Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 11
DOI : 10.15672/hujms.xx

Research Article

Further characterizations of m-weak group inverses in a proper $*$-ring

Yukun Zhou*[1], Jianlong Chen[2]

[1] *School of Physical and Mathematical Sciences, Nanjing Tech University, Nanjing 211816, China*
[2] *School of Mathematics, Southeast University, Nanjing 210096, China*

Abstract

Let R be a proper $*$-ring, $a, b \in R$ and $m \in \mathbb{N}$. It is proved that a is m-weak group invertible if and only if a is right hybrid $(a^k, (a^k)^* a^m)$-invertible for some $k \in \mathbb{N}^+$. Several new characterizations of m-weak group inverses are presented by means of right ideal and right annihilator. Under the assumption that a has the m-weak group inverse $a^{ⓦ_m}$, we present some sufficient and necessary conditions which guarantee the additive property to hold for m-weak group inverses, namely $(a+b)^{ⓦ_m} = (1 + a^{ⓦ_m} b)^{-1} a^{ⓦ_m}$.

Mathematics Subject Classification (2020). 15A09, 16U90

Keywords. m-weak group inverse, weak group inverse, hybrid (b, c)-inverse, pseudo core inverse

1. Introduction

Pseudo core inverses [11] and weak group inverses [21] are two kinds of significant generalized inverses. They play important roles in different research fields. In 2021, as a common generalization of these two generalized inverses, a new generalized inverse called m-weak group inverse [26] was introduced.

Let R be a ring with an involution $*$. Recall that R is a proper $*$-ring if $a^* a = 0$ implies $a = 0$ for any $a \in R$. Throughout this article, R is always a proper $*$-ring. The symbols $\mathbb{N}$ and $\mathbb{N}^+$ denote the sets of all nonnegative integers and positive integers, respectively. Let $a \in R$ and $m \in \mathbb{N}$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that

$$xa^{k+1} = a^k, \qquad ax^2 = x, \qquad (a^k)^* a^{m+1} x = (a^k)^* a^m,$$

then x is called the m-weak group inverse of a. The m-weak group inverse of a is unique if it exists, and it is denoted by $a^{ⓦ_m}$. When m is taken as a different value, the m-weak group inverse coincides with other generalized inverses.

(i) 0-weak group inverse of a is exactly its pseudo core inverse;
(ii) 1-weak group inverse of a is exactly its weak group inverse;
(iii) 2-weak group inverse of a is exactly its generalized group inverse [8].

*Corresponding Author.
Email addresses: ykzhou187@163.com (Y. Zhou), jlchen@seu.edu.cn (J. Chen)
Received: 29.11.2024; Accepted: 23.03.2025

The m-weak group inverse is closely related to different generalized inverses, such as DMP inverses [14], weak core inverses [7], m-weak core inverses [9], Moore-Penrose m-WGI [19]. For more results of m-weak group inverses, readers can refer to [6, 8, 10, 16–18, 22].

In 2012, Drazin [5] introduced notions of (b, c)-inverse and hybrid (b, c)-inverse. The (b, c)-inverse unifies many known generalized inverses. For instance, given $a \in R$,

(i) the group inverse of a is its (a, a)-inverse;
(ii) the Drazin inverse of a is its (a^k, a^k)-inverse for some $k \in \mathbb{N}^+$;
(iii) the pseudo core inverse of a is its $(a^k, (a^k)^*)$-inverse for some $k \in \mathbb{N}^+$.

As far as the authors know, few known generalized inverses coincide with hybrid (b, c)-inverses but not (b, c)-inverses. In Section 3, the relations among m-weak group inverse, (b, c)-inverse and hybrid (b, c)-inverse are established. It is proved that a is m-weak group invertible if and only if a is right hybrid $(a^k, (a^k)^*a^m)$-invertible for some $k \in \mathbb{N}^+$. Remark 3.7 below illustrates that the weak group inverse of a is exactly right hybrid $(a^k, (a^k)^*a)$-inverse of a but not $(a^k, (a^k)^*a)$-inverse. In addition, by means of right ideal and right annihilator, several new characterizations of m-weak group inverses are obtained.

Let $A \in M_n(\mathbb{C})$ be invertible and $E \in M_n(\mathbb{C})$. As is well known, if $I + A^{-1}E$ is invertible, then $A + E$ is invertible with

$$(A + E)^{-1} = (I + A^{-1}E)^{-1}A^{-1}.$$

This formula was extended to the cases of different generalized inverses. For instance, Castro-González et al. [2] gave some sufficient and necessary conditions for $(A + E)^D = (I + A^D E)^{-1}A^D$. In Section 4, we consider the equivalent conditions under which additive properties of m-weak group inverses hold, namely $(a + b)^{Ⓦ_m} = (1 + a^{Ⓦ_m} b)^{-1}a^{Ⓦ_m}$.

2. Preliminaries

In this section, we present some necessary definitions and lemmas.

Definition 2.1. [4] Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad xa = ax,$$

then a is said to be Drazin invertible and x is called the Drazin inverse of a. Such an x is unique and denoted by a^D. In particular, x is called the group inverse of a when $k = 1$.

If k is the smallest positive integer such that above equations hold, then k is called the Drazin index of a and denoted by $\mathrm{i}(a)$.

The core-EP inverse of a complex matrix [15] was extended to the pseudo core inverse of an element in a ring with involution by Gao and Chen [11].

Definition 2.2. [11] Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (ax)^* = ax,$$

then a is said to be pseudo core invertible and x is called the pseudo core inverse of a. Such an x is unique and denoted by $a^{\textcircled{D}}$. In particular, when $k = 1$, x is called the core inverse [1, 20] of a.

In 2018, Wang and Chen [21] defined the weak group inverse of a complex matrix. In 2020, Zhou et al. [23] generalized this notion to the weak group inverse of an element in a proper $*$-ring.

