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Research Article

Contributions to the Fractional Hardy Integral Inequality

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Abstract

This article makes three contributions to the fractional Hardy integral inequality. First, we refine an existing result in the literature by improving the main constant and relaxing some assumptions on the parameters. We then propose a fractional-type Hardy integral inequality for an under-studied case, with a significant adaptation of the existing general proof. Finally, a version of this result is established when the integral domain is finite. The proofs are given in detail, with the exact expression of the constants involved at each step. We also mention that almost no intermediate results are used.

1. Introduction

The study of integral inequalities is a central topic in mathematics, particularly in real and functional analysis. It is used to determine the possible values of complex integrals, i.e., integrals that cannot be evaluated exactly using the usual techniques. Let us take a brief look at the most famous integral inequalities. The Hölder integral inequality connects integrals of products, the Minkowski integral inequality generalizes the triangular inequality to integrals, the Cauchy-Schwarz integral inequality, which can be seen as a special case of the Hölder integral inequality, gives a fundamental bound for the integral version of the inner product, the Jensen integral inequality applies to convex functions and integrals, the Grönwall integral inequality estimates solutions to differential inequalities, the Sobolev integral inequality relates function norms in Sobolev spaces, the Chebyshev integral inequality bounds probabilities using integrals, the Young integral inequality helps with convolution estimates, and the Hardy integral inequality gives bounds on weighted integrals and establishes key relationships in functional spaces. These results have many applications in physics, probability and optimization. The mathematical details of them can be found in the following books: [1–5].

For the purposes of this article, we will emphasize the Hardy integral inequality. A brief review of the results used for the purposes of the article is given below; the full historical facts and details can be found in [6, 7]. The classical Hardy integral inequality states that, for $p \in (1,+\infty)$ and $f : (0,+\infty) \mapsto \mathbb{R}$ such that

$$\int_0^{+\infty} |f(x)|^p dx < +\infty,$$

we have

$$\int_0^{+\infty} \left[\frac{1}{x}\int_0^x |f(t)|dt\right]^p dx \leq C_p \int_0^{+\infty} |f(x)|^p dx,$$

where

$$C_p = \left(\frac{p}{p-1}\right)^p. \tag{1.1}$$

See [1, 8]. A finite integration interval version was also established, attributed to [9]. It states that, for $(a,b) \in (0,+\infty)^2 \cup \{\pm\infty\}^2$ with $a < b$, $p \in (1,+\infty)$ and $f : (a,b) \mapsto \mathbb{R}$ such that

$$\int_a^b |f(x)|^p dx < +\infty,$$

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we have

$$\int_a^b \left[\frac{1}{x}\int_a^x |f(t)|dt\right]^p dx \leq C_p \int_a^b |f(x)|^p dx,$$

where C_p is given in Equation (1.1). For more information, we refer to [6, Section 1.5].
Another famous variant of the Hardy integral inequality is the fractional Hardy integral inequality, which states that, for $p \in [1,+\infty)$, $\lambda \in (0,+\infty)\backslash\{1\}$ and $f:(0,+\infty)\mapsto \mathbb{R}$ such that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty,$$

we have

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq D \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

where D is a certain constant (to be discussed later). We also refer to [6, Chapter 5] for the relevant historical background and developments. There are other versions of these integral inequalities that extend their applicability. They are used in harmonic analysis, partial differential equations, mathematical physics and probability theory. In addition to the classic books [6, 7], a selection of articles on the subject is given below: [10–29].
In particular, in [23], some results and proofs have attracted our attention. The fractional Hardy integral inequality is demonstrated in an original and simple way, with clear assumptions and exact constants. In particular, we emphasize two results, described below.

