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Research Article

An expression for zeta values and a summation formula via hyperbolic secant random variables

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Abstract

The aim of this paper is to derive a summation formula for the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}$ and an expression for $\zeta(2n+2)$ by using hyperbolic secant random variables. These identities involve Euler numbers and are obtained by computing the moments of the random variable and the moments of the sum of two independent such random variables.

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1. Introduction

The aim of this paper is twofold. Firstly, we assume that X is the hyperbolic secant random variable. Then we derive a summation formula for $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}$ by computing the moments $E[X^{2n}]$ in two different ways. Secondly, we assume that X and Y are independent hyperbolic secant random variables. Then we obtain the expression for $\zeta(2n+2)$ by computing the moments $E[(X+Y)^{2n}]$ in two different ways. We note here that both of these identities involve the Euler numbers (see (1.1)).

In more detail, the outline of this paper is as follows. In Section 1, we remind the reader of the facts that are needed throughout this paper. Among other things, we mention that Kim derived the summation formula in (1.8) from the generating function of Euler numbers and the Fourier series of sine function (see [4]). Section 2 contains the main results of this paper. Let X be the hyperbolic secant random variable (see (2.1)). We determine the moment generating function of X by using the beta function and the reflection formula of the gamma function. This yields an expression for $E[X^{2n}]$ (see Theorem 2.1). In another way, we compute the moment $E[X^{2n}]$ directly from definition. By equating these two, we get the summation formula for $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}$ in Theorem 2.2, which is the same as the one in (1.8). Assume that X and Y are independent hyperbolic secant random variables. On the one hand, we derive the moment generating function of $X+Y$ from those of X and Y. Thereby we obtain an expression of $E[(X+Y)^{2n}]$. On the other hand, we compute $E[(X+Y)^{2n}]$ directly from the definition. Now, equating these two gives

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us an expression for $\zeta(2n+2)$, $(n=0,1,2,\dots)$, in Theorem 2.4. It is noteworthy that these special values of the zeta function at even positive integers possess an alternative representation involving Euler numbers, in contrast to their conventional expression in terms of Bernoulli numbers (see [6, 13]). In Section 3, we apply the central limit theorem to a sequence of independent hyperbolic secant random variables to show a lemma.

In recent years, much work has been done for probabilisitc extensions of many special polynomials and numbers and their applications. Let Y be a random variable satisfying suitable moment condition. Indeed, probabilisitc Stirling numbers associated with Y are introduced in [1], probabilisitic Bernoulli and Euler polynomials associated with Y are studied in [7] and probabilistic Bell polynomials associated with Y are investigated in [11]. Spivey's type recurrence relation is shown for probabilistic r-Bell polynomials associated with Y in [5], some dimorphic properties associated with Bernoulli random variables are explored in [9] and some identities related to Poisson and uniform random variables are considered in [8]. For further details on these, the reader may refer to [1, 5, 7–9, 11] and the references therein. In the rest of this section, we recall the facts that are needed throughout this paper.

The Euler numbers are defined by

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |x| < \frac{\pi}{2}. \tag{1.1}$$

From (1.1), we get

$$\begin{aligned} &E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ E_{10} = -50521, \cdots; \\ &E_{2k+1} = 0, \text{ for } \ k = 0,1,2,\cdots, \quad (\text{see [4]}). \end{aligned} \tag{1.2}$$

Euler's formula states that, for any real number x,

$$e^{ix} = \cos x + i \sin x, \quad \text{where } i = \sqrt{-1}, \quad (\text{see [13]}). \tag{1.3}$$

From (1.3), we note that

$$\cos x = \frac{1}{2}(e^{ix}+e^{-ix}), \quad (\text{see [13]}). \tag{1.4}$$

Thus, by (1.1) and (1.4), we get

$$\begin{aligned} \sec x &= \frac{2}{e^{ix}+e^{-ix}} = \operatorname{sech}(ix) = \sum_{n=0}^{\infty} \frac{i^n E_n}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n+1}}{(2n+1)!} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}. \end{aligned} \tag{1.5}$$

From (1.5), we have

$$x \sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n+1}, \quad (|x| < \frac{\pi}{2}). \tag{1.6}$$

From the Fourier series of $f(x) = \sin ax$ on $[-\pi,\pi]$, Kim derived the following formula

$$\frac{\pi a}{2} \sec\Big(\frac{\pi a}{2}\Big) = \sum_{k=0}^{\infty} 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} a^{2k+1}, \quad (\text{see [4]}). \tag{1.7}$$

