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On a Hardy-Type Integral Inequality under Convexity and Submultiplicativity Assumptions

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Abstract

The focus of this article is a Hardy-type integral inequality published in 2012, which has the particularity of combining convexity and submultiplicativity assumptions. A revision is carried out in several steps. First, a counterexample to this result is given, questioning its validity. An alternative statement is then proposed, together with a detailed proof. Among the facts revealed is the relaxation of the assumption on a key parameter of the integral norm. Finally, two examples of the revised result are presented, showing new modifications of the Hardy integral inequality. Their originality takes advantage of the combined convexity and submultiplicativity assumptions, which remain an underexplored framework.

Keywords: *Convexity; Counter-example; Hardy-type integral inequality; Integral norms; Submultiplicativity.*
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1. Introduction

Integral inequalities involving the norms of functions are fundamental to mathematics. In particular, they are useful tools in functional analysis, probability theory, and differential equations. Their main aim is to give tractable upper and/or lower bounds on sophisticated integrals, possibly defined with complex functions. Basic applications include solving optimization problems, analyzing the convergence of integrals, bounding various moment measures defined with the expectation operator, and proving the existence of solutions to certain equations. Important examples of such inequalities are the Cauchy-Schwarz, Hölder, Minkowski, Hilbert, and Hardy integral inequalities. This topic is central to the following books: [16, 6, 28, 5, 30]. To underline the continuing relevance of the subject, recent references from 2024-2025 include [11, 17, 12, 13, 8, 10].

For the purposes of this article, we will focus on a specific modification of the Hardy integral inequality published in 2012 in [26]. To introduce this modification, it is necessary to recall the classical Hardy integral inequality. First, it was established in [15] and can be stated as follows: For any $p \in (1,+\infty)$ and $f : [0,+\infty) \mapsto [0,+\infty)$, i.e., a positive function defined on $[0,+\infty)$ denoted f, by considering the primitive of f given by

$$F(x) = \int_0^x f(t)dt,$$

with $x \in [0,+\infty)$, the following holds:

$$\int_0^{+\infty} \frac{F^p(x)}{x^p} dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1.1}$$

provided that the integrals exist. The factor constant $[p/(p-1)]^p$ is optimal in the sense that if a smaller constant was used, there are functions f for which the inequality no longer holds. The Hardy integral inequality has attracted much attention, especially in functional analysis and operator theory. It has been adapted to various mathematical frameworks, leading to numerous extensions and modifications. Examples include the weighted, fractional, dynamic, and multidimensional Hardy-type integral inequalities. The most notable of these can be found in [18, 26, 25, 29, 7, 31, 23, 24, 27, 22, 19, 2, 20, 4, 1, 3, 9].

This article provides a revision of a particular Hardy-type integral inequality presented in [26], which is referred to as [26, Theorem 2.5]. This theorem has two important features. First, it modifies the functional structure of the Hardy integral inequality by using an intermediate function that leads to an innovative upper bound. Second, it combines certain convexity and submultiplicativity assumptions in its proof. To the best of our knowledge, it is one of the few integral inequality results obtained under these combined assumptions. For the sake of clarity, we formulate [26, Theorem 2.5] below, with minor ”stylistic changes” in the notation and an auxiliary note on some key assumptions.

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Theorem 1.1. *[26, Theorem 2.5] Let $p \in (1,+\infty)$, $a \in (0,+\infty)$, $f : [0,a] \mapsto [0,+\infty)$ and $\phi : [0,+\infty) \mapsto [0,+\infty)$. We assume that ϕ is twice differentiable, convex and submultiplicative, with $\phi(0) = 0$.*
Auxiliary note on some of these assumptions:

- *The convexity assumption on ϕ means that, for any $\lambda \in [0,1]$ and $x,y \in [0,+\infty)$, the following holds:*

$$\phi[\lambda x + (1-\lambda)y] \leq \lambda\phi(x) + (1-\lambda)\phi(y),$$

which is equivalent to $\phi''(x) \geq 0$ because ϕ is also assumed to be twice differentiable.
- *The submultiplicativity assumption on ϕ means that, for any $x,y \in [0,+\infty)$, the following holds:*

$$\phi(xy) \leq \phi(x)\phi(y).$$

We consider the primitive of f given by

$$F(x) = \int_0^x f(t)dt, \tag{1.2}$$

with $x \in [0,a]$. Then the following holds:

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx \leq \frac{1}{p-1}\int_0^a x^{2-p}\frac{\phi[f(x)]}{\phi(x)}dx,$$

provided that the integrals exist.

