_1(x)t} - \frac{OF_{p,q,1}(x)-\alpha_1(x)OF_{p,q,0}(x)}{1-\alpha_2(x)t}\right].$$

□

**Lemma 2.3.** *Let $F_{p,q,n}(x)$ and $L_{p,q,n}(x)$ be the $(p,q)-$Fibonacci and Lucas polynomials respectively. We have*
1.

$$F_{p,q,k+1}(x)-\alpha_2(x)F_{p,q,k}(x)=\alpha_1^k(x)$$
$$F_{p,q,k+1}(x)-\alpha_1(x)F_{p,q,k}(x)=\alpha_2^k(x)$$

2.

$$\frac{L_{p,q,k+1}(x)-\alpha_2(x)L_{p,q,k}(x)}{\alpha_1(x)-\alpha_2(x)}=\alpha_1^k(x)$$
$$\frac{\alpha_1(x)L_{p,q,k}(x)-L_{p,q,k+1}(x)}{\alpha_1(x)-\alpha_2(x)}=\alpha_2^k(x).$$

*Proof.* 1.We prove it by induction. Let $k=1$

$$F_{p,q,2}(x)-\alpha_2(x)F_{p,q,1}(x)=p(x)-\alpha_2(x)=\alpha_1(x).$$

So the hypothesis is right for $k=1$. Let us assume that the equation is $F_{p,q,n}(x)-\alpha_2(x)F_{p,q,n-1}(x)=\alpha_1^{n-1}(x)$ for $k=n-1$. For $k=n$ it becomes

$$\begin{aligned}\alpha_1^n(x)&=\alpha_1^{n-1}(x)\alpha_1(x)\\&=(F_{p,q,n}(x)-\alpha_2(x)F_{p,q,n-1}(x))\alpha_1(x)\\&=\alpha_1(x)F_{p,q,n}(x)-\alpha_1(x)\alpha_2(x)F_{p,q,n-1}(x)\\&=(p(x)-\alpha_2(x))F_{p,q,n}(x)-(-q(x))F_{p,q,n-1}(x)\\&=p(x)F_{p,q,n}(x)+q(x)F_{p,q,n-1}(x)-\alpha_2(x)F_{p,q,n}(x)\\&=F_{p,q,n+1}(x)-\alpha_2(x)F_{p,q,n}(x).\end{aligned}$$

So we get the desired result for the $(p,q)-$Fibonacci polynomials. 2. The $(p,q)-$Lucas polynomials can be proved similarly. □

To derive the Binet Formulas for $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$, we can give the following theorems.

**Theorem 2.4.** *For $n\geq 0$ , the Binet formula for the $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ and also $OL_{p,q,n}(x)$ is as follows*

$$OF_{p,q,n}(x)=\frac{\alpha_1^*(x)\alpha_1^n(x)-\alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x)-\alpha_2(x)}$$
$$OL_{p,q,n}(x)=\alpha_1^*(x)\alpha_1^n(x)+\alpha_2^*(x)\alpha_2^n(x)$$

where $\alpha_1^*(x)=\sum\limits_{k=0}^{7}\alpha_1^k(x)e_k$ and $\alpha_2^*(x)=\sum\limits_{k=0}^{7}\alpha_2^k(x)e_k$.

*Proof.* From Lemma 2.1, we get

$$\begin{aligned}g_{OF}(t)&=\frac{1}{\alpha_1(x)-\alpha_2(x)}[(OF_{p,q,1}(x)-\alpha_2(x)OF_{p,q,0}(x))\\&\sum_{n=0}^{\infty}\alpha_1^n(x)t^n-(OF_{p,q,1}(x)-\alpha_1(x)OF_{p,q,0}(x))\sum_{n=0}^{\infty}\alpha_2^n(x)t^n]\\&=\frac{1}{\alpha_1(x)-\alpha_2(x)}\left\{\begin{array}{l}\sum_{k=0}^{7}(F_{p,q,1+k}(x)-\alpha_2(x)F_{p,q,k}(x))e_k\sum_{n=0}^{\infty}\alpha_1^n(x)t^n\\-\sum_{k=0}^{7}(F_{p,q,1+k}(x)-\alpha_1(x)F_{p,q,k}(x))e_k\sum_{n=0}^{\infty}\alpha_1^n(x)t^n\end{array}\right\}\\&=\frac{1}{\alpha_1(x)-\alpha_2(x)}\left[\sum_{k=0}^{7}\alpha_1^k(x)e_k\sum_{n=0}^{\infty}\alpha_1^n(x)t^n-\sum_{k=0}^{7}\alpha_2^k(x)e^k\sum_{n=0}^{\infty}\alpha_2^n(x)t^n\right]\\&=\sum_{n=0}^{\infty}\frac{\alpha_1^*(x)\alpha_1^n(x)-\alpha_2^*(x)\alpha_2^n(x)}{\alpha_1(x)-\alpha_2(x)}t^n.\end{aligned}$$