- The result in [23, Lemma 2] is formulated below, modulo some minor changes in the presentation. Let $p \in [1,+\infty)$, $\lambda \in (0,+\infty)\backslash\{1\}$ (note that the value $\lambda = 1$ is excluded) and $f:(0,+\infty)\mapsto \mathbb{R}$ such that

  $$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty.$$

  Then the following integral inequality holds:

  $$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq D_{\alpha,p,\lambda} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

  where

  $$D_{\alpha,p,\lambda} := \frac{2^p}{\alpha}\{\max[|1-\alpha|,|2\alpha-1|]\}^{1+\lambda}$$

  and $\alpha \in (0,+\infty)$ can be arbitrarily chosen such that

  $$\frac{2^{p-1}}{\lambda}(2^\lambda-1)\alpha^{\lambda-1} \leq \frac{1}{2}. \tag{1.2}$$

- In the same framework and under the same assumptions, but with a finite and $f:(0,a)\mapsto \mathbb{R}$, [23, Corollary 1] ensures that

  $$\int_0^a \frac{|f(x)|^p}{x^\lambda} dx \leq D_{\alpha,p,\lambda} \int_0^a\int_0^a \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy.$$

The proofs of [23, Lemma 2] and [23, Corollary 1] are based on the triangular inequality, a thorough decomposition of the integral, the use of the Fubini-Tonelli integral theorem, integral calculus and basic arithmetic. The fact that these comprehensible developments are combined with the exact expressions of the constants involved is a real plus for a deeper understanding of these inequalities.
This article contributes to the topic in three related ways. First, we generalize the result in [23, Lemma 2]. In particular, we extend it to $p \in (0,+\infty)$, including the new case $p \in (0,1]$, and we slightly relax the assumption on α described in Equation (1.2). Second, the fractional Hardy integral inequality in [23, Lemma 2] is not valid for $\lambda = 1$. So we have no upper bound on the term

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx,$$

which we will call "the case $\lambda = 1$" for our purposes.
This raises the following question: Can we prove a fractional-type Hardy integral inequality for this case? We provide an answer to this question by modifying the proof of [23, Lemma 2] on several crucial points. The case $p \in (0,+\infty)$ is also considered and the constants involved are given. Third, a similar question arises for the term

$$\int_0^a \frac{|f(x)|^p}{x} dx,$$

where a is finite. An answer is also given. Thus, new integral inequalities are established, considering understudied cases of the fractional Hardy integral inequality. The proofs are detailed for accuracy and reproducibility.
The rest of the article is divided into four sections: Section 2 is devoted to the generalization of [23, Lemma 2]. Fractional-type Hardy integral inequalities for "the case $\lambda = 1$" and for infinite and finite intervals are considered in Sections 3 and 4, respectively. A conclusion is proposed in Section 5.

2. Generalization of an Existing Result

The first proposition offers a generalized version of [23, Lemma 2]. In particular, the points below are developed.

- The condition on p, i.e., $p \in (1,+\infty)$, can be relaxed as $p \in (0,+\infty)$, with a slight modification of a constant.
- The condition on α recalled in Equation (1.2) can be slightly relaxed with little mathematical effort.

These modifications give more flexibility to the constant in the factor in the main inequality.

Proposition 2.1. *Let $p \in (0,+\infty)$, $\lambda \in (0,+\infty)\backslash\{1\}$ and $f : (0,+\infty) \mapsto \mathbb{R}$ such that*

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty.$$

Then the following integral inequality holds:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \le F_{\beta,p,\lambda} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

where

$$F_{\beta,p,\lambda} := \left[1 - \frac{\max(2^{p-1},1)}{\lambda}(2^\lambda - 1)\beta^{\lambda-1}\right]^{-1} \frac{\max(2^{p-1},1)}{\beta}\left\{\max\left[|1-\beta|,|2\beta-1|\right]\right\}^{1+\lambda} \tag{2.1}$$

and $\beta \in (0,+\infty)$ can be arbitrarily chosen such that

$$\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda - 1)\beta^{\lambda-1} < 1.$$

Proof. The proof revisits that of [23, Lemma 2]. The details are given below. First, for any $p \in (0,+\infty)$ and $(u,v) \in \mathbb{R}^2$, we have

$$|u+v|^p \le \max(2^{p-1},1)[|u|^p + |v|^p]. \tag{2.2}$$

See [30, Chapter 1]. This inequality applied to $u = f(y)$ and $v = f(x) - f(y)$ gives

$$|f(x)|^p = |f(y) + [f(x)-f(y)]|^p \le \max(2^{p-1},1)|f(y)|^p + \max(2^{p-1},1)|f(x)-f(y)|^p.$$