The proof in [26, Page 522] is built on the following elements, in order: the use of the submultiplicativity assumption on ϕ, the Jensen integral inequality based on the convexity assumption on ϕ, a change of the order of integration, an auxiliary lemma on the non-increasing property of $x/\phi(x)$, i.e., [26, Lemma 2.4], and an integral calculus on the power function.
Despite the originality of the approach, a thorough investigation shows that Theorem 1.1 and the corresponding proof contain some gaps. This article proposes to fill them. First, we give an understandable counterexample to the integral inequality stated in the theorem. Then we revise it by proposing a new statement and giving a detailed proof. This revised work is important because Theorem 1.1 can legitimately be considered solid, thanks to its decades-long existence, the reputation of the corresponding journal, i.e., *Applied Mathematics Letters*, and the deserved high impact of the numerous advances in [26], which undeniably contains several key results in the field of integral inequalities. For these reasons, it can be mistakenly used as a benchmark for broader perspectives. Moreover, research on integral inequalities under submultiplicativity assumptions has recently regained interest in modern mathematical scenarios. The reference [21], published in 2023, supports this claim. As an unexpected fact of our revised theorem, the condition on p is relaxed; it is proved to be valid for $p \in (0,+\infty)$. The study is completed with some new examples of integral inequalities based on a particular function ϕ, which open new perspectives for applications.
The rest of the article is organized as follows: Section 2 is devoted to a counterexample to Theorem 1.1. A new statement and proof of this result are given in Section 3. Based on this progress, several examples are presented in Section 4. Some final remarks are proposed in Section 5.

2. Counterexample to Theorem 1.1

The counterexample to Theorem 1.1 mentioned above is highlighted in the theorem below.

Theorem 2.1. *Let $a \in (0,+\infty)$, $\lambda \in (0,1)$ and $f : (0,a] \mapsto (0,+\infty)$ defined by*

$$f(x) = \frac{\lambda}{x^2}e^{-\lambda/x},$$

with $x \in [0,a]$. By considering the corresponding primitive F as given in Equation (1.2), the following holds:

$$\int_0^a \frac{F(x)}{x^3}dx > \int_0^a \frac{f(x)}{x}dx.$$

This contradicts the strict application of Theorem 1.1 with $p = 2$ and $\phi(x) = x$, which is obviously twice differentiable, convex and submultiplicative, with $\phi(0) = 0$.

Proof. Surprisingly, the proof does not require the calculus of integrals, which explains our flexibility in choosing the upper integration bound a. For any $x \in (0,a]$, we have

$$F(x) = \int_0^x f(t)dt = \int_0^x \frac{\lambda}{t^2}e^{-\lambda/t}dt = \left[e^{-\lambda/t}\right]_{t\to 0}^{t=x} = e^{-\lambda/x} - 0 = e^{-\lambda/x}.$$

We therefore have

$$\int_0^a \frac{F(x)}{x^3}dx = \int_0^a \frac{e^{-\lambda/x}}{x^3}dx,$$

which is a convergent integral. Let us denote it by A. On the other hand, by the sole definition of f, we have

$$\int_0^a \frac{f(x)}{x}dx = \int_0^a \frac{1}{x} \times \frac{\lambda}{x^2}e^{-\lambda/x}dx = \lambda\int_0^a \frac{e^{-\lambda/x}}{x^3}dx = \lambda A.$$

Since λ is chosen such that $\lambda \in (0,1)$, we clearly have

$$\int_0^a \frac{F(x)}{x^3}dx = A > \lambda A = \int_0^a \frac{f(x)}{x}dx.$$

We complete this development by noticing that, taking $p=2$ and $\phi(x)=x$ in the framework of Theorem 1.1, we have

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx = \int_0^a \frac{F(x)}{x^3}dx$$

and

$$\frac{1}{p-1}\int_0^a x^{2-p}\frac{\phi[f(x)]}{\phi(x)}dx = \int_0^a \frac{f(x)}{x}dx.$$

We have thus established a counterexample to Theorem 1.1. This concludes the proof of Theorem 2.1. □

After further investigation, for the same example of the function f, a wide range of values for p, and power functions for ϕ with a wide range of exponent values, i.e., $\phi(x)=x^\alpha$ with $\alpha \in (1,+\infty)$, a numerical study shows that we can often find values for λ that contradict the strict application of Theorem 1.1.
The gap is thus deeper than the construction of a "very special artificial counterexample", as one might think in the statement of Theorem 2.1. This motivates a revision work, as developed in the next section.