Similarly, it can be also proved that $OL_{p,q,n}(x)=\alpha_1^*(x)\alpha_1^n(x)+\alpha_2^*(x)\alpha_2^n(x)$. □

**Theorem 2.5.** *(Catalan identity) Let the $(p,q)-$Fibonacci and Lucas octonion polynomials $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$. For $n$ and $\alpha$, nonnegative integer numbers, such that $\alpha\leq n$ , we have*

$$OF_{p,q,n+r}(x)OF_{p,q,n-r}(x)-OF_{p,q,n}^2(x)=\frac{(-1)^{r+n+1}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1^r(x)-\alpha_2^r(x))^2}{(\alpha_1(x)-\alpha_2(x))^2}$$
$$OL_{p,q,n+r}(x)OL_{p,q,n-r}(x)-OL_{p,q,n}^2(x)=(-1)^{r+n}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1^r(x)-\alpha_2^r(x))^2.$$

*Proof.* Using the identity (1.3), Lemma 2.2 and Theorem 2.2, we have

$$\begin{aligned}
&OF_{p,q,n+r}(x)OF_{p,q,n-r}(x)-OF^2_{p,q,n}(x)\\
&=\left(\frac{\alpha_1^*(x)\alpha_1^{n+r}(x)-\alpha_2^*(x)\alpha_2^{n+r}(x)}{\alpha_1(x)-\alpha_2(x)}\right)\left(\frac{\alpha_1^*(x)\alpha_1^{n-r}(x)-\alpha_2^*(x)\alpha_2^{n-r}(x)}{\alpha_1(x)-\alpha_2(x)}\right)\\
&-\left(\frac{\alpha_1^*(x)\alpha_1^{n}(x)-\alpha_2^*(x)\alpha_2^{n}(x)}{\alpha_1(x)-\alpha_2(x)}\right)^2\\
&=\frac{\left\{\begin{array}{c}-\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n-r}(x)\alpha_2^{n+r}(x)\\ -\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n+r}(x)\alpha_2^{n-r}(x)\\ +2\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n}(x)\alpha_2^{n}(x)\end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{-\alpha_1^*(x)\alpha_2^*(x)\alpha_1^{n}(x)\alpha_2^{n}(x)\left[\left(-\frac{\alpha_2^2(x)}{q(x)}\right)^r+\left(-\frac{\alpha_1^2(x)}{q(x)}\right)^r-2\frac{(\alpha_1(x)\alpha_2(x))^r}{q^r(x)}\right]}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{(-1)^{r+n+1}\alpha_1^*(x)\alpha_2^*(x)q^{n-r}(x)(\alpha_1^r(x)-\alpha_2^r(x))^2}{(\alpha_1(x)-\alpha_2(x))^2}.
\end{aligned}$$

The other case can be proved similarly. □

**Theorem 2.6.** *(Cassini identity) For the $(p,q)-$Fibonacci octonion polynomials $OF_{p,q,n}(x)$ and $(p,q)-$Lucas octonion polynomials $OL_{p,q,n}(x)$, we have*

$$\begin{aligned}
&OF_{p,q,n+1}(x)OF_{p,q,n-1}(x)-OF^2_{p,q,n}(x)=(-1)^n\alpha_1^*(x)\alpha_2^*(x)q^{n-1}(x)\\
&OL_{p,q,n+1}(x)OL_{p,q,n-1}(x)-OL^2_{p,q,n}(x)=(-1)^{1+n}\alpha_1^*(x)\alpha_2^*(x)q^{n-1}(x)(\alpha_1(x)-\alpha_2(x))^2
\end{aligned}$$

*for any natural number n .*

**Theorem 2.7.** *Let $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$ be the $(p,q)-$Fibonacci and Lucas octonion polynomials respectively. Then for $n\geq 0$, we have*

*1.*

$$\begin{aligned}
q(x)(OF_{p,q,n}(x))^2+(OF_{p,q,n+1}(x))^2&=\frac{(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x)-(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)}{\alpha_1(x)-\alpha_2(x)}\\
q(x)(OL_{p,q,n}(x))^2+(OL_{p,q,n+1}(x))^2&=(\alpha_1(x)-\alpha_2(x))(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x)-(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)
\end{aligned}$$