Thus, by (1.6) and (1.7), we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n \frac{1}{2} \frac{E_{2n}}{(2n)!} \Big(\frac{\pi}{2}\Big)^{2n+1} = \frac{1}{2} \frac{|E_{2n}|}{(2n)!} \Big(\frac{\pi}{2}\Big)^{2n+1}, \ (\text{see [4]}), \tag{1.8}$$

where n is a nonnegative integer. Here we note that Euler considered the numbers $|E_{2n}|$ in connection with sums like (1.8). Later, in 1851, Raabe introduced the term "Euler numbers," (see [12]).

For $s \in \mathbb{C}$ with $Re(s) > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{(see [2, 13])}. \tag{1.9}$$

From (1.8) and (1.9), we note that

$$\zeta(2n+1, \frac{1}{4}) + 2^{2n}(1 - 2^{2n+1})\zeta(2n+1) = (-1)^n \frac{E_{2n}}{2(2n)!}\pi^{2n+1}2^{2n}, \tag{1.10}$$

where $\zeta(s,a)$ is Hurwitz zeta function given by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad (Re(s) > 1,\ a \neq 0, -1, -2, \cdots), \quad \text{(see [13])}.$$

In addition, by (1.9), we get

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{4(2n-1)!(1-4^n)} E^*_{2n-1}, \quad \text{(see [4, 6])},$$

where E^*_n is defined by $\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E^*_n \frac{t^n}{n!}$.

Let X and Y be independent random variables such that $f(x)$ and $g(x)$ are their respective probability density functions. We recall that the cumulative distribution function of the random variable X is defined by

$$F_X(a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx, \quad \text{(see [10])}.$$

Assume that $f_{X+Y}(a)$ is the probability density function of $X+Y$. Then it is given by the convolution of $f(x)$ and $g(x)$ as in the following:

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} g(y)f(a-y)dy, \quad \text{(see [10])}. \tag{1.11}$$

By (1.5), we easily get

$$\begin{aligned} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\cosh x}dx &= \frac{2}{\pi}\int_{0}^{\infty} \frac{1}{\cosh x}dx = \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{e^x(1+e^{-2x})}dx \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k \int_0^{\infty} e^{-(2k+1)x}dx = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{4}{\pi}\frac{\pi}{4} = 1. \end{aligned} \tag{1.12}$$

A random variable X is the hyperbolic secant random variable if the probability density function is given by

$$f(x) = \frac{1}{\pi}\operatorname{sech} x = \frac{1}{\pi}\frac{1}{\cosh x}, \quad (x \in (-\infty, \infty)), \quad \text{(see [3])}. \tag{1.13}$$

For $\alpha > 0$, the gamma function is defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt, \quad \text{(see [13])}. \tag{1.14}$$

For $\alpha, \beta > 0$, the Beta function is defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \text{(see [13])}. \tag{1.15}$$

Thus, by (1.15), we get

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^{\infty} \frac{t^{\beta-1}}{(1+t)^{\alpha+\beta}}dt, \quad \text{(see [13])}. \tag{1.16}$$

For $|t|<1$, the Euler's reflection formula of the gamma function is given by

$$\Gamma(t)\Gamma(1-t)=\frac{\pi}{\sin \pi t}, \quad \text{(see [13]).} \tag{1.17}$$

2. An expression for zeta values and a summation formula via hyperbolic secant random variables

In this section, we assume that X is the hyperbolic secant random variable. Then the probability density function of X is given by

$$f(x)=\frac{1}{\pi}\operatorname{sech} x=\frac{1}{\pi}\frac{1}{\cosh x}, \quad (x\in(-\infty,\infty)). \tag{2.1}$$

First, we consider the moment generating function of X.
For $|t|<1$, we have

$$\begin{aligned} E[e^{Xt}]&=\int_{-\infty}^{\infty}e^{xt}f(x)dx=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{xt}\frac{1}{\cosh x}dx \\ &=\frac{2}{\pi}\int_{-\infty}^{\infty}e^{xt}\frac{1}{e^{x}+e^{-x}}dx=\frac{2}{\pi}\int_{-\infty}^{\infty}e^{xt}\frac{e^{-x}}{e^{2x}+1}e^{2x}dx. \end{aligned} \tag{2.2}$$