Now, since $\beta \in (0,+\infty)$ and $x^{1+\lambda} \in (0,+\infty)$, dividing by $\beta x^{1+\lambda}$, we have

$$\frac{|f(x)|^p}{\beta x^{1+\lambda}} \le \max(2^{p-1},1)\frac{|f(y)|^p}{\beta x^{1+\lambda}} + \max(2^{p-1},1)\frac{|f(x)-f(y)|^p}{\beta x^{1+\lambda}}.$$

Integrating with respect to $y \in (\beta x, 2\beta x)$, we get

$$\int_{\beta x}^{2\beta x} \frac{|f(x)|^p}{\beta x^{1+\lambda}} dy \le \max(2^{p-1},1)\int_{\beta x}^{2\beta x} \frac{|f(y)|^p}{\beta x^{1+\lambda}} dy + \max(2^{p-1},1)\int_{\beta x}^{2\beta x} \frac{|f(x)-f(y)|^p}{\beta x^{1+\lambda}} dy.$$

For the left term, we have

$$\int_{\beta x}^{2\beta x} \frac{|f(x)|^p}{\beta x^{1+\lambda}} dy = \frac{|f(x)|^p}{\beta x^{1+\lambda}} \int_{\beta x}^{2\beta x} dy = \frac{|f(x)|^p}{x^\lambda}.$$

Using this and integrating with respect to $x \in (0,+\infty)$, we find that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \le \max(2^{p-1},1)P + \max(2^{p-1},1)Q, \tag{2.3}$$

where

$$P := \int_0^{+\infty}\int_{\beta x}^{2\beta x} \frac{|f(y)|^p}{\beta x^{1+\lambda}} dydx$$

and

$$Q := \int_0^{+\infty}\int_{\beta x}^{2\beta x} \frac{|f(x)-f(y)|^p}{\beta x^{1+\lambda}} dydx.$$

Let us now bound P and Q successively.
For P, the Fubini-Tonelli integral theorem ensures the change in the order of integration, which gives

$$\begin{aligned} P &= \int_0^{+\infty}\int_{y/(2\beta)}^{y/\beta} \frac{|f(y)|^p}{\beta x^{1+\lambda}} dxdy = \int_0^{+\infty} |f(y)|^p \left[\int_{y/(2\beta)}^{y/\beta} \frac{1}{\beta x^{1+\lambda}} dx\right] dy \\ &= \int_0^{+\infty} |f(y)|^p \frac{1}{\beta\lambda}\left[\left(\frac{2\beta}{y}\right)^\lambda - \left(\frac{\beta}{y}\right)^\lambda\right] dy = \frac{1}{\lambda}(2^\lambda - 1)\beta^{\lambda-1}\int_0^{+\infty} \frac{|f(y)|^p}{y^\lambda} dy. \end{aligned} \tag{2.4}$$

On the other hand, for Q, with a suitable decomposition of the integrated term, we have

$$\begin{aligned} Q &= \frac{1}{\beta}\int_0^{+\infty}\int_{\beta x}^{2\beta x}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}}\times\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx \\ &\le \frac{1}{\beta}\left[\sup_{x\in(0,+\infty)}\sup_{y\in(\beta x,2\beta x)}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}}\right]\int_0^{+\infty}\int_{\beta x}^{2\beta x}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx \\ &\le \frac{1}{\beta}\left[\sup_{x\in(0,+\infty)}\sup_{y\in(\beta x,2\beta x)}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}}\right]\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx. \end{aligned}$$

The term in square brackets can be developed as follows:

$$\begin{aligned} \sup_{x\in(0,+\infty)}\sup_{y\in(\beta x,2\beta x)}\frac{|x-y|^{1+\lambda}}{x^{1+\lambda}} &= \sup_{x\in(0,+\infty)}\max\left[\frac{|x-\beta x|^{1+\lambda}}{x^{1+\lambda}},\frac{|x-2\beta x|^{1+\lambda}}{x^{1+\lambda}}\right] \\ &= \{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}. \end{aligned}$$

We therefore obtain

$$Q\le\frac{1}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx. \tag{2.5}$$

Combining Equations (2.3), (2.4) and (2.5) together, we get

$$\begin{aligned} \int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx &\le \frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx \\ &\quad+\frac{\max(2^{p-1},1)}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx. \end{aligned}$$