*2.*

$$\begin{aligned}
OF_{p,q,1}(x)-\alpha_1(x)QF_{p,q,0}(x)&=\alpha_2^*(x)\\
OF_{p,q,1}(x)-\alpha_2(x)QF_{p,q,0}(x)&=\alpha_1^*(x)
\end{aligned}$$

*and*

$$\begin{aligned}
OL_{p,q,1}(x)-\alpha_1(x)OL_{p,q,0}(x)&=(\alpha_1(x)-\alpha_2(x))\alpha_2^*(x)\\
OL_{p,q,1}(x)-\alpha_2(x)OL_{p,q,0}(x)&=(\alpha_1(x)-\alpha_2(x))\alpha_1^*(x).
\end{aligned}$$

*Proof.* Let us prove the identity 1.. From Theorem 2.2

$$\begin{aligned}
q(x)(OF_{p,q,n}(x))^2+(OF_{p,q,n+1}(x))^2&=q(x)\left(\frac{\alpha_1^*(x)\alpha_1^{n}(x)-\alpha_2^*(x)\alpha_2^{n}(x)}{\alpha_1(x)-\alpha_2(x)}\right)^2+\left(\frac{\alpha_1^*(x)\alpha_1^{n+1}(x)-\alpha_2^*(x)\alpha_2^{n+1}(x)}{\alpha_1(x)-\alpha_2(x)}\right)^2\\
&=\frac{\left\{\begin{array}{c}q(x)(\alpha_1^*)^2(x)\alpha_1^{2n}(x)-2q(x)\alpha_1^*(x)\alpha_1^{n}(x)\alpha_2^*(x)\alpha_2^{n}(x)\\ +q(x)(\alpha_2^*)^2(x)\alpha_2^{2n}(x)+(\alpha_1^*)^2(x)\alpha_1^{2n+2}(x)\\ -2\alpha_1^*(x)\alpha_1^{n+1}(x)\alpha_2^*(x)\alpha_2^{n+1}(x)+(\alpha_2^*)^2(x)\alpha_2^{2n+2}(x)\end{array}\right\}}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{(\alpha_1^*)^2(x)\alpha_1^{2n}(x)\left(q(x)-q(x)\frac{\alpha_1(x)}{\alpha_2(x)}\right)+(\alpha_2^*)^2(x)\alpha_2^{2n}(x)\left(q(x)-q(x)\frac{\alpha_2(x)}{\alpha_1(x)}\right)}{(\alpha_1(x)-\alpha_2(x))^2}\\
&=\frac{(\alpha_1^*)^2(x)\alpha_1^{2n+1}(x)-(\alpha_2^*)^2(x)\alpha_2^{2n+1}(x)}{\alpha_1(x)-\alpha_2(x)}.
\end{aligned}$$

Also the proof of the identity 2. is similar to 1.. □

**Theorem 2.8.** *For the $(p,q)-$Fibonacci and Lucas octonion polynomials $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$, $n\geq 0$ we have following binomial sum formula for odd and even terms,* *1.*

$$OF_{p,q,2n}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OF_{p,q,m}(x)$$
$$OF_{p,q,2n+1}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OF_{p,q,m+1}(x)$$

*2.*

$$OL_{p,q,2n}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OL_{p,q,m}(x)$$
$$OL_{p,q,2n+1}(x) = \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OL_{p,q,m+1}(x).$$

*Proof.* For 1. from (1.3) and Binet formulas, we get

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m OF_{p,q,m}(x) \\
&= \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} p(x)^m \frac{\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)}{\alpha_1(x) - \alpha_2(x)} \\
&= \frac{\alpha_1^*(x)}{\alpha_1(x) - \alpha_2(x)} \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} (p(x)\alpha_1(x))^m \\
&- \frac{\alpha_2^*(x)}{\alpha_1(x) - \alpha_2(x)} \sum_{m=0}^{n} \binom{n}{m} q(x)^{n-m} (p(x)\alpha_2(x))^m \\
&= \frac{\alpha_1^*(x)}{\alpha_1(x) - \alpha_2(x)} (q(x) + p(x)\alpha_1(x))^n - \frac{\alpha_2^*(x)}{\alpha_1(x) - \alpha_2(x)} (q(x) + p(x)\alpha_2(x))^n \\
&= \frac{\alpha_1^*(x)\alpha_1^{2n}(x) - \alpha_2^*(x)\alpha_2^{2n}(x)}{\alpha_1(x) - \alpha_2(x)} \\
&= OF_{p,q,2n}(x).
\end{aligned}$$

