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Some Results of Nearly Cosymplectic Manifolds with Schouten-Van Kampen Connection

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Abstract

In this paper, we study conharmonic curvature tensor and concircular curvatur tensor of nearly cosymplectic manifolds with Schouten-Van Kampen connection, and we give a conharmonically flat and concircularly flat nearly cosymplectic manifold with generalized SVK connection.

Keywords: *Nearly cosymplectic manifolds, Schouten-Van Kampen connection.*
2010 Mathematics Subject Classification: *53C15, 53C25.*

1. Introduction

In more recent times, the geometry of cosymplectic manifolds has received increasing attention, and many researchers have studied the properties and applications of almost cosymplectic manifolds ([10, 19, 20]).
In addition, many mathematicians have begun to define nearly structures on various manifolds by mentioning the properties of new curvatures. One of the nearly structures is nearly cosymplectic manifolds, which we will study.
Cosymplectic manifold is an odd-dimensional analogue of Kähler manifold, as defined by Blair ([1]). Olszak's with Endo also studied the geometry of nearly cosymplectic manifolds ([3, 10]). An almost contact metric structure (ϕ,ξ,η,g) is nearly cosymplectic if $(\nabla_X\phi)X=0$ ([2]). Many authors have studied nearly cosymplectic manifolds later ([16, 17] and [18]).
The concept of concircular curvature C was introduced by J. A. Schouten in 1940, focusing on curvature properties preserved under specific transformations in Riemannian manifolds. If C vanishes, we say that the manifold is concircularly flat.
Also, the concept of conharmonic curvature K was introduced by T. Y. Thomas in 1938 as part of this study on conformal geometry and conharmonic transformations. It plays a significant role in differential geometry by analyzing how certain transformations affect the curvature tensor in Riemannian manifolds. If the conharmonic curvature tensor vanishes, we say that the manifold is conharmonically flat. Thus, this tensor represents the deviation of the manifold from conharmonic flatness ([1]).
Scouten-Van Kampen connection SVK is a mathematical tool developed by Jan Arnoldus Schouten and Egbert Van Kampen in the early 20th century ([14]). It combines Schouten's contributions to differential geometry and tensor analysis with Van Kampen's contributions to algebraic topology. The connection is used to study the behavior of differential forms and tensors on manifolds and is an important tool for understanding homotopy types of topological spaces. There are many studies on this connection; for example, Ghosh has studied the Schouten-Van Kampen connection associated with a Sasakian structure ([6]), Olszak has studied the Schouten-Van Kampen connection associated with an almost contact or paracontact metric structure ([5]), and many other studies ([7, 8, 9]).
This study is organized as follows: After presenting the properties of a nearly cosymplectic manifold with SVK connection, we introduced the conharmonic and concircular curvatures on this manifold. Furthermore, we investigated the conditions under which the manifold is flat in both cases.

2. Preliminaries

Let (M,ϕ,η,ξ,g) be a $(2n+1)-$dimensional differentiable manifold M is called an almost contact Riemannian manifold, where ϕ is a $(1,1)-$tensor field, ξ is the structure vector field, η is a $1-$form, and g is the Riemannian metric. This structure satisfies the following

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conditions:

$$\phi^2 X = -X + \eta(X)\xi \tag{2.1}$$

$$\eta(\xi) = 1, \phi\xi = 0, \eta \cdot \phi = 0 \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X,\xi) = \eta(X), \tag{2.4}$$

for any vector fields X and Y on M ([10]).
With condition (2,5) an almost contact Riemannian manifold M is said to be a nearly cosymplectic manifold

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0 \text{ and } (\nabla_X \phi)X = 0. \tag{2.5a}$$

It is said to be nearly cosymplectic manifold if the following conditions are satisfying:

$$\nabla_X \xi = HX, g(\nabla_X \xi, Y) + g(Y, \nabla_X \xi) = 0 \text{ and } (\nabla_X \phi)\xi = -\phi HX, \tag{2.6a}$$

where ∇ denotes the Levi-Civita connection ([10]), and H is a skew-symmetric tensor field.
In addition, on a nearly cosymplectic manifold M, the following relations hold:

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = g(HX,Y), \tag{2.7}$$

$$R(\xi,X,Y,Z) = -g((\nabla_X H)Y,Z) = \eta(Y)g(H^2X,Z) - \eta(Z)g(H^2X,Y), \tag{2.8}$$

$$\eta(R(Y,Z)X) = g((\nabla_X H)Y,Z), \tag{2.9}$$

$$S(X,\xi) = -\eta(X)tr(H^2), \tag{2.10}$$

$$S(X,Y) = g(QX,Y) \tag{2.11}$$

where R is stated as the Riemannian curvature tensor, S is shown as the Ricci tensor, Q is the Ricci operator, and H is a skew-symmetric tensor field ([10]).