From (2.2), by making change of the variable $y=e^{2x}$, and using (1.16) and (1.17), we get

$$\begin{aligned} E[e^{Xt}]&=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{xt}\frac{e^{-x}}{e^{2x}+1}2e^{2x}dx=\frac{1}{\pi}\int_{0}^{\infty}\frac{y^{\frac{t}{2}+\frac{1}{2}-1}}{1+y}dy \\ &=\frac{1}{\pi}B\Big(\frac{t}{2}+\frac{1}{2},1-\frac{t}{2}-\frac{1}{2}\Big)=\frac{1}{\pi}\frac{\Gamma(\frac{t}{2}+\frac{1}{2})\Gamma(1-\frac{t}{2}-\frac{1}{2})}{\Gamma(1)} \\ &=\frac{1}{\pi}\frac{\pi}{\sin(\frac{t}{2}+\frac{1}{2})\pi}=\frac{1}{\cos\frac{\pi t}{2}}. \end{aligned} \tag{2.3}$$

From (1.5) and (2.3), we note that

$$\begin{aligned} \sum_{n=0}^{\infty}E[X^n]\frac{t^n}{n!}&=E[e^{Xt}]=\frac{1}{\cos\frac{\pi t}{2}}=\frac{1}{\cosh\frac{\pi i t}{2}} \\ &=\sum_{n=0}^{\infty}E_{2n}\Big(\frac{\pi}{2}\Big)^{2n}(-1)^n\frac{t^{2n}}{(2n)!}, \quad (|t|<1). \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.1. *Let X be the hyperbolic secant random variable whose probability density function is given by $f(x)=\frac{1}{\pi}\frac{1}{\cosh x}$. Then, for $n\geq 0$, we have*

$$E[X^{2n}]=E_{2n}\Big(\frac{\pi}{2}\Big)^{2n}(-1)^n, \tag{2.5}$$

and

$$E[X^{2n+1}]=0.$$

We recall that the variance of X is given by

$$\sigma^2=E[(X-E[X])^2]=E[X^2]-(E[X])^2. \tag{2.6}$$

From (2.5) and (2.6), we note that

$$\sigma^2=E[X^2]-(E[X])^2=\Big(\frac{\pi}{2}\Big)^2E_2(-1)=\Big(\frac{\pi}{2}\Big)^2, \quad \mu=E[X]=0. \tag{2.7}$$

On the other hand, by (1.13), we get

$$\begin{aligned} E[X^{2n}] &= \int_{-\infty}^{\infty} x^{2n} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} x^{2n}\frac{1}{\cosh x}dx \\ &= \frac{4}{\pi}\int_{0}^{\infty} x^{2n}\frac{1}{e^{x}+e^{-x}}dx = \frac{4}{\pi}\int_{0}^{\infty} x^{2n}\frac{e^{-x}}{1+e^{-2x}}dx \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\int_{0}^{\infty} x^{2n+1}e^{-(2k+1)x}\frac{dx}{x} \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}(2n)!. \end{aligned} \tag{2.8}$$

From (2.5) and (2.8), we have the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{(-1)^n}{(2n)!}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^{2n+1}E_{2n}.$$

In particular, for $n=1$, we have the following result.

Corollary 2.3.

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{3}} = -\frac{1}{4}\Big(\frac{\pi}{2}\Big)^{3}E_2 = \frac{1}{32}\pi^3.$$

Assume that X and Y are independent hyperbolic secant random variables. Then, by (1.11), we get

$$\begin{aligned} f_{X+Y}(a) &= \int_{-\infty}^{\infty}\frac{1}{\pi}\frac{1}{\cosh(y)}\frac{1}{\pi}\frac{1}{\cosh(a-y)}dy \\ &= \frac{4}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{e^{y}+e^{-y}}\frac{1}{e^{a-y}+e^{y-a}}dy \\ &= \frac{4}{\pi^2}\int_{-\infty}^{\infty}\frac{e^{y}}{e^{2y}+1}\frac{e^{-a+y}}{e^{-2a+2y}+1}dy. \end{aligned} \tag{2.9}$$