Also the other cases for $OL_{p,q,n}(x)$ can be done similarly. □

**Theorem 2.9.** *The sums of the first $n-$terms of the sequences $OF_{p,q,n}(x)$ and $OL_{p,q,n}(x)$ are given by*

$$\sum_{m=0}^{n} OF_{p,q,m}(x) = \frac{-q(x)OF_{p,q,n}(x) - OF_{p,q,n+1}(x) + OF_{p,q,0}(x) - \frac{\alpha_1^*(x)\alpha_2(x) - \alpha_2^*(x)\alpha_1(x)}{\alpha_1(x) - \alpha_2(x)}}{(\alpha_1(x) - 1)(\alpha_2(x) - 1)}$$

*and*

$$\sum_{m=0}^{n} OL_{p,q,m}(x) = \frac{-q(x)OL_{p,q,n}(x) - OL_{p,q,n+1}(x) + OL_{p,q,0}(x) - \left[\alpha_1^*(x)\alpha_2(x) + \alpha_2^*(x)\alpha_1(x)\right]}{(\alpha_1(x) - 1)(\alpha_2(x) - 1)}$$

*respectively.*

*Proof.* Using Binet formulas and the roots $\alpha_1(x)$, $\alpha_2(x)$, we get

$$\begin{aligned}
\sum_{m=0}^{n} OF_{p,q,m}(x) &= \frac{\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)}{\alpha_1(x) - \alpha_2(x)} \\
&= \frac{1}{\alpha_1(x) - \alpha_2(x)} \sum_{m=0}^{n} (\alpha_1^*(x)\alpha_1^m(x) - \alpha_2^*(x)\alpha_2^m(x)) \\
&= \frac{1}{\alpha_1(x) - \alpha_2(x)} (\alpha_1^*(x) \sum_{m=0}^{n} \alpha_1^m(x) - \alpha_2^*(x) \sum_{m=0}^{n} \alpha_2^m(x)) \\
&= \frac{1}{\alpha_1(x) - \alpha_2(x)} (\alpha_1^*(x) \frac{\alpha_1^{n+1}(x) - 1}{\alpha_1(x) - 1} - \alpha_2^*(x) \frac{\alpha_2^{n+1}(x) - 1}{\alpha_2(x) - 1}) \\
&= \frac{\alpha_1^*(x)(\alpha_1^{n+1}(x) - 1)(\alpha_2(x) - 1) - \alpha_2^*(x)(\alpha_2^{n+1}(x) - 1)(\alpha_1(x) - 1)}{(\alpha_1(x) - \alpha_2(x))(\alpha_1(x) - 1)(\alpha_2(x) - 1)} \\
&= \frac{\left\{ \begin{array}{c} (\alpha_1^*(x)\alpha_1^{n+1}(x)\alpha_2(x)) - (\alpha_1^*(x)\alpha_1^{n+1}(x)) - (\alpha_1^*(x)\alpha_2(x)) + \alpha_1^*(x) \\ -\alpha_2^*(x) - \alpha_2^*(x)\alpha_2^{n+1}(x)\alpha_1(x) + \alpha_2^*(x)\alpha_2^{n+1}(x) + \alpha_2^*(x)\alpha_2(x) \end{array} \right\}}{(\alpha_1(x) - \alpha_2(x))(\alpha_1(x) - 1)(\alpha_2(x) - 1)} \\
&= \frac{-q(x)OF_{p,q,n}(x) - OF_{p,q,n+1}(x) + OF_{p,q,0}(x) - \frac{\alpha_1^*(x)\alpha_2(x) - \alpha_2^*(x)\alpha_1(x)}{\alpha_1(x) - \alpha_2(x)}}{(\alpha_1(x) - 1)(\alpha_2(x) - 1)}.
\end{aligned}$$

The other cases for $OL_{p,q,n}(x)$ can be done similarly. □

## 3. Conclusion

Octonions have great importance as they are used in quantum physics, applied mathematics, graph theory. In this work, we introduce the $(p,q)-$Fibonacci and Lucas octonion polynomials and formulate the Binet-style formula, the generating function and some identities of the $(p,q)-$Fibonacci octonion and Lucas octonion polynomial sequence. Thus, in our future studies we plan to examine different quaternion and octonion polynomials and their key features.

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