3. Schouten-Van Kampen Connection On Nearly Cosymplectic Manifolds

([14])The Schouten-Van Kampen connection $\overset{svk}{\nabla}$ on almost contact metric manifold M is defined by

$$\overset{svk}{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{3.1}$$

for all vectors X and Y, where ∇ is the Levi-Civita connection on M.
If we use (2.6) and (2.7) in (3.1), we get,

$$\overset{svk}{\nabla}_X Y = \nabla_X Y - \eta(Y)HX + g(HX,Y)\xi, \tag{3.2}$$

for all vector fields X and Y.
By taking $Y = \xi$ in (3.2) and using (2.2) and (2.6), we obtain

$$\overset{svk}{\nabla}_X \xi = 0. \tag{3.3}$$

Thus we can state that the characteristic vector field of a nearly cosymplectic manifold is parallel to a Schouten-Van Kampen connection. Let M be a $2n+1-$dimensional nearly cosymplectic manifold. The curvature tensor R of M with respect to the connection ∇ is given by

$$\overset{svk}{R}(X,Y)Z = \overset{svk}{\nabla}_X \overset{svk}{\nabla}_Y Z - \overset{svk}{\nabla}_Y \overset{svk}{\nabla}_X Z - \overset{svk}{\nabla}_{[X,Y]} Z. \tag{3.4}$$

Then, after a long computation in a nearly cosymplectic manifold, we have

$$\overset{svk}{R}(X,Y)Z = \quad R(X,Y)Z - g(Z,HX)HY + g(Z,HY)HX - \eta(Z)\eta(X)H^2Y + \eta(Z)\eta(Y)H^2X - \eta(Y)g(H^2X,Z)\xi + \eta(X)g(H^2Y,Z)\xi. \tag{3.5}$$

By taking $Z = \xi$ in (3.5), we get

$$\overset{svk}{R}(X,Y)\xi = R(X,Y)\xi - \eta(X)H^2Y + \eta(Y)H^2X. \tag{3.6}$$

The Ricci tensor $\overset{svk}{S}$ and the scalar curvature $\overset{svk}{r}$ of a nearly cosymplectic manifold M with respect to the connection $\overset{svk}{\nabla}$ are given by

$$\overset{svk}{S}(Y,Z) = S(Y,Z) + g(Z,HY)tr(H) + \eta(Z)\eta(Y)tr(H^2), \tag{3.7}$$

$$\overset{svk}{r} = r + (2n+1)tr(H^2), \overset{svk}{r} = 0. \tag{3.8}$$

By taking $Z = \xi$ in (3.7), we get,

$$\overset{svk}{S}(Y,\xi) = S(Y,\xi) + \eta(Y)tr(H^2), \overset{svk}{S}(Y,\xi) = 0. \tag{3.9}$$

This gives

$$\overset{svk}{Q}Y = QY + tr(H^2)\xi, \overset{svk}{Q}Y = 0. \tag{3.10}$$

Theorem 3.1. *For a nerly cosymplectic manifold M with Schouten-Van Kampen connection* $\overset{svk}{\nabla}$ *,*

1. $\overset{svk}{R}(X,Y)Z = -\overset{svk}{R}(Y,X)Z$
2. $g(\overset{svk}{R}(X,Y)Z,W) = -g(\overset{svk}{R}(X,Y)W,Z)$
3. $g(\overset{svk}{R}(X,Y)Z,W) = g(\overset{svk}{R}(Z,W)X,Y)$
4. $\overset{svk}{R}(X,Y)Z + \overset{svk}{R}(Y,Z)X + \overset{svk}{R}(Z,X)Y = -2\,[g(Z,HZ)HY + g(Z,HY)HX + g(Y,HX)HZ]$
5. The Ricci tensor $\overset{svk}{S}$ is not symmetric.