This is equivalent to the following inequality:

$$\left[1-\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]\int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx\le\frac{\max(2^{p-1},1)}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dydx.$$

Since $\beta\in(0,+\infty)$ is chosen such that

$$\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}<1,$$

we also have

$$\begin{aligned} \int_0^{+\infty}\frac{|f(x)|^p}{x^\lambda}dx &\le \left[1-\frac{\max(2^{p-1},1)}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]^{-1}\frac{\max(2^{p-1},1)}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}\times\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dxdy \\ &= F_{\beta,p,\lambda}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}dxdy, \end{aligned}$$

where $F_{\beta,p,\lambda}$ is given in Equation (2.1). The proof of Proposition 2.1 ends. □

This result shows greater flexibility than [23, Lemma 2], on the same mathematical basis. Indeed, the case $p\in(0,1]$ is now considered, and β can be chosen more flexibly than α in [23, Lemma 2]. Note that, for any $p\in(0,1]$, based on Equation (2.1), we have

$$F_{\beta,p,\lambda}=\left[1-\frac{1}{\lambda}(2^\lambda-1)\beta^{\lambda-1}\right]^{-1}\frac{1}{\beta}\{\max[|1-\beta|,|2\beta-1|]\}^{1+\lambda}.$$

The presence of p in the index of $F_{\beta,p,\lambda}$ is due to the fact that β may depend on p.
These aspects have a positive effect on the constant factor in the main inequality.

3. A New Fractional-Type Hardy Integral Inequality

The proposition below fills a gap in [23, Lemma 2] and in the literature on integral inequalities in general. It provides a valuable fractional-type inequality for "the case $\lambda=1$", i.e., for the following integral as the left term:

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx,$$

which was avoided in [23, Lemma 2]. As in Proposition 2.1, we assume that $p\in(0,+\infty)$ and reuse some techniques from the proof of that proposition.

Proposition 3.1. *Let $p\in(0,+\infty)$ and $f:(0,+\infty)\mapsto\mathbb{R}$ such that*

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx<+\infty.$$

Then we have

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx\leq G_{\theta,p,\alpha}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx,$$

where

$$G_{\theta,p,\alpha}:=\frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta-\max(2^{p-1},1)\right]}\left\{\max\left[|1-\alpha|,|2\alpha-1|\right]\right\}^{1+1/\theta}, \tag{3.1}$$

$\theta>\max(2^{p-1},1)$ *and* $\alpha\in(0,+\infty)$ *(it is completely arbitrary).*

Proof. A significant modification of the proof of [23, Lemma 2] is necessary, where the new constant θ plays an important role. First, it follows from Equation (2.2) applied to $u=f(y)$ and $v=f(x)-f(y)$ that

$$|f(x)|^p=|f(y)+[f(x)-f(y)]|^p\leq\max(2^{p-1},1)|f(y)|^p+\max(2^{p-1},1)|f(x)-f(y)|^p.$$

We now divide the above terms by $\alpha x^{1+\theta}$, activating the parameter θ, which is positive. This gives us

$$\frac{|f(x)|^p}{\alpha x^{1+\theta}}\leq\max(2^{p-1},1)\frac{|f(y)|^p}{\alpha x^{1+\theta}}+\max(2^{p-1},1)\frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}}.$$

Integrating with respect to $y\in(\alpha x^\theta,2\alpha x^\theta)$, we get

$$\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(x)|^p}{\alpha x^{1+\theta}}dy\leq\max(2^{p-1},1)\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(y)|^p}{\alpha x^{1+\theta}}dy+\max(2^{p-1},1)\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}}dy.$$

Noticing that

$$\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(x)|^p}{\alpha x^{1+\theta}}dy=\frac{|f(x)|^p}{\alpha x^{1+\theta}}\int_{\alpha x^\theta}^{2\alpha x^\theta}dy=\frac{|f(x)|^p}{x},$$

and integrating with respect to $x\in(0,+\infty)$, we find that

$$\int_0^{+\infty}\frac{|f(x)|^p}{x}dx\leq\max(2^{p-1},1)U+\max(2^{p-1},1)V, \tag{3.2}$$

where

$$U:=\int_0^{+\infty}\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(y)|^p}{\alpha x^{1+\theta}}dydx$$

and

$$V:=\int_0^{+\infty}\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}}dydx.$$