Definition 2.3. [23] Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (a^k)^*a^2x = (a^k)^*a,$$

then a is said to be weak group invertible and x is called the weak group inverse of a. Such an x is unique and denoted by $a^{Ⓦ}$.

The m-weak group index is defined similarly as the Drazin index. If a is m-weak group invertible, then a is Drazin invertible, and the m-weak group index of a is equal to its Drazin index. Therefore, we still use $\mathrm{i}(a)$ to represent the m-weak group index of a.

The symbols R^D, $R^{\textcircled{\tiny D}}$, $R^{\textcircled{\tiny W}}$ and $R^{\textcircled{\tiny W}_m}$ denote sets of all Drazin invertible, pseudo core invertible, weak group invertible and m-weak group invertible elements in R, respectively.

Lemma 2.4. [11] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x,$$

then

(1) *$ax = a^m x^m$ for arbitrary positive integer m;*
(2) *$xax = x$;*
(3) *a is Drazin invertible, $a^D = x^{k+1}a^k$ and $\mathrm{i}(a) \le k$.*

In [25], the authors denote

$$T_l(a) = \{x \in R : \ xa^{k+1} = a^k, \ ax^2 = x \text{ for some positive integer } k\}.$$

Lemma 2.5. [25] *Let $a \in R^D$, $k_1, ..., k_n, s_1, ..., s_n \in \mathbb{N}$ and $x_1, ..., x_n \in T_l(a)$. If $s_n \neq 0$, then*

$$\prod_{i=1}^{n} a^{k_i} x_i^{s_i} = a^k x_n^s, \text{ where } k = \sum_{i=1}^{n} k_i \text{ and } s = \sum_{i=1}^{n} s_i.$$

Lemma 2.6. [4] *Let $a \in R$. Then $a \in R^D$ if and only if $a^k = a^{k+1}x = ya^{k+1}$ for some $x, y \in R$ and $k \in \mathbb{N}^+$. In this case, $a^D = a^k x^{k+1} = y^{k+1}a^k$.*

Lemma 2.7. [26] *Let $a \in R$. Then $a \in R^{\textcircled{\tiny W}}$ if and only if there exists $x \in R$ such that $(a^D)^*a = (a^D)^*a^Dx$. In this case, $a^{\textcircled{\tiny W}} = (a^D)^3x$.*

Now, recall the notions of (b, c)-inverse and right hybrid (b, c)-inverse.

Definition 2.8. [5] Let $a, b, c \in R$. If there exists $y \in R$ such that

$$yay = y, \quad yR = bR, \quad Ry = Rc,$$

then a is said to be (b, c)-invertible. Such an element y is unique and called the (b, c)-inverse of a.

For arbitrary $y \in R$, the right annihilator of y is defined, as usual, by $y^\circ = \{r \in R : \ yr = 0\}$.

Definition 2.9. [5] Let $a, b, c \in R$. If there exists $y \in R$ such that

$$yay = y, \quad yR = bR, \quad y^\circ = c^\circ,$$

then a is said to be right hybrid (b, c)-invertible. Such an element y is unique and called the right hybrid (b, c)-inverse of a.

Lemma 2.10. [27] *Let $a, b, c, y \in R$. Then the following conditions are equivalent:*

(1) *y is the right hybrid (b, c)-inverse of a;*
(2) *$yab = b$, $cay = c$, $yR \subseteq bR$ and $c^\circ \subseteq y^\circ$.*

Recall that an element $a \in R$ is called $\{1, 3\}$-invertible if there exists $x \in R$ such that $axa = a$, $(ax)^* = ax$. The symbol $R^{\{1,3\}}$ denotes the set of all $\{1, 3\}$-invertible elements in R.

Lemma 2.11. [12] *Let $a \in R$, then $Ra = Ra^*a$ if and only if $a \in R^{\{1,3\}}$.*

3. New characterizations of the m-weak group inverse

In this section, we establish the relations among m-weak group inverse, (b,c)-inverse and hybrid (b,c)-inverse. Now, we establish the relation between weak group inverse and right hybrid (b,c)-inverse.

Theorem 3.1. *Let $a \in R$. Then $a \in R^{\textcircled{w}}$ if and only if a is right hybrid $(a^k,(a^k)^*a)$-invertible for some $k \in \mathbb{N}^+$. In this case, $a^{\textcircled{w}}$ is the right hybrid $(a^k,(a^k)^*a)$-inverse of a.*

Proof. Suppose that $a \in R^{\textcircled{w}}$ with $\mathrm{i}(a) = k$. Now, we claim that $a^{\textcircled{w}}$ is the right hybrid $(a^k,(a^k)^*a)$-inverse of a. It follows, from Lemma 2.4, that $a^{\textcircled{w}}aa^{\textcircled{w}} = a^{\textcircled{w}}$ and $a^{\textcircled{w}} = a^k(a^{\textcircled{w}})^{k+1} \in a^kR$. This together with $a^k = a^{\textcircled{w}}a^{k+1} \in a^{\textcircled{w}}R$ implies $a^{\textcircled{w}}R = a^kR$. Then, for any $t \in (a^{\textcircled{w}})^\circ$, we get that $(a^k)^*at = (a^k)^*a^2a^{\textcircled{w}}t = 0$. That is, $(a^{\textcircled{w}})^\circ \subseteq ((a^k)^*a)^\circ$. Let $r \in ((a^k)^*a)^\circ$. Since R is a proper $*$-ring and $(a^k)^*a^{k+1}(a^{\textcircled{w}})^kr = (a^k)^*a^2a^{\textcircled{w}}r = (a^k)^*ar = 0$, it follows that $a^{k+1}(a^{\textcircled{w}})^kr = 0$. So, we conclude that $a^{\textcircled{w}}r = a^Da^k(a^{\textcircled{w}})^kr = 0$ by Lemmas 2.4 and 2.5. That is, $((a^k)^*a)^\circ \subseteq (a^{\textcircled{w}})^\circ$.