Proof. The theorem is proven at length by switching vector fields, the inner product and metric properties, and also using (3.5) and (3.7). □

3.1. Conharmonic Curvature Tensor Of Nearly Cosymplectic Manifolds With $\overset{svk}{\nabla}$ Connection

In a nearly cosymplectic manifold M of dimension $n > 2$, the conharmonic curvature tensor K with respect to the Schouten-van Kampen connection $\overset{svk}{\nabla}$ is given by ([15])

$$\overset{svk}{K}(X,Y)Z = \overset{svk}{R}(X,Y)Z - \frac{1}{n-2}\left[\overset{svk}{S}(Y,Z)X - \overset{svk}{S}(X,Z)Y + g(Y,Z)\overset{svk}{Q}X - g(X,Z)\overset{svk}{Q}Y\right], \tag{3.11}$$

for all vector fields X, Y, and Z on M, where $\overset{svk}{R}$, $\overset{svk}{S}$, and $\overset{svk}{Q}$ are the Riemannian curvature tensor, Ricci tensor, and Ricci operator, respectively, with respect to the connection $\overset{svk}{\nabla}$.
Using (3.5), (3.7), (3.9), and (3.10) in (3.11), we get

$$\begin{aligned}\overset{svk}{K}(X,Y)Z = \;& R(X,Y)Z - g(Z,HX)HY + g(Z,HY)HX \\ & -\eta(Z)\eta(X)H^2Y + \eta(Z)\eta(Y)H^2X \\ & -\eta(Y)g(H^2X,Z)\xi + \eta(X)g(H^2Y,Z)\xi \\ & -\frac{1}{n-2}\quad [S(Y,Z)X + g(Z,HY)tr(H)X \\ & \qquad +\eta(Z)\eta(Y)tr(H^2)X - S(X,Z)Y \\ & \qquad -g(Z,HX)tr(H)Y - \eta(Z)\eta(X)tr(H^2)Y \\ & \qquad +g(Y,Z)QX + g(Y,Z)tr(H^2)X \\ & \qquad -g(X,Z)QY - g(X,Z)tr(H^2)Y],\end{aligned} \tag{3.12}$$

and thus,

$$\begin{aligned}\overset{svk}{K}(X,Y)Z = \;& K(X,Y)Z - g(Z,HX)HY + g(Z,HY)HX \\ & -\eta(Z)\eta(X)H^2Y + \eta(Z)\eta(Y)H^2X \\ & -\eta(Y)g(H^2X,Z)\xi + \eta(X)g(H^2Y,Z)\xi \\ & -\frac{1}{n-2}\quad [g(Z,HY)tr(H)X + \eta(Z)\eta(Y)tr(H^2)X \\ & \qquad -g(Z,HX)tr(H)Y - \eta(z)\eta(X)tr(H^2)Y \\ & \qquad +g(Y,Z)tr(H^2)X - g(X,Z)tr(H^2)Y].\end{aligned} \tag{3.13}$$

By taking $X = \xi$ and using (2.2), (3.7) and (3.9) in (3.11), we get

$$\begin{aligned}\overset{svk}{K}(\xi,Y)Z = -\frac{1}{n-2}\quad & [S(Y,Z)\xi + g(Z,HY)tr(H)\xi \\ & \eta(Z)\eta(Y)tr(H^2)\xi - \eta(Z)tr(H^2)Y \\ & -\eta(Z)QY],\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\eta\left(\overset{svk}{K}(X,Y)Z\right) = -\frac{1}{n-2}\quad & [S(Y,Z)\eta(X) + g(Z,HY)tr(H)\eta(X) \\ & -S(X,Z)\eta(Y) - g(Z,HX)tr(H)\eta(Y)].\end{aligned} \tag{3.15}$$

3.2. Conharmonically Flat Nearly Cosymplectic Manifolds With $\overset{svk}{\nabla}$ Connection

Assume that M is a conharmonically flat nearly cosymlectic manifold with respect to the connection $\overset{svk}{\nabla}$. That is, $\overset{svk}{K}=0$.
Then, from (3.11), we have

$$\overset{svk}{R}(X,Y)Z=\frac{1}{n-2}\left[\overset{svk}{S}(Y,Z)X-\overset{svk}{S}(X,Z)Y+g(Y,Z)\overset{svk}{Q}X-g(X,Z)\overset{svk}{Q}Y\right]. \tag{3.16}$$