From (2.9), by making change of the variable $e^y = x$, we have

$$\begin{aligned} f_{X+Y}(a) &= \frac{4}{\pi^2}\int_{0}^{\infty}\frac{x}{1+x^2}\frac{e^{-a}}{1+x^2e^{-2a}}dx \\ &= \frac{4}{\pi^2(e^{a}-e^{-a})}\int_{0}^{\infty}\Big(\frac{x}{1+x^2}-\frac{xe^{-2a}}{1+x^2e^{-2a}}\Big)dx \\ &= \frac{4}{\pi^2(e^{a}-e^{-a})}\Big[\frac{\log(1+x^2)-\log(1+e^{-2a}x^2)}{2}\Big]_0^{\infty} \\ &= \frac{4}{\pi^2(e^{a}-e^{-a})}\Big[\frac{1}{2}\log\big(\frac{1+x^2}{1+x^2e^{-2a}}\big)\Big]_0^{\infty} \\ &= \frac{4}{\pi^2(e^{a}-e^{-a})}\frac{1}{2}\log(e^{2a}) = \frac{4a}{\pi^2(e^{a}-e^{-a})}, \end{aligned} \tag{2.10}$$

where $a \in (-\infty,\infty)$.

Thus, the probability density function of $Z = X+Y$ is given by

$$f_Z(a) = f_{X+Y}(a) = \frac{4a}{\pi^2(e^{a}-e^{-a})}, \quad (a \in (-\infty,\infty)). \tag{2.11}$$

Since X and Y are independent random variables,

$$E[e^{t(X+Y)}] = E[e^{tX}]E[e^{tY}] = \frac{1}{\cos\frac{\pi}{2}t}\frac{1}{\cos\frac{\pi}{2}t}. \tag{2.12}$$

Thus, by (1.5) and (2.12), we get

$$\begin{aligned} E[e^{t(X+Y)}] &= \sum_{m=0}^{\infty} E_{2m}\Big(\frac{\pi}{2}\Big)^{2m}(-1)^m\frac{t^{2m}}{(2m)!}\sum_{j=0}^{\infty}E_{2j}\Big(\frac{\pi}{2}\Big)^{2j}(-1)^j\frac{t^{2j}}{(2j)!} \\ &= \sum_{n=0}^{\infty}\bigg((-1)^n\Big(\frac{\pi}{2}\Big)^{2n}\sum_{l=0}^{n}\binom{2n}{2l}E_{2l}E_{2n-2l}\bigg)\frac{t^{2n}}{(2n)!}, \end{aligned} \tag{2.13}$$

From (2.11), we note that

$$\begin{aligned} E[(X+Y)^{2n}] &= E[Z^{2n}] \\ &= \int_{-\infty}^{\infty} x^{2n} f_Z(x)dx = \int_{-\infty}^{\infty} x^{2n}\frac{4x}{\pi^2}\frac{1}{e^x-e^{-x}}dx \\ &= \frac{8}{\pi^2}\int_0^{\infty}\frac{x^{2n+1}}{e^x-e^{-x}}dx = \frac{8}{\pi^2}\int_0^{\infty}\frac{e^{-x}x^{2n+1}}{1-e^{-2x}}dx \\ &= \frac{8}{\pi^2}\sum_{k=0}^{\infty}\int_0^{\infty}x^{2n+2}e^{-(2k+1)x}\frac{dx}{x} \\ &= \frac{8}{\pi^2}(2n+1)!\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}. \end{aligned} \tag{2.14}$$

By (2.13) and (2.14), we get

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}} &= (-1)^n\Big(\frac{\pi}{2}\Big)^{2n}\frac{\pi^2}{8}\frac{1}{(2n+1)!}\sum_{l=0}^{n}\binom{2n}{2l}E_{2l}E_{2n-2l} \\ &= (-1)^n\Big(\frac{\pi}{2}\Big)^{2n+2}\frac{1}{2}\frac{1}{(2n+1)!}\sum_{l=0}^{n}\binom{2n}{2l}E_{2l}E_{2n-2l}. \end{aligned} \tag{2.15}$$

By (1.9), we have

$$\begin{aligned} \zeta(2n+2) &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}} + \sum_{k=1}^{\infty}\frac{1}{(2k)^{2n+2}} \\ &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}} + \frac{1}{2^{2n+2}}\zeta(2n+2). \end{aligned} \tag{2.16}$$

Thus, (2.16), we get

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}} = \Big(1-\frac{1}{2^{2n+2}}\Big)\zeta(2n+2). \tag{2.17}$$