Conversely, assume that there exists $k \in \mathbb{N}^+$ such that y is the right hybrid $(a^k,(a^k)^*a)$-inverse of a. It follows, from Lemma 2.10, that $ya^{k+1} = a^k$, $(a^k)^*a^2y = (a^k)^*a$, $y \in a^kR$ and $((a^k)^*a)^\circ = y^\circ$. Multiplying $(a^k)^*a^2y = (a^k)^*a$ by a^k on the right gives $(a^k)^*a^2ya^k = (a^k)^*a^{k+1}$. Since $y \in a^kR$ and R is a proper $*$-ring, we conclude that $a^2ya^k = a^{k+1} \in a^{k+2}R$. By Lemma 2.6, a is Drazin invertible, which together with $y \in a^kR$ implies $aa^Dy = y$. So, $ay^2 = ayaa^Dy = aya^{k+1}(a^D)^{k+1}y = a^{k+1}(a^D)^{k+1}y = y$. Hence, y is the weak group inverse of a. $\square$

Using a similar method as in Theorem 3.1, we can prove the main result of this section as follows.

Theorem 3.2. *Let $a \in R$ and $m \in \mathbb{N}$. Then $a \in R^{\textcircled{w}_m}$ if and only if a is right hybrid $(a^k,(a^k)^*a^m)$-invertible for some $k \in \mathbb{N}^+$. In this case, $a^{\textcircled{w}_m}$ is the right hybrid $(a^k,(a^k)^*a^m)$-inverse of a.*

In the above theorem, the corresponding result on pseudo core inverses can be presented by taking m $= 0$, which recovers a result by Zhu et al. [28].

Recall that $a \in R$ is called a weak group element if $a \in R^{\textcircled{w}}$ and $a^{\textcircled{w}} = a^D$.

Corollary 3.3. *Let $a \in R$. Then a is a weak group element if and only if a is right hybrid $(a^k,(a^k)^*a)$-invertible for some $k \in \mathbb{N}^+$ and $((a^k)^*a)^\circ = (a^k)^\circ$.*

It is natural to consider the $(a^k,(a^k)^*a)$-invertibility of a.

Proposition 3.4. *Let $a \in R$. Then $a \in R^{\textcircled{D}}$ if and only if a is $(a^k,(a^k)^*a)$-invertible for some $k \in \mathbb{N}^+$.*

Proof. Suppose that $a \in R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$. It follows, from [23, Remark 4.4] and Theorem 3.1, that $a \in R^{\textcircled{w}}$ and $a^{\textcircled{w}} = (a^{\textcircled{D}})^2a$ is the right hybrid $(a^k,(a^k)^*a)$-inverse of a. So, it suffices to prove that $R(a^k)^*a = Ra^{\textcircled{w}}$. By Lemmas 2.4 and 2.5, we have that

$$\begin{aligned} a^{\textcircled{w}} &= (a^{\textcircled{D}})^2a = (a^{\textcircled{D}})^2aa^{\textcircled{D}}a = (a^{\textcircled{D}})^2(aa^{\textcircled{D}})^*a \\ &= (a^{\textcircled{D}})^2(a^k(a^{\textcircled{D}})^k)^*a = (a^{\textcircled{D}})^2((a^{\textcircled{D}})^k)^*(a^k)^*a \in R(a^k)^*a. \end{aligned}$$

This together with $(a^k)^*a = (a^k)^*a^2a^{\textcircled{w}} \in Ra^{\textcircled{w}}$ implies $R(a^k)^*a = Ra^{\textcircled{w}}$.

Conversely, assume that there exists $k \in \mathbb{N}^+$ such that a is $(a^k,(a^k)^*a)$-invertible. By Theorem 3.1, we get that $a \in R^{\textcircled{w}} \subseteq R^D$ with $\mathrm{i}(a) \leq k$ and $R(a^k)^*a = Ra^{\textcircled{w}}$. So, there exists $t \in R$ such that $t(a^k)^*a = a^2a^{\textcircled{w}}$. Multiplying by a^{k+1} on the right, we obtain that $t(a^k)^*a^{k+2} = a^{k+2}$. Hence, $a^{k+2} = t((a^D)^*)^2(a^{k+2})^*a^{k+2} \in R(a^{k+2})^*a^{k+2}$. Then, it follows, from Lemma 2.11, that $a^{k+2} \in R^{\{1,3\}}$, which implies $a \in R^{\textcircled{D}}$ by [11, Theorem 2.3]. $\square$

Combining Theorem 3.1 and Proposition 3.4, we can recover a result in [22]: for $A \in M_n(\mathbb{C})$, the weak group inverse of A is exactly its $(A^k, (A^k)^*A)$-inverse for some $k \in \mathbb{N}^+$.

Remark 3.5. Let S be a ring with involution (not necessarily be a proper $*$-ring). Recall that an element $b \in S$ is left $*$-cancellable if $b^*bx = b^*by$ implies $bx = by$ for any $x, y \in R$. An immediate fact is that a ring is a proper $*$-ring if and only if each element is left $*$-cancellable. Noting that an element is left $*$-cancellable if it is $\{1,3\}$-invertible, we can conclude a result below.

Let $a \in S^D$ with $\mathrm{i}(a) = k$. Then $a \in S^{\textcircled{D}}$ if and only if a is $(a^k, (a^k)^*a)$-invertible. Indeed, noting that a^k is left $*$-cancellable if $a^k \in R^{\{1,3\}}$, the rest proof is immediate from the proof of Theorem 3.1 and Proposition 3.4.

Corollary 3.6. *Let $a \in R$ and $m \in \mathbb{N}$. Then $a \in R^{\textcircled{D}}$ if and only if a is $(a^k, (a^k)^*a^m)$-invertible for some $k \in \mathbb{N}^+$.*

Remark 3.7. Let $R = M_2(\mathbb{Z})$ and take the involution as the transpose. From [23, Example 4.5], we know that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is weak group invertible but not pseudo core invertible. That is, in a proper $*$-ring, a is right hybrid (a, a^*a)-invertible but not (a, a^*a)-invertible.