Taking the inner product with ξ in (3.16), then

$$\begin{aligned} g(\overset{svk}{R}(X,Y)Z,\xi)=\tfrac{1}{n-2} \quad & [\overset{svk}{S}(Y,Z)\eta(X)-\overset{svk}{S}(X,Z)\eta(Y) \\ & +g(Y,Z)\overset{svk}{S}(X,\xi)-g(X,Z)\overset{svk}{S}(Y,\xi)]. \end{aligned} \tag{3.17}$$

This gives

$$\begin{aligned} \overset{svk}{R}(X,Y,Z,\xi)=\tfrac{1}{n-2} \quad & [\overset{svk}{S}(Y,Z)\eta(X)-\overset{svk}{S}(X,Z)\eta(Y) \\ & +g(Y,Z)\overset{svk}{S}(X,\xi)-g(X,Z)\overset{svk}{S}(Y,\xi)]. \end{aligned} \tag{3.18}$$

Using (3.7) in (3.18), we get

$$\begin{aligned} \overset{svk}{R}(X,Y,Z,\xi)=\tfrac{1}{n-2} \quad & [S(Y,Z)\eta(X)+g(Z,HY)tr(H)\eta(X) \\ & -S(X,Z)\eta(Y)-g(Z,HX)tr(H)\eta(Y)]. \end{aligned} \tag{3.19}$$

Using (2.8) and (3.6) in (3.19) and taking $X=\xi$, we have

$$S(Y,Z)=-g(Z,HY)tr(H)-\eta(Y)\eta(Z)tr(H^2), \tag{3.20}$$

and $r=-(2n+1)tr(H^2)$.
Thus,wecanstatethefollowing:

Theorem 3.2. *For a conharmonically flat nearly cosymplectic manifold with Schouten-Van Kampen connection, the scalar curvature is* $-(2n+1)tr(H^2)$.

3.3. Concircular Curvature Tensor Of Nearly Cosymplectic Manifolds With $\overset{svk}{\nabla}$ Connection

In a nearly cosymplectic manifold M of dimension $n>2$, the concircular curvature tensor C with respect to the Schouten-van Kampen ([15])

$$\overset{svk}{C}(X,Y)Z=\overset{svk}{R}(X,Y)Z-\frac{\overset{svk}{r}}{2n(2n+1)}\{g(Y,Z)X-g(X,Z)Y\}, \tag{3.21}$$

for all vector fields X, Y, and Z on M, where $\overset{svk}{R}$ and $\overset{svk}{r}$ are the Riemannian curvature tensor and scalar curvature, respectively, with respect to the connection $\overset{svk}{\nabla}$.
Using (3.5) and (3.8) in (3.22), we get

$$\begin{aligned} \overset{svk}{C}(X,Y)Z= \quad & R(X,Y)Z-g(Z,HX)HY+g(Z,HY)HX \\ & -\eta(Z)\eta(X)H^2Y+\eta(Z)\eta(Y)H^2X \\ & -\eta(Y)g(H^2X,Z)\xi+\eta(X)g(H^2Y,Z)\xi \\ & -\tfrac{r+(2n+1)tr(H^2)}{2n(2n+1)} \quad \{g(Y,Z)X-g(X,Z)Y\}, \end{aligned} \tag{3.22}$$

and thus

$$\begin{aligned} \overset{svk}{C}(X,Y)Z= \quad & C(X,Y)Z-g(Z,HX)HY+g(Z,HY)HX \\ & -\eta(Z)\eta(X)H^2Y+\eta(Z)\eta(Y)H^2X \\ & -\eta(Y)g(H^2X,Z)\xi+\eta(X)g(H^2Y,Z)\xi \\ & -\tfrac{tr(H^2)}{2n} \quad \{g(Y,Z)X-g(X,Z)Y\}. \end{aligned} \tag{3.23}$$