3. Revision of the theorem

A revised statement of Theorem 1.1 is given below, with all details to ensure completeness.

Theorem 3.1. *Let $p \in (0,+\infty)$, $a \in (0,+\infty)$, $f:[0,a] \mapsto [0,+\infty)$ and $\phi:[0,+\infty) \mapsto [0,+\infty)$. We assume that ϕ is twice differentiable, convex and submultiplicative, with $\phi(0)=0$. We consider the primitive of f given by*

$$F(x) = \int_0^x f(t)dt,$$

with $x \in [0,a]$. Then the following holds:

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx \le \frac{1}{p}\int_0^a x^{1-p}\frac{\phi\left[f(x)\right]}{\phi(x)}dx.$$

Proof. The detailed proof of the theorem is given below, with each step explained for clarity, in the spirit of revising an existing result. We have

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx = \int_0^a x^{1-p}\frac{1}{\phi^2(x)}\phi\left[x\frac{F(x)}{x}\right]dx.$$

Using the submultiplicativity assumption on ϕ, we obtain

$$\int_0^a x^{1-p}\frac{1}{\phi^2(x)}\phi\left[x\frac{F(x)}{x}\right]dx \le \int_0^a x^{1-p}\frac{1}{\phi^2(x)}\phi(x)\phi\left[\frac{F(x)}{x}\right]dx = \int_0^a x^{1-p}\frac{1}{\phi(x)}\phi\left[\frac{1}{x}\int_0^x f(t)dt\right]dx.$$

The convexity assumption on ϕ allows us to apply the Jensen integral inequality with the probability measure $\mu(\mathscr{S}) = \int_{\mathscr{S}}(1/x)dt$, with $\mathscr{S} \subseteq [0,x]$. This gives

$$\int_0^a x^{1-p}\frac{1}{\phi(x)}\phi\left[\frac{1}{x}\int_0^x f(t)dt\right]dx \le \int_0^a x^{1-p}\frac{1}{\phi(x)}\left\{\frac{1}{x}\int_0^x \phi\left[f(t)\right]dt\right\}dx = \int_0^a\int_0^x x^{-p}\frac{1}{\phi(x)}\phi\left[f(t)\right]dtdx.$$

Since the main integrated term is positive, we can use the Fubini-Tonelli integral theorem to change the order of integration. We thus obtain

$$\int_0^a\int_0^x x^{-p}\frac{1}{\phi(x)}\phi\left[f(t)\right]dtdx = \int_0^a\int_t^a x^{-p}\frac{1}{\phi(x)}\phi\left[f(t)\right]dxdt = \int_0^a \phi\left[f(t)\right]\left[\int_t^a x^{-p-1}\frac{x}{\phi(x)}dx\right]dt.$$

Based on [26, Lemma 2.4], since ϕ is positive, twice differentiable, convex and satisfies $\phi(0)=0$, $x/\phi(x)$ is non-increasing. Note that we do not need to use the submultiplicativity assumption on ϕ for this result, contrary to what is stated in [26, Lemma 2.4], but we need the twice differentiability assumption since ϕ'' is involved. So we have $\sup_{x\in[t,a]} x/\phi(x) = t/\phi(t)$, which implies that

$$\int_0^a \phi\left[f(t)\right]\left[\int_t^a x^{-p-1}\frac{x}{\phi(x)}dx\right]dt \le \int_0^a \phi\left[f(t)\right]\frac{t}{\phi(t)}\left[\int_t^a x^{-p-1}dx\right]dt.$$

Performing simple integral calculus on the power function, taking into account that $p \in (0,+\infty)$, we get

$$\int_0^a \phi\left[f(t)\right]\frac{t}{\phi(t)}\left[\int_t^a x^{-p-1}dx\right]dt = \int_0^a \phi\left[f(t)\right]\frac{t}{\phi(t)}\left[\frac{1}{p}(t^{-p}-a^{-p})\right]dt = \frac{1}{p}\int_0^a t^{1-p}\frac{\phi\left[f(t)\right]}{\phi(t)}dt - \frac{a^{-p}}{p}\int_0^a t\frac{\phi\left[f(t)\right]}{\phi(t)}dt.$$

Since the second main term of the previous expression is negative (due to the minus in factor of a positive integral term), the following holds:

$$\frac{1}{p}\int_0^a t^{1-p}\frac{\phi\left[f(t)\right]}{\phi(t)}dt - \frac{a^{-p}}{p}\int_0^a t\frac{\phi\left[f(t)\right]}{\phi(t)}dt \le \frac{1}{p}\int_0^a t^{1-p}\frac{\phi\left[f(t)\right]}{\phi(t)}dt.$$