The forthcoming example shows that the condition R being a proper $*$-ring is not superfluous for Theorem 3.1. In other words, when R is not a proper $*$-ring, the condition $a \in R^{\textcircled{w}}$ is not equivalent to the condition that a is right hybrid $(a^k, (a^k)^*a)$-invertible.

Example 3.8. Let $G = \{e, a, b, c\}$ be a non-cyclic group. Let $S = \mathbb{Z}_3G$ be equipped with an involution $*$: $(x_1e + x_2a + x_3b + x_4c)^* = x_1e + x_2b + x_3a + x_4c$. Take $t = 2e + a$. From [26, Example 3.8], we know that each element in S has at most one weak group inverse and t is not left $*$-cancellable. It is easy to verify that t is idempotent and hence is weak group invertible. However, t is not right hybrid (t, t^*t)-invertible. Assume that t is right hybrid (t, t^*t)-invertible. Since S is commutative, we get that t^*t is idempotent. Then t is (t, t^*t)-invertible by [27, Proposition 2.5]. From Remark 3.5, we have that t is core invertible and hence is left $*$-cancellable. It is a contraction. Therefore, t is not right hybrid (t, t^*t)-invertible.

Inspired by the characterizations of pseudo core inverses in [28], we give the following characterizations by means of right ideal and right annihilator.

Proposition 3.9. *Let $a \in R^D$ with $\mathrm{i}(a) = k$. Then the following conditions are equivalent:*

(1) *a is weak group invertible;*
(2) *$R = a^kR \oplus ((a^k)^*a)^\circ$;*
(3) *$R = a^kR + ((a^k)^*a)^\circ$;*
(4) *$a^*aR = a^*a^kR \oplus a^*a((a^k)^*a)^\circ$;*
(5) *$a^*aR = a^*a^kR + a^*a((a^k)^*a)^\circ$.*

Proof. (1)$\Rightarrow$(2): It is clear by Theorem 3.1 and [27, Theorem 2.4].

(2)$\Rightarrow$(3)$\Rightarrow$(5) and (4)$\Rightarrow$(5): They are obvious.

(1)$\Rightarrow$(4): Suppose that a is weak group invertible. If there exist $x \in R$ and $y \in ((a^k)^*a)^\circ$ such that $a^*a^kx = a^*ay$, then $(a^k)^*a^kx = (a^k)^*ay = 0$. Since R is a proper $*$-ring, it follows that $a^kx = 0$. Consequently, $a^*a^kx = a^*ay = 0$. That is, $a^*a^kR \cap a^*a((a^k)^*a)^\circ = 0$. In view of $a^*a^kR \oplus a^*a((a^k)^*a)^\circ \subseteq a^*aR$, it suffices to prove that $a^*aR \subseteq a^*a^kR \oplus a^*a((a^k)^*a)^\circ$. It follows, from Lemma 2.4, that $a^*a^2a^{\textcircled{w}} = a^*a^{k+1}(a^{\textcircled{w}})^k \in a^*a^kR$. This together with $1 - aa^{\textcircled{w}} \in ((a^k)^*a)^\circ$ implies that $a^*a = a^*a^2a^{\textcircled{w}} + a^*a(1 - aa^{\textcircled{w}}) \in a^*a^kR \oplus a^*a((a^k)^*a)^\circ$. Thus, $a^*aR = a^*a^kR \oplus a^*a((a^k)^*a)^\circ$.

(5)$\Rightarrow$(1): There exists $r \in R$ and $t \in ((a^k)^*a)^\circ$ such that $a^*a = a^*a^kr + a^*at$. Multiplying by $(a^k)^*$ on the left, we get $(a^{k+1})^*a = (a^{k+1})^*a^kr + (a^{k+1})^*at = (a^{k+1})^*a^kr$. Then, Multiplying by $((a^D)^{k+2})^*$ on the left, we can get $(a^D)^*a = (a^D)^*a^k r = (a^D)^*a^D a^{k+1} r$. It follows, from Lemma 2.7, that a is weak group invertible. $\square$

Proposition 3.10. *Let $m \in \mathbb{N}^+$ and $a \in R^D$ with $\mathrm{i}(a) = k$. Then the following conditions are equivalent:*

(1) *a is m-weak group invertible;*
(2) $R = a^k R \oplus ((a^k)^* a^m)^\circ$;
(3) $R = a^k R + ((a^k)^* a^m)^\circ$;
(4) $(a^m)^* a^m R = (a^m)^* a^k R \oplus (a^m)^* a^m ((a^k)^* a^m)^\circ$;
(5) $(a^m)^* a^m R = (a^m)^* a^k R + (a^m)^* a^m ((a^k)^* a^m)^\circ$.

4. Additive properties of m-weak group inverses

In this section, we consider the equivalent conditions under which additive properties of m-weak group inverses hold, namely $(a+b)^{ⓦ_m} = (1 + a^{ⓦ_m} b)^{-1} a^{ⓦ_m}$.

Lemma 4.1. [24] *Let $a \in R^D$ with $x \in T_l(a)$ and $s \in \mathbb{N}^+$. Suppose that $1 + xb$ is invertible for some $b \in R$. Take $f = a + b$ and $f_0 = (1 + xb)^{-1} x$. Then the following statements are equivalent:*

(1) $(1 - ax)bax = 0$ *and* $(1 - ax) f^s (1 - ax) = 0$;
(2) *f is Drazin invertible with $\mathrm{i}(f) \leqslant s$ and $f_0 \in T_l(f)$.*

In this case, $f f_0 = ax$ and $T_l(f) = \alpha T_l(a)$, where $\alpha = (1 + xb)^{-1}$.

Using Lemma 4.1, we give additive properties of weak group inverses in next two theorems.