By taking $X=\xi$ and using (2.2), (2.10), and (2.11) in (3.24), we get

$$\begin{aligned} \overset{svk}{C}(\xi,Y)Z= \quad & C(\xi,Y)Z-R(\xi,Y)Z \\ & +\tfrac{1}{2n} \quad \{S(\xi,Z)Y-g(Y,Z)tr(H^2)\xi\}, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \eta(\overset{svk}{C}(X,Y)Z)= \quad & \eta(C(X,Y)Z)-\eta(R(X,Y)Z) \\ & +\tfrac{1}{2n} \quad \{S(X,\xi)g(Y,Z)-S(Y,\xi)g(X,Z)\}. \end{aligned} \tag{3.25}$$

3.4. Concircularly Flat Nearly Cosymplectic Manifolds With $\overset{svk}{\nabla}$ Connection

Assume that M is a concircularly flat nearly cosymplectic manifold with respect to the connection $\overset{svk}{\nabla}$, i.e., $\overset{svk}{C}=0$. Then from (3.22),

$$\overset{svk}{R}(X,Y)Z=\frac{\overset{svk}{r}}{2n(2n+1)}\left\{g(Y,Z)X-g(X,Z)Y\right\}. \tag{3.26}$$

Taking the inner product of the above equation with ξ, we have

$$g(\overset{svk}{R}(X,Y)Z,\xi)=\frac{\overset{svk}{r}}{2n(2n+1)}\left\{g(Y,Z)g(X,\xi)-g(X,Z)g(Y,\xi)\right\}. \tag{3.27}$$

Using (2.1), (2.9), and (3.6) in (3.28), we get

$$g((\nabla_Z H)X,Y)+\left[\eta(X)g(H^2Y,Z)-\eta(Y)g(H^2X,Z)\right]=\frac{\overset{svk}{r}}{2n(2n+1)}\left\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right\}. \tag{3.28}$$

Taking $X=\xi$ in (3.29) yields

$$g((\nabla_Z H)\xi,Y)+g(H^2Y,Z)=\frac{\overset{svk}{r}}{2n(2n+1)}\left\{g(Y,Z)-\eta(Z)\eta(Y)\right\}. \tag{3.29}$$

Using (2.6) in (3.30), we get

$$\frac{\overset{svk}{r}}{2n(2n+1)}\left\{g(Y,Z)-\eta(Z)\eta(Y)\right\}=0. \tag{3.30}$$

Replacing Y by QY in (3.31), we get

$$\frac{\overset{svk}{r}}{2n(2n+1)}\left\{g(QY,Z)-\eta(Z)\eta(QY)\right\}=0. \tag{3.31}$$

Using (2.10) and (3.8) in (3.32), we get

$$\frac{r+tr(H^2)(2n+1)}{2n(2n+1)}\left\{S(X,Z)+\eta(Z)\eta(Y)tr(H^2)\right\}. \tag{3.32}$$

This implies either $r=-tr(H^2)(2n+1)$ or $S(X,Z)=-tr(H^2)\eta(X)\eta(Z)$.

Theorem 3.3. *Either the scalar curvature is $-(2n+1)tr(H^2)$ or the manifold is a special type of $\eta-$Eintein manifold for a $\xi-$concircularly flat nearly cosymplectic manifold with Schouten-Van Kampen connection.*

Example 3.4. *Considering $M=\{(x,y,z)\in R^3\}$with $(x,y,z)\in R^3$ standard coordinates,*

$$F_1=e^{-z}\frac{\partial}{\partial y},F_2=e^{-z}\frac{\partial}{\partial x},F_3=\frac{\partial}{\partial z}$$

there are linear independent vector fields at every point of M. Riemannian metric g is defined as

$g(F_1,F_1)=g(F_2,F_2)=g(F_3,F_3)=1$
$g(F_1,F_2)=g(F_1,F_3)=g(F_2,F_3)=0$
In this case , the g metric is
$g=\frac{dx^2+dy^2+dz^2}{z^2}$
Let $1-$form η be defined as $\eta(Z)=g(Z,F_3)$ for $\forall Z\in\chi(M)$ and ϕ be a tensor field of type $(1,1)$ defined as $\phi(F_1)=F_2$, $\phi(F_2)=-F_1$, $\phi(F_3)=0$.