Let us now bound U and V successively.
For U, the Fubini-Tonelli integral theorem ensures the change in the order of integration, which gives

$$\begin{aligned}U&=\int_0^{+\infty}\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}}\frac{|f(y)|^p}{\alpha x^{1+\theta}}dxdy=\int_0^{+\infty}|f(y)|^p\left[\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}}\frac{1}{\alpha x^{1+\theta}}dx\right]dy\\&=\int_0^{+\infty}|f(y)|^p\frac{1}{\alpha\theta}\left[\left(\frac{2\alpha}{y}\right)-\left(\frac{\alpha}{y}\right)\right]dy=\frac{1}{\theta}\int_0^{+\infty}\frac{|f(y)|^p}{y}dy.\end{aligned} \tag{3.3}$$

Note that the resulting term no longer depends on α.
On the other hand, for V, we have

$$\begin{aligned}V&=\frac{1}{\alpha}\int_0^{+\infty}\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|x^\theta-y|^{1+1/\theta}}{x^{1+\theta}}\times\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx\\&\leq\frac{1}{\alpha}\left[\sup_{x\in(0,+\infty)}\sup_{y\in(\alpha x^\theta,2\alpha x^\theta)}\frac{|x^\theta-y|^{1+1/\theta}}{x^{1+\theta}}\right]\int_0^{+\infty}\int_{\alpha x^\theta}^{2\alpha x^\theta}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx\\&\leq\frac{1}{\alpha}\left[\sup_{x\in(0,+\infty)}\sup_{y\in(\alpha x^\theta,2\alpha x^\theta)}\frac{|x^\theta-y|^{1+1/\theta}}{x^{1+\theta}}\right]\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|x^\theta-y|^{1+1/\theta}}dydx.\end{aligned}$$

Thanks to the introduction of the key term $|x^\theta - y|^{1+1/\theta}$, we have

$$\sup_{x\in(0,+\infty)} \sup_{y\in(\alpha x^\theta, 2\alpha x^\theta)} \frac{|x^\theta - y|^{1+1/\theta}}{x^{1+\theta}} = \sup_{x\in(0,+\infty)} \max\left[\frac{|x^\theta - \alpha x^\theta|^{1+1/\theta}}{x^{1+\theta}}, \frac{|x^\theta - 2\alpha x^\theta|^{1+1/\theta}}{x^{1+\theta}}\right]$$
$$= \{\max[|1-\alpha|, |2\alpha - 1|]\}^{1+1/\theta}.$$

We therefore obtain

$$V \leq \frac{1}{\alpha}\{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx. \tag{3.4}$$

It follows from Equations (3.2), (3.3) and (3.4) that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)}{\theta}\int_0^{+\infty} \frac{|f(x)|^p}{x} dx + \frac{\max(2^{p-1},1)}{\alpha}\{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx.$$

This is equivalent to

$$\left[1 - \frac{\max(2^{p-1},1)}{\theta}\right]\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)}{\alpha}\{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx.$$

Since $\theta > \max(2^{p-1},1)$, we get

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta - \max(2^{p-1},1)\right]}\{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx$$
$$= G_{\theta,p,\alpha}\int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,$$

where $G_{\theta,p,\alpha}$ is given in Equation (3.1). The proof of Proposition 3.1 ends. □

This result is thus a proposal of a fractional-type Hardy integral inequality for an under-explored case. The key point was the use of an adaptable parameter θ, which activates numerous intermediate terms, including $|x^\theta - y|^{1+1/\theta}$. We should also mention the presence of the parameter α, which can be set to any positive value.
Note that, when $p \in (0,1]$, the constant $G_{\theta,p,\alpha}$ in Equation (3.1) is reduced to

$$G_{\theta,p,\alpha} = \frac{\theta}{\alpha(\theta - 1)}\{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta},$$

which is independent of p.
Furthermore, for $p \in (0,+\infty)$ taking $\alpha = 1/2$, we find that

$$G_{\theta,p,\alpha} = \frac{\max(2^{p-1},1)\theta}{(1/2)\left[\theta - \max(2^{p-1},1)\right]}\left\{\max\left[\left|1-\frac{1}{2}\right|, \left|2\times\frac{1}{2} - 1\right|\right]\right\}^{1+1/\theta}$$
$$= \frac{\max(2^{p-1},1)\theta 2^{-1/\theta}}{\theta - \max(2^{p-1},1)},$$

and the inequality in Proposition 3.1 gives

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \frac{\max(2^{p-1},1)\theta 2^{-1/\theta}}{\theta - \max(2^{p-1},1)}\int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,$$

with $\theta > \max(2^{p-1},1)$. Especially, for $p \in (0,1]$, we have

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \frac{\theta 2^{-1/\theta}}{\theta - 1}\int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx.$$