Theorem 4.2. *Let $a \in R^{ⓦ}$ and $s \in \mathbb{N}^+$. Suppose that $1 + a^{ⓦ} b$ is invertible for some $b \in R$. Take $f = a + b$ and $f_0 = (1 + a^{ⓦ} b)^{-1} a^{ⓦ}$. Then the following statements are equivalent:*

(1) $(aa^{ⓦ})^* b (1 - aa^{ⓦ}) \in (aa^{ⓦ})^* aa^{ⓦ} R$, $(1 - aa^{ⓦ}) baa^{ⓦ} = 0$ *and* $(1 - aa^{ⓦ}) f^s (1 - aa^{ⓦ}) = 0$;
(2) *f is weak group invertible with $\mathrm{i}(f) \leqslant s$ and $f_0 \in T_l(f)$.*

In this case, $f^{ⓦ} = f_0 + f_0^2 t (1 - aa^{ⓦ})$, where $(aa^{ⓦ})^* b (1 - aa^{ⓦ}) = (aa^{ⓦ})^* aa^{ⓦ} t (1 - aa^{ⓦ})$.

Proof. Assume $\mathrm{i}(a) = k$. Since

$$\begin{aligned}(aa^{ⓦ})^* a (1 - aa^{ⓦ}) &= [a^k (a^{ⓦ})^k]^* a (1 - aa^{ⓦ}) \\ &= ((a^{ⓦ})^k)^* (a^k)^* a (1 - aa^{ⓦ}) = 0,\end{aligned}$$

it follows that $(aa^{ⓦ})^* b (1 - aa^{ⓦ}) = (aa^{ⓦ})^* f (1 - aa^{ⓦ})$.

(1)⇒(2): It follows, from Lemma 4.1, that f is Drazin invertible and

$$(1 + a^{ⓦ} b)^{-1} a^{ⓦ} \in T_l(f), \ \mathrm{i}(f) \leqslant s.$$

According to Lemma 4.1, $f f_0 = aa^{ⓦ}$. Since $(f f_0)^* f (1 - f f_0) = (aa^{ⓦ})^* f (1 - aa^{ⓦ}) \in (aa^{ⓦ})^* aa^{ⓦ} R$ and $f^2 f_0 \in f f_0 R$, we conclude that $(f f_0)^* f \in (f f_0)^* f f_0 R$, which implies $(f^D)^* f \in (f^D)^* f^D R$ by Lemma 2.5. Thus, f is weak group invertible according to Lemma 2.7.

(2)⇒(1): From Lemma 4.1, we have that there exists $s \in \mathbb{N}^+$ such that $(1 - aa^{ⓦ}) baa^{ⓦ} = 0$, $(1 - aa^{ⓦ}) f^s (1 - aa^{ⓦ}) = 0$. It follows, from Lemma 2.5, that $(f f_0)^* f = (f f_0)^* f^2 f^{ⓦ} = (f f_0)^* f f_0 f^2 f^{ⓦ} \in (f f_0)^* f f_0 R$. That is, $(aa^{ⓦ})^* f \in (aa^{ⓦ})^* aa^{ⓦ} R$. Thus,

$$(aa^{ⓦ})^* f (1 - aa^{ⓦ}) \in (aa^{ⓦ})^* aa^{ⓦ} R.$$

In this case, suppose that $(aa^{ⓦ})^*b(1-aa^{ⓦ})=(aa^{ⓦ})^*aa^{ⓦ}t(1-aa^{ⓦ})$. By direct computation, we have

$$\begin{aligned}(aa^{ⓦ})^*f &= (aa^{ⓦ})^*faa^{ⓦ}+(aa^{ⓦ})^*f(1-aa^{ⓦ})\\ &= (aa^{ⓦ})^*faa^{ⓦ}+(aa^{ⓦ})^*b(1-aa^{ⓦ})\\ &= (aa^{ⓦ})^*faa^{ⓦ}+(aa^{ⓦ})^*aa^{ⓦ}t(1-aa^{ⓦ})\\ &= (ff_0)^*f^2f_0+(ff_0)^*ff_0t(1-ff_0)\\ &= (ff_0)^*f^{s+1}f_0^s+(ff_0)^*f^sf_0^st(1-ff_0)\\ &= (ff_0)^*f^s[ff_0^s+f_0^st(1-ff_0)].\end{aligned}$$

That is, $(f^D)^*f=(f^D)^*f^Df^{s+1}[ff_0^s+f_0^st(1-ff_0)]\in(f^D)^*f^DR$. Then, by Lemmas 2.5 and 2.7, $f^{ⓦ}=(f^D)^3f^{s+1}[ff_0^s+f_0^st(1-ff_0)]=f_0+f_0^2t(1-aa^{ⓦ})$. □

Theorem 4.3. *Let $a\in R^{ⓦ}$ and $s\in\mathbb{N}^+$. Suppose that $1+a^{ⓦ}b$ is invertible for some $b\in R$. Take $f=a+b$ and $f_0=(1+a^{ⓦ}b)^{-1}a^{ⓦ}$. Then the following statements are equivalent:*

(1) $(aa^{ⓦ})^*b(1-aa^{ⓦ})=0$, $(1-aa^{ⓦ})baa^{ⓦ}=0$ *and* $(1-aa^{ⓦ})f^s(1-aa^{ⓦ})=0$;
(2) f *is weak group invertible with* $\mathrm{i}(f)\leqslant s$ *and* $f^{ⓦ}=f_0$.

Proof. (1)⇒(2): It follows, from Theorem 4.2, that f is weak group invertible and $f^{ⓦ}=f_0+f_0^2t(1-aa^{ⓦ})$, where $(aa^{ⓦ})^*b(1-aa^{ⓦ})=(aa^{ⓦ})^*aa^{ⓦ}t(1-aa^{ⓦ})$. Since $0=(aa^{ⓦ})^*b(1-aa^{ⓦ})$ and R is a proper $*$-ring, it follows $aa^{ⓦ}t(1-aa^{ⓦ})=0$. Thus, $f_0^2t(1-aa^{ⓦ})=f_0^2ff_0t(1-aa^{ⓦ})=0$, which implies $f^{ⓦ}=f_0$.