For $\forall X,Y\in\chi(M)$, using the linearity of ϕ and g,
$\eta(F_3)=1$, $\phi^2X=-X+\eta(X)F_3$, $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$.
So for $F_3=\xi$, the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M.
Let's ∇ be the Levi Civita connection with the metric g. In this case, we get
$[F_1,F_2]=0,[F_1,F_3]=F_1,[F_2,F_3]=F_2$.
In the Koszul formula, the ∇ Riemannian connection of the g metric is given as
$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y])$.
Using this formula, the following equations are obtained:
$\nabla_{F_1}F_1=0,\nabla_{F_1}F_2=0,\nabla_{F_1}F_3=HF_1$,
$\nabla_{F_2}F_1=0,\nabla_{F_2}F_2=0,\nabla_{F_2}F_3=HF_2$,
$\nabla_{F_3}F_1=0,\nabla_{F_3}F_2=0,\nabla_{F_3}F_3=0$.
In this case, with $F_3=\xi$, M provides the relation $\nabla_X\xi=HX$. In addition, the following equations are obtained:
$\overset{svk}{\nabla}_{F_1}F_1=g(HF_1,F_1)F_3,\overset{svk}{\nabla}_{F_1}F_2=0,\overset{svk}{\nabla}_{F_1}F_3=0$

$\overset{svk}{\nabla}_{F_2}F_1 = 0, \overset{svk}{\nabla}_{F_2}F_2 = g(HF_2,F_2)F_3, \overset{svk}{\nabla}_{F_2}F_3 = 0$
$\overset{svk}{\nabla}_{F_3}F_1 = 0, \overset{svk}{\nabla}_{F_3}F_2 = 0, \overset{svk}{\nabla}_{F_3}F_3 = 0$

Again, in this case, M provides the relation $\overset{svk}{\nabla}_X \xi = 0$, with $F_3 = \xi$. So the following equations are reached:
$\overset{svk}{R}(F_1,F_1)F_1 = \overset{svk}{R}(F_2,F_2)F_2 = \overset{svk}{R}(F_3,F_3)F_3 = 0,$
$\overset{svk}{R}(F_1,F_1)F_2 = \overset{svk}{R}(F_1,F_1)F_3 = \overset{svk}{R}(F_1,F_2)F_3 = 0,$
$\overset{svk}{R}(F_1,F_3)F_2 = \overset{svk}{R}(F_2,F_1)F_3 = 0,$
$\overset{svk}{R}(F_2,F_2)F_1 = \overset{svk}{R}(F_2,F_2)F_3 = \overset{svk}{R}(F_2,F_3)F_1 = 0,$
$\overset{svk}{R}(F_3,F_1)F_2 = \overset{svk}{R}(F_3,F_2)F_1 = \overset{svk}{R}(F_3,F_3)F_1 = 0,$
$\overset{svk}{R}(F_3,F_3)F_2 = \overset{svk}{R}(F_3,F_2)F_3 = \overset{svk}{R}(F_3,F_1)F_3 = 0,$
$\overset{svk}{R}(F_1,F_3)F_3 = \overset{svk}{R}(F_2,F_3)F_3 = 0,$
$\overset{svk}{R}(F_1,F_2)F_1 = HF_2\ , \overset{svk}{R}(F_2,F_1)F_1 = -HF_2,$
$\overset{svk}{R}(F_2,F_1)F_2 = HF_1\ , \overset{svk}{R}(F_1,F_2)F_2 = -HF_1,$
$\overset{svk}{R}(F_1,F_3)F_1 = -g(HF_1,F_1)F_3,$
$\overset{svk}{R}(F_2,F_3)F_2 = -g(HF_2,F_2)F_3,$
$\overset{svk}{R}(F_3,F_2)F_2 = g(HF_2,F_2)F_3,$
$\overset{svk}{R}(F_3,F_1)F_1 = g(HF_1,F_1)F_3.$

4. Conclusion

In this study, we have examined the fundamental curvature properties of nearly cosymplectic manifolds equipped with the Schouten-Van Kampen connection. Based on these properties, we introduced the conharmonic and concircular curvature tensors and investigated their role in characterizing flatness. Additionally, further analysis can be conducted on certain symmetry conditions, as well as $\phi-$concircular flatness and pseudo-concircular flatness. Finally, we provided an illustrative example to support our findings.

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