Besides filling a gap in the literature, this inequality has the merit of having a simple and original constant in the factor.
In the general case, we can also note that the change of variables $z = x^\theta$ gives

$$\int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx = \frac{1}{\theta}\int_0^{+\infty}\int_0^{+\infty} \frac{|f(z^{1/\theta})-f(y)|^p}{|z - y|^{1+1/\theta}} z^{1/\theta - 1} dydz.$$

Therefore, the inequality in Proposition 3.1 with the denominator term $|x-y|^{1+1/\theta}$ can be reformulated as follows:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq \frac{1}{\theta} G_{\theta,p,\alpha}\int_0^{+\infty}\int_0^{+\infty} \frac{|f(x^{1/\theta})-f(y)|^p}{|x - y|^{1+1/\theta}} x^{1/\theta - 1} dydx.$$

From this point of view, the inequality can be seen as a special type of fractional Hardy integral inequality.

4. A New Fractional-Type Hardy Integral Inequality on a Finite Interval

The proposition below fills a gap in [23, Corollary 1] and in the literature on integral inequalities in general. It provides a valuable fractional-type inequality for "the case $\lambda = 1$" when the integration domain is finite, i.e., for the following integral as the left term:

$$\int_0^a \frac{|f(x)|^p}{x} dx,$$

where a is finite. This case was avoided in [23, Corollary 1]. We also assume that $p \in (0,+\infty)$, including the not yet considered case $p \in (0,1]$.

Proposition 4.1. *Let $a \in (0,+\infty)$, $p \in (0,+\infty)$ and $f : (0,a) \mapsto \mathbb{R}$ such that*

$$\int_0^a \frac{|f(x)|^p}{x} dx < +\infty.$$

Then we have

$$\int_0^a \frac{|f(x)|^p}{x} dx \leq H_{\theta,p,\alpha} \int_0^a \int_0^a \frac{|f(x)-f(y)|^p}{|x^\theta - y|^{1+1/\theta}} dydx,$$

where

$$H_{\theta,p,\alpha} := \frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta - \max(2^{p-1},1)\right]} \{\max[|1-\alpha|, |2\alpha-1|]\}^{1+1/\theta}, \tag{4.1}$$

$\theta > \max(2^{p-1},1)$ and $\alpha \in (0,+\infty)$ is such that

$$\alpha \leq \frac{1}{2} a^{1-\theta}. \tag{4.2}$$

Note that α may depend on a.

Proof. The proof is similar to that of Proposition 3.1, but a special treatment of the integration interval has to be done at several strategic points. We detail it to understand the assumption made on α in Equation (4.2), which depends on a. First, the inequality in Equation (2.2) gives

$$|f(x)|^p = |f(y) + [f(x) - f(y)]|^p \leq \max(2^{p-1},1)|f(y)|^p + \max(2^{p-1},1)|f(x)-f(y)|^p.$$

Dividing the above terms by $\alpha x^{1+\theta}$, which is positive, we get

$$\frac{|f(x)|^p}{\alpha x^{1+\theta}} \leq \max(2^{p-1},1) \frac{|f(y)|^p}{\alpha x^{1+\theta}} + \max(2^{p-1},1) \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}}.$$

Integrating with respect to $y \in (\alpha x^\theta, 2\alpha x^\theta)$ (with $x \in (0,a)$), we obtain

$$\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)|^p}{\alpha x^{1+\theta}} dy \leq \max(2^{p-1},1) \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dy + \max(2^{p-1},1) \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}} dy.$$