(2)⇒(1): By direct computation,

$$\begin{aligned}(aa^{ⓦ})^*f(1-aa^{ⓦ}) &= (ff_0)^*f(1-ff_0)\\ &= (f^kf_0^k)^*f-(f^kf_0^k)^*f^2f_0\\ &= (f^kf_0^k)^*f-(f_0^k)^*(f^k)^*f^2f_0=0.\end{aligned}$$

□

The next example illustrates that the condition

$$(aa^{ⓦ})^*b(1-aa^{ⓦ})\in(aa^{ⓦ})^*aa^{ⓦ}R$$

is necessary in Theorem 4.2.

Example 4.4. In $M_4(\mathbb{Z})$, take transpose as the involution. Let $a=\begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&0&2\\0&0&0&0\end{pmatrix}$, $b=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$, $f=a+b$. It is clear that $a^{ⓦ}=a^D=a^2=\begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$. It is easy to verify that $(1-aa^{ⓦ})baa^{ⓦ}=0$, $(1-aa^{ⓦ})f^3(1-aa^{ⓦ})=0$ and $(aa^{ⓦ})^*b(1-aa^{ⓦ})\notin(aa^{ⓦ})^*aa^{ⓦ}M_4(\mathbb{Z})$. Therefore, by Theorem 4.2, f is not weak group invertible.

Compared with Theorem 4.2, the condition $(aa^{ⓦ})^*b(1-aa^{ⓦ})=0$ in Theorem 4.3 is necessary.

Example 4.5. In $M_4(\mathbb{R})$, take transpose as the involution. Let $a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $f = a+b$. It is clear that $(1-aa^{\textcircled{w}})baa^{\textcircled{w}} = 0$, $(1-aa^{\textcircled{w}})f^3(1-aa^{\textcircled{w}}) = 0$ and $(aa^{\textcircled{w}})^*b(1-aa^{\textcircled{w}}) \in (aa^{\textcircled{w}})^*aa^{\textcircled{w}}M_4(\mathbb{R})$. Thus, according to Theorem 4.2, f is weak group invertible. However, $(aa^{\textcircled{w}})^*b(1-aa^{\textcircled{w}}) \neq 0$. It follows, from Theorem 4.3, that $f^{\textcircled{w}} \neq (1+a^{\textcircled{w}}b)^{-1}a^{\textcircled{w}}$.

Now, we obtain corresponding results of m-weak group inverses. Their proofs are omitted, since they are similar to those of Theorems 4.2 and 4.3.

Theorem 4.6. *Let $m \in \mathbb{N}$, $a \in R^{\textcircled{w}_m}$ and $s \in \mathbb{N}^+$. Suppose that $1+a^{\textcircled{w}_m}b$ is invertible for some $b \in R$. Take $f = a+b$ and $f_0 = (1+a^{\textcircled{w}_m}b)^{-1}a^{\textcircled{w}_m}$. Then the following statements are equivalent:*

(1) *$(aa^{\textcircled{w}_m})^*f^m(1-aa^{\textcircled{w}_m}) \in (aa^{\textcircled{w}_m})^*aa^{\textcircled{w}_m}R$, $(1-aa^{\textcircled{w}_m})baa^{\textcircled{w}_m} = 0$ and $(1-aa^{\textcircled{w}_m})f^s(1-aa^{\textcircled{w}_m}) = 0$;*

(2) *f is m-weak group invertible with $\mathrm{i}(f) \leqslant s$ and $f_0 \in T_l(f)$.*

In this case, $f^{\textcircled{w}_m} = f_0 + f_0^{m+1}t(1-aa^{\textcircled{w}_m})$, where $(aa^{\textcircled{w}_m})^*f^m(1-aa^{\textcircled{w}_m}) = (aa^{\textcircled{w}_m})^*aa^{\textcircled{w}_m}t(1-aa^{\textcircled{w}_m})$.

Theorem 4.7. *Let $m \in \mathbb{N}$, $a \in R^{\textcircled{w}_m}$ and $s \in \mathbb{N}^+$. Suppose that $1+a^{\textcircled{w}_m}b$ is invertible for some $b \in R$. Take $f = a+b$ and $f_0 = (1+a^{\textcircled{w}_m}b)^{-1}a^{\textcircled{w}_m}$. Then the following statements are equivalent:*

(1) *$(aa^{\textcircled{w}_m})^*f^m(1-aa^{\textcircled{w}_m}) = 0$, $(1-aa^{\textcircled{w}_m})baa^{\textcircled{w}_m} = 0$ and $(1-aa^{\textcircled{w}_m})f^s(1-aa^{\textcircled{w}_m}) = 0$;*

(2) *f is m-weak group invertible with $\mathrm{i}(f) \leqslant s$ and $f^{\textcircled{w}_m} = f_0$.*

Take $m = 0$ in the above result. Since $(aa^{\textcircled{D}})^*(1-aa^{\textcircled{D}}) = 0$, we can recover the main result in [3].

Corollary 4.8. [3] *Let $a \in R^{\textcircled{D}}$ and $s \in \mathbb{N}^+$. Suppose that $1+a^{\textcircled{D}}b$ is invertible for some $b \in R$. Take $f = a+b$ and $f_0 = (1+a^{\textcircled{D}}b)^{-1}a^{\textcircled{D}}$. Then the following statements are equivalent:*

(1) *$(1-aa^{\textcircled{D}})baa^{\textcircled{D}} = 0$ and $(1-aa^{\textcircled{D}})f^s(1-aa^{\textcircled{D}}) = 0$;*

(2) *f is pseudo core invertible with $\mathrm{i}(f) \leqslant s$ and $f^{\textcircled{D}} = f_0$.*

Under the assumption of Theorem 4.3, the next result presents the conditions which guarantee f being a weak group element.