From (2.15) and (2.17), we have the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\zeta(2n+2) = \frac{(-1)^n}{\Big(1-\frac{1}{2^{2n+2}}\Big)(2n+1)!}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^{2n+2}\sum_{l=0}^{n}\binom{2n}{2l}E_{2l}E_{2n-2l}. \tag{2.18}$$

To illustrate our results, by using (1.2) and (2.18) we compute

$$\begin{aligned}
\zeta(2) &= \frac{4}{3\cdot 1}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^2 = \frac{\pi^2}{6},\\
\zeta(4) &= \frac{-16}{15\cdot 6}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^4(-1-1) = \frac{\pi^4}{90},\\
\zeta(6) &= \frac{64}{63\cdot 120}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^6(5+6(-1)(-1)+5) = \frac{\pi^6}{945},\\
\zeta(8) &= \frac{-256}{255\cdot 5040}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^8\big(-61+15(-1)5+15\cdot 5(-1)-61\big) = \frac{\pi^8}{9450},\\
\zeta(10) &= \frac{1024}{1023\cdot 362880}\frac{1}{2}\Big(\frac{\pi}{2}\Big)^{10}(1385+28(-1)(-61)\\
&\qquad +70\cdot 5\cdot 5+28(-61)(-1)+1385) = \frac{\pi^{10}}{93555}.
\end{aligned}$$

3. Further remark

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu$, $Var(X_i) = \sigma^2$, $(i = 1, 2, \ldots)$. Then the cental limit theorem (see [10]) states that

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \sim N(0, 1).$$

That is,

$$P\Big\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\Big\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{x^2}{2}}\,dx, \tag{3.1}$$

as $n \to \infty$.

Lemma 3.1. *Let $X_1, X_2, \ldots$ be a sequence of independent hyperbolic secant random variables, and let $f_{X_1+X_2+\cdots+X_n}(x)$ be the probability density function of $X_1 + X_2 + \cdots + X_n$. Then*

$$\frac{\pi}{2}\sqrt{n} f_{X_1+X_2+\cdots+X_n}\big(\frac{\pi}{2}\sqrt{n}\, y\big) \to \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}},$$

as $n \to \infty$.

Proof. By (2.7), we get

$$\sigma^2 = Var(X_i) = \Big(\frac{\pi}{2}\Big)^2,\ \mu = E[X_i] = 0,$$

where $i = 1, 2, \ldots$. By central limit theorem (see (3.1)), we have

$$P\Big\{\frac{X_1 + X_2 + \cdots + X_n}{\frac{\pi}{2}\sqrt{n}} \le y\Big\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx, \tag{3.2}$$

as $n \to \infty$. Then (3.2) is the same as saying that

$$\int_{-\infty}^{y} \frac{\pi}{2}\sqrt{n} f_{X_1+X_2+\cdots+X_n}\big(\frac{\pi}{2}\sqrt{n}\, x\big)dx \to \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx,$$

as $n \to \infty$. □

4. Conclusion

In this paper, among other things, we showed the summation formula

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = (-1)^n \frac{1}{2}\frac{E_{2n}}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n+1} = \frac{1}{2}\frac{|E_{2n}|}{(2n)!}\Big(\frac{\pi}{2}\Big)^{2n+1},$$

by evaluating even moments of the hyperbolic secant random variable in two different ways. As we mentioned in the Introduction, this formula was known to Euler. So the contribution of the present paper would be that it gives a probabilistic and simple proof of the above summation formula.

We would like to note that the Euler numbers in (1.1) has interesting connections with Euler zigzag numbers defined by the Taylor series, which is given by

$$\sec x + \tan x = \tan\Big(\frac{\pi}{4}+\frac{x}{2}\Big) = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}.$$

Indeed, one can show that $A_{2n} = (-1)^n E_{2n} = |E_{2n}|,\ (n = 0, 1, 2, \dots)$. Interesting combinatorial interpretations for the zigzag numbers can be found in the stanley's recent talk given at the 14th Ramanujan Colloquium, which was held at University of Florida in 2023, (see [12]). For example, A_n is equal to the number of permutations in the symmetric group S_n that are alternationg. Here a sequence $a_1, a_2, \dots, a_n$ of distinct integers is defined to be *alternating* if $a_1 > a_2 < a_3 > a_4 < \cdots$. For example, $A_4 = E_4 = 5$, since 2143, 3142, 3241, 4132, 4231 are the alternating permutations in S_4.

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