Noticing that

$$\int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)|^p}{\alpha x^{1+\theta}} dy = \frac{|f(x)|^p}{\alpha x^{1+\theta}} \int_{\alpha x^\theta}^{2\alpha x^\theta} dy = \frac{|f(x)|^p}{x},$$

and integrating with respect to $x \in (0,a)$, we find that

$$\int_0^a \frac{|f(x)|^p}{x} dx \leq \max(2^{p-1},1)W + \max(2^{p-1},1)Z, \tag{4.3}$$

where

$$W := \int_0^a \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dydx$$

and

$$Z := \int_0^a \int_{\alpha x^\theta}^{2\alpha x^\theta} \frac{|f(x)-f(y)|^p}{\alpha x^{1+\theta}} dydx.$$

Let us now bound W and Z successively.

For W, the Fubini-Tonelli integral theorem ensures the change in the order of integration, but we need to adjust the bounds of the integral by taking into account a. We find that

$$W = \int_0^{2\alpha a^\theta} \int_{[y/(2\alpha)]^{1/\theta}}^{\min\left[(y/\alpha)^{1/\theta},a\right]} \frac{|f(y)|^p}{\alpha x^{1+\theta}} dxdy.$$

Using $\min\left[(y/\alpha)^{1/\theta},a\right]\leq(y/\alpha)^{1/\theta}$, Equation (4.2) which gives $2\alpha a^{\theta}\leq a$ and the fact that the integrated term is non-negative, we have

$$\begin{aligned}W&\leq\int_{0}^{2\alpha a^{\theta}}\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}}\frac{|f(y)|^{p}}{\alpha x^{1+\theta}}dxdy\leq\int_{0}^{a}\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}}\frac{|f(y)|^{p}}{\alpha x^{1+\theta}}dxdy\\&=\int_{0}^{a}|f(y)|^{p}\left[\int_{[y/(2\alpha)]^{1/\theta}}^{(y/\alpha)^{1/\theta}}\frac{1}{\alpha x^{1+\theta}}dx\right]dy\\&=\int_{0}^{a}|f(y)|^{p}\frac{1}{\alpha\theta}\left[\left(\frac{2\alpha}{y}\right)-\left(\frac{\alpha}{y}\right)\right]dy=\frac{1}{\theta}\int_{0}^{a}\frac{|f(y)|^{p}}{y}dy.\end{aligned}\tag{4.4}$$

On the other hand, for Z, using again Equation (4.2) which gives $y\in(\alpha x^{\theta},2\alpha x^{\theta})\subseteq(0,a)$, we get

$$\begin{aligned}Z&=\frac{1}{\alpha}\int_{0}^{a}\int_{\alpha x^{\theta}}^{2\alpha x^{\theta}}\frac{|x^{\theta}-y|^{1+1/\theta}}{x^{1+\theta}}\times\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx\\&\leq\frac{1}{\alpha}\left[\sup_{x\in(0,a)}\sup_{y\in(\alpha x^{\theta},2\alpha x^{\theta})}\frac{|x^{\theta}-y|^{1+1/\theta}}{x^{1+\theta}}\right]\int_{0}^{a}\int_{\alpha x^{\theta}}^{2\alpha x^{\theta}}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx\\&\leq\frac{1}{\alpha}\left[\sup_{x\in(0,a)}\sup_{y\in(\alpha x^{\theta},2\alpha x^{\theta})}\frac{|x^{\theta}-y|^{1+1/\theta}}{x^{1+\theta}}\right]\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx.\end{aligned}$$

We have

$$\begin{aligned}\sup_{x\in(0,a)}\sup_{y\in(\alpha x^{\theta},2\alpha x^{\theta})}\frac{|x^{\theta}-y|^{1+1/\theta}}{x^{1+\theta}}&=\sup_{x\in(0,a)}\max\left[\frac{|x^{\theta}-\alpha x^{\theta}|^{1+1/\theta}}{x^{1+\theta}},\frac{|x^{\theta}-2\alpha x^{\theta}|^{1+1/\theta}}{x^{1+\theta}}\right]\\&=\{\max\left[|1-\alpha|,|2\alpha-1|\right]\}^{1+1/\theta}.\end{aligned}$$

We therefore obtain

$$Z\leq\frac{1}{\alpha}\{\max\left[|1-\alpha|,|2\alpha-1|\right]\}^{1+1/\theta}\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx.\tag{4.5}$$