Proposition 4.9. *Let $s \in \mathbb{N}^+$ and $a \in R^{\textcircled{w}}$. Suppose that $1+a^{\textcircled{w}}b$ is invertible for some $b \in R$. Take $f = a+b$ and $f_0 = (1+a^{\textcircled{w}}b)^{-1}a^{\textcircled{w}}$. Then the following statements are equivalrnt:*

(1) *$(aa^{\textcircled{w}})^*b(1-aa^{\textcircled{w}}) = 0$, $(1-aa^{\textcircled{w}})baa^{\textcircled{w}} = 0$ and $f^s(1-aa^{\textcircled{w}}) = 0$;*

(2) *$(aa^{\textcircled{w}})^*b(1-aa^{\textcircled{w}}) = 0$, $faa^{\textcircled{w}} = aa^{\textcircled{w}}f$ and $(1-aa^{\textcircled{w}})f^s = 0$;*

(3) *f is a weak group element with $\mathrm{i}(f) \leqslant s$ and $f^{\textcircled{w}} = f_0$.*

Proof. (1)$\Rightarrow$(3): According to Lemma 4.1 and Theorem 4.3, f is weak group invertible with $\mathrm{i}(f) \leqslant s$ and $f^{\textcircled{w}} = (1+a^{\textcircled{w}}b)^{-1}a^{\textcircled{w}}$, $ff^{\textcircled{w}} = aa^{\textcircled{w}}$. This implies $f^s(1-ff^{\textcircled{w}}) = 0$. Thus, $f^{s+1}f^{\textcircled{w}} = f^s$. By [26, Lemma 2.7], we have $f^D = f^{\textcircled{w}}$.

(3)$\Rightarrow$(2): It follows, from Lemma 4.1 and Theorem 4.3, that $(aa^{\textcircled{w}})^*b(1-aa^{\textcircled{w}}) = 0$, $ff^{\textcircled{w}} = aa^{\textcircled{w}}$ and $(1-aa^{\textcircled{w}})f^s(1-aa^{\textcircled{w}}) = 0$. Since $f^{\textcircled{w}} = f^D$, we conclude that $faa^{\textcircled{w}} = f^2f^D = ff^Df = aa^{\textcircled{w}}f$ and $(1-aa^{\textcircled{w}})f^s = (1-ff^D)f^s = 0$.

(2)⇒(1): It is clear that $f^s(1-aa^{ⓦ}) = (1-aa^{ⓦ})f^s = 0$. Since $faa^{ⓦ} = aa^{ⓦ}f$, we have $baa^{ⓦ} = aa^{ⓦ}f - a^2a^{ⓦ}$. This implies $baa^{ⓦ} \in aa^{ⓦ}R$. Hence, $(1-aa^{ⓦ})baa^{ⓦ} = 0$. □

Let $A,\ B \in M_n(\mathbb{C})$ be invertible. The absorption law is known as $A^{-1}(A+B)B^{-1} = A^{-1} + B^{-1}$. This property is extended to cases of generalized inverses such as the group inverse, the Drazin inverse, and the pseudo core inverse. As an application of Theorem 4.7, we obtain the next result about the absorption law for m-weak group inverses.

Proposition 4.10. *Let $m \in \mathbb{N}$ and $a, c \in R^{ⓦ_m}$ with $\mathrm{i}(c) = s$. Take $b = c - a$. Then the following statements are equivalent:*

(1) $a^{ⓦ_m}(a+c)c^{ⓦ_m} = a^{ⓦ_m} + c^{ⓦ_m}$;
(2) $aa^{ⓦ_m} = cc^{ⓦ_m}$;
(3) $1 + a^{ⓦ_m}b$ *is invertible,* $(aa^{ⓦ_m})^*c^m(1-aa^{ⓦ_m}) = 0$, $(1-aa^{ⓦ_m})baa^{ⓦ_m} = 0$ *and* $(1-aa^{ⓦ_m})c^s(1-aa^{ⓦ_m}) = 0$;
(4) $1 + a^{ⓦ_m}b$ *is invertible,* $c^{ⓦ_m} = (1+a^{ⓦ_m}b)^{-1}a^{ⓦ_m}$.

Proof. (1)⇔(2): It is obvious by [13, Theorem 4.10].

(2)⇒(4): Since $aa^{ⓦ_m} = cc^{ⓦ_m}$, it follows, from Lemmas 2.4 and 2.5, that

$$\begin{aligned}
a^{ⓦ_m}bc^{ⓦ_m} &= a^{ⓦ_m}cc^{ⓦ_m} - a^{ⓦ_m}ac^{ⓦ_m} = a^{ⓦ_m}aa^{ⓦ_m} - a^{ⓦ_m}ac^{ⓦ_m} \\
&= a^{ⓦ_m} - a^{ⓦ_m}ac^{ⓦ_m} = a^{ⓦ_m} - a^{ⓦ_m}a(cc^{ⓦ_m})c^{ⓦ_m} \\
&= a^{ⓦ_m} - a^{ⓦ_m}a(aa^{ⓦ_m})c^{ⓦ_m} = a^{ⓦ_m} - a^2(a^{ⓦ_m})^2c^{ⓦ_m} \\
&= a^{ⓦ_m} - (aa^{ⓦ_m})c^{ⓦ_m} = a^{ⓦ_m} - (cc^{ⓦ_m})c^{ⓦ_m} \\
&= a^{ⓦ_m} - c^{ⓦ_m}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& (1+a^{ⓦ_m}b)(1-c^{ⓦ_m}b) \\
=\ & 1 + a^{ⓦ_m}b - c^{ⓦ_m}b - a^{ⓦ_m}bc^{ⓦ_m}b \\
=\ & 1 + a^{ⓦ_m}b - c^{ⓦ_m}b - (a^{ⓦ_m} - c^{ⓦ_m})b = 1.
\end{aligned}$$

Similarly, $(1-c^{ⓦ_m}b)(1+a^{ⓦ_m}b) = 1$, hence $1+a^{ⓦ_m}b$ is invertible and $(1+a^{ⓦ_m}b)^{-1} = 1 - c^{ⓦ_m}b$. Then, $(1+a^{ⓦ_m}b)^{-1}a^{ⓦ_m} = (1-c^{ⓦ_m}b)a^{ⓦ_m} = a^{ⓦ_m} - c^{ⓦ_m}ba^{ⓦ_m} = a^{ⓦ_m} - (a^{ⓦ_m} - c^{ⓦ_m}) = c^{ⓦ_m}$.