It follows from Equations (4.3), (4.4) and (4.5) that

$$\int_{0}^{a}\frac{|f(x)|^{p}}{x}dx\leq\frac{\max(2^{p-1},1)}{\theta}\int_{0}^{a}\frac{|f(x)|^{p}}{x}dx+\frac{\max(2^{p-1},1)}{\alpha}\{\max\left[|1-\alpha|,|2\alpha-1|\right]\}^{1+1/\theta}\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx,$$

which is equivalent to

$$\left[1-\frac{\max(2^{p-1},1)}{\theta}\right]\int_{0}^{a}\frac{|f(x)|^{p}}{x}dx\leq\frac{\max(2^{p-1},1)}{\alpha}\{\max\left[|1-\alpha|,|2\alpha-1|\right]\}^{1+1/\theta}\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx.$$

Since $\theta>\max(2^{p-1},1)$, we get

$$\begin{aligned}\int_{0}^{a}\frac{|f(x)|^{p}}{x}dx&\leq\frac{\max(2^{p-1},1)\theta}{\alpha\left[\theta-\max(2^{p-1},1)\right]}\{\max\left[|1-\alpha|,|2\alpha-1|\right]\}^{1+1/\theta}\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx\\&=H_{\theta,p,\alpha}\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx,\end{aligned}$$

where $H_{\theta,p,\alpha}$ is given in Equation (4.1). The proof of Proposition 4.1 ends. □

Proposition 4.1 thus completes Proposition 3.1 by considering a finite integration interval. It shows that, contrary to Proposition 3.1, when considering the integration interval $(0,a)$, α cannot be chosen arbitrarily; the assumption in Equation (4.2) must be satisfied. In particular, if we take $a=1$, we can choose $\alpha=1/2$, so that the constant in Equation (4.1) becomes

$$H_{\theta,p,\alpha}=\frac{\max(2^{p-1},1)\theta2^{-1/\theta}}{\theta-\max(2^{p-1},1)}$$

and the inequality in Proposition 4.1 gives

$$\int_{0}^{1}\frac{|f(x)|^{p}}{x}dx\leq\frac{\max(2^{p-1},1)\theta2^{-1/\theta}}{\theta-\max(2^{p-1},1)}\int_{0}^{1}\int_{0}^{1}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx.$$

In the general case, note that the change of variables $z=x^{\theta}$ gives

$$\int_{0}^{a}\int_{0}^{a}\frac{|f(x)-f(y)|^{p}}{|x^{\theta}-y|^{1+1/\theta}}dydx=\frac{1}{\theta}\int_{0}^{a^{\theta}}\int_{0}^{a}\frac{|f(z^{1/\theta})-f(y)|^{p}}{|z-y|^{1+1/\theta}}z^{1/\theta-1}dydz.$$

Therefore, the inequality in Proposition 4.1 can be reformulated with the denominator term $|x-y|^{1+1/\theta}$ as follows:

$$\int_{0}^{a}\frac{|f(x)|^{p}}{x}dx\leq\frac{1}{\theta}H_{\theta,p,\alpha}\int_{0}^{a^{\theta}}\int_{0}^{a}\frac{|f(x^{1/\theta})-f(y)|^{p}}{|x-y|^{1+1/\theta}}x^{1/\theta-1}dydx.$$

We can note that the integral of integration with respect to x is now $(0,a^{\theta})$.

5. Conclusion

In this article, we have contributed to two key results in [23], which are [23, Lemma 2] and [23, Corollary 1]. In particular,

- we have refined the main constant in the main inequality in [23, Lemma 2], and with $p \in (0,+\infty)$ instead of $p \in (1,+\infty)$,
- we have provided solutions to open problems corresponding to the establishment of fractional-type inequalities with the following integrals as the left term:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx$$

or

$$\int_0^{a} \frac{|f(x)|^p}{x} dx,$$

where a is finite.

These inequalities have been proved with some significant modifications of the proof proposed in [23], which are more adaptable to the particular case under consideration. The article thus fills a theoretical gap and, in a sense, completes [23, Lemma 2] and [23, Corollary 1]. The techniques developed can certainly be reused to solve complex mathematical inequalities. This is the logical perspective of the article.

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