(4)⇒(2): It follows, from Lemma 4.1, that $cc^{ⓦ_m} = aa^{ⓦ_m}$.

(3)⇔(4): It is clear by Theorem 4.7. □

Acknowledgements

We would like to thank the editor and reviewers sincerely for their constructive comments and suggestions that have improved the quality of the paper.

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This work is supported by the National Natural Science Foundation of China (Grant Nos. 12171083, 12071070) and the Postgraduate Research & Practice Innovation Program of Jiangsu Province (Grant No. KYCX22_0231).

Data availability. No data was used for the research described in the article.

References

[1] O.M. Baksalary, G. Trenkler, *Core inverse of matrices*, Linear Multilinear A. 58(6), 681-697, 2010.

[2] N. Castro-González, J.J. Koliha, Y.M. Wei, *Perturbation of the Drazin inverse for matrices with equal eigenprojections at zero*, Linear Algebra Appl. 312, 181-189, 2000.

[3] J.L. Chen, X.F. Chen, D.G. Wang, *Additive properties for the pseudo core inverse of morphisms in an additive category*, J. Algebra Appl. 22(1), No. 2350030, 14 pp., 2023.

[4] M.P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly 65, 506-514, 1958.

[5] M.P. Drazin, *A class of outer generalized inverses*, Linear Algebra Appl. 436(7), 1909-1923, 2012.

[6] D.E. Ferreyra, F.E. Levis, N. Thome, *Characterizations of k-commutative equalities for some outer generalized inverses*, Linear Multilinear A. 68(1), 177-192, 2018.

[7] D.E. Ferreyra, F.E. Levis, A.N. Priori, N. Thome, *The weak core inverse*, Aequat. Math. 95(2), 351-373, 2021.

[8] D.E. Ferreyra, S.B. Malik, *A generalization of the group inverse*, Quaest. Math. 46(10), 2129-2145, 2023.

[9] D.E. Ferreyra, S.B. Malik, *The m-weak core inverse*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. 118(1), No.41, 17pp., 2024.

[10] D.E. Ferreyra, V. Orquera, N. Thome, *A weak group inverse for rectangular matrices*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM 113(4), 3727-3740, 2019.

[11] Y.F. Gao, J.L. Chen, *Pseudo core inverses in rings with involution*, Comm. Algebra 46(1), 38-50, 2018.

[12] R.E. Hartwig, *Block generalized inverses*, Arch. Rational Mech. Anal. 61, 197-251, 1976.

[13] W.D. Li, J.L. Chen, Y.K. Zhou, *Characterizations and properties of weak core inverses in rings with involution*, Rocky Mountain J. Math., 54(3), 793-807, 2024.

[14] S.B. Malik, N. Thome, *On a new generalized inverse for matrices of an arbitrary index*, Appl. Math. Comput. 226, 575-580, 2014.

[15] K. Manjunatha Prasad, K.S. Mohana, *Core-EP inverse*, Linear Multilinear A. 62, 792-802, 2014.

[16] D. Mosić, P.S. Stanimirović, L.A. Kazakovtsev, *The m-weak group inverse for rectangular matrices*, Electron. Res. Arch. 32(3), 1822-1843, 2024.

[17] D. Mosić, P.S. Stanimirović, L.A. Kazakovtsev, *Application of m-weak group inverse in solving optimization problems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. 118(1), No. 13, 21 pp., 2024.

[18] D. Mosić, D.C. Zhang, *New representations and properties of the m-weak group inverse*, Results Math. 78(3), No. 97, 19 pp., 2023.

[19] D. Mosić, D.C. Zhang, P.S. Stanimirović, *An extension of the MPD and MP weak group inverses*, Appl. Math. Comput. 465, No. 128429, 16 pp., 2024.

[20] D.S. Rakić, N.Č. Dinčić, D.S. Djordjević, *Group, Moore-Penrose, core and dual core inverse in rings with involution*, Linear Algebra Appl. 463, 115-133, 2014.

[21] H.X. Wang, J.L. Chen, *Weak group inverse*, Open Math. 16, 1218-1232, 2018.

[22] H. Yan, H.X. Wang, K.Z. Zuo, Y. Chen, *Further characterizations of the weak group inverse of matrices and the weak group matrix*, AIMS Math. 6(9), 9322-9341, 2021.

[23] M.M. Zhou, J.L. Chen, Y.K. Zhou, *Weak group inverses in proper $*$-rings*, J. Algebra Appl. 19(12), No. 2050238, 14 pp., 2020.

[24] Y.K. Zhou, *The research on the m-weak group inverse*, Master's thesis, Southeast University, China, 2021. (in Chinese)

[25] Y.K. Zhou, J.L. Chen, *Weak core inverses and pseudo core inverses in a ring with involution*, Linear Multilinear A. 70(21), 6876-6890, 2022.

[26] Y.K. Zhou, J.L. Chen, M.M. Zhou, *m-weak group inverses in a ring with involution*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM. 115(1), No.2, 14 pp., 2021.

[27] H.H. Zhu, *Further results on several types of generalized inverses*, Comm. Algebra 46(8), 3388-3396, 2018.

[28] H.H. Zhu, P. Patrício, *Characterizations for pseudo core inverses in a ring with involution,* Linear Multilinear A. 67(6), 1109-1120, 2019.