If we combine the above inequalities in the order in which they are treated, we obtain the integral inequality we want. This completes the proof of Theorem 3.1. □

If we compare the proof of Theorem 3.1 with that of Theorem 1.1 available in [26, Page 522], there is an extra "x" in line 5, which is mistakenly taken into account in the rest of the development, giving an incorrect result. We see two main changes in the upper bound: the constant $1/(p-1)$ becomes $1/p$ and the power function x^{2-p} has its exponent revised as x^{1-p}.
As an analytical test, let us consider the example of the function proposed in Theorem 2.1, with the same configuration, i.e., $p=2$ and $\phi(x)=x$. After some integral calculations, omitting the details for the sake of brevity, we obtain

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx=\int_0^a \frac{F(x)}{x^3}dx=\int_0^a \frac{e^{-\lambda/x}}{x^3}dx=\frac{e^{-\lambda/a}(a+\lambda)}{a\lambda^2}$$

and

$$\frac{1}{p}\int_0^a x^{1-p}\frac{\phi\left[f(x)\right]}{\phi(x)}dx=\frac{1}{2}\int_0^a \frac{f(x)}{x^2}dx=\frac{\lambda}{2}\int_0^a \frac{e^{-\lambda/x}}{x^4}dx=\frac{e^{-\lambda/a}(2a^2+2a\lambda+\lambda^2)}{2a^2\lambda^2},$$

so that

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx-\frac{1}{p}\int_0^a x^{1-p}\frac{\phi\left[f(x)\right]}{\phi(x)}dx=\frac{e^{-\lambda/a}(a+\lambda)}{a\lambda^2}-\frac{e^{-\lambda/a}(2a^2+2a\lambda+\lambda^2)}{2a^2\lambda^2}=-\frac{e^{-\lambda/a}}{2a^2}<0.$$

This is the expected result, according to that of Theorem 3.1. So, the example that was a counterexample to Theorem 1.1 is no longer a counterexample to Theorem 3.1.
As a notable remark, the inequality in Theorem 1.1 holds for $p\in(0,+\infty)$, not just for $p\in(1,+\infty)$. This is of some interest, since the case of $p\in(0,1)$ is often not considered in the context of Hardy-type integral inequalities.
If we analyze the proof of Theorem 3.1, in the penultimate step of the development, we can deduce the following:

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx\leq\frac{1}{p}\int_0^a x^{1-p}\frac{\phi\left[f(x)\right]}{\phi(x)}dx-\frac{a^{-p}}{p}\int_0^a x\frac{\phi\left[f(x)\right]}{\phi(x)}dx,$$

which can be seen as a refined form of the inequality presented in Theorem 3.1. We have omitted this refinement in the statement of Theorem 3.1 because its purpose was to have a point of comparison with that of Theorem 1.1. However, this result may be of interest for further work in this direction.
We also mention that the case $a\to+\infty$ can be included without mathematical effort. The given inequality becomes

$$\int_0^{+\infty} x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx\leq\frac{1}{p}\int_0^{+\infty} x^{1-p}\frac{\phi\left[f(x)\right]}{\phi(x)}dx.$$

In particular, if we take $\phi(x)=x$, which is obviously twice differentiable, convex and submultiplicative with $\phi(0)=0$, we get

$$\int_0^{+\infty}\frac{F(x)}{x^{p+1}}dx\leq\frac{1}{p}\int_0^{+\infty}\frac{f(x)}{x^p}dx.$$

This can be seen as a simple variant of the Hardy integral inequality, where a parameter p only affects the weight of the integral norm and the factor constant.

4. Some special examples

In this section, we give two new examples, all derived from the application of Theorem 3.1. They use concrete convex and submultiplicative functions for ϕ.

Example 1: For any $\alpha\in[1,+\infty)$, $\beta\in[1,+\infty)$ and $\gamma\in[1,+\infty)$, the function $\phi(x)=x^\alpha(\gamma+x)^\beta$ is twice differentiable, convex because of the product of two basic convex non-decreasing functions, i.e., x^α and $(\gamma+x)^\beta$, and submultiplicative because of the product of two submultiplicative functions, i.e., x^α and $(\gamma+x)^\beta$. Indeed, $\psi(x)=(\gamma+x)^\beta$ is submultiplicative because we have

$$\psi(xy)=(\gamma+xy)^\beta\leq(\gamma^2+xy)^\beta\leq(\gamma^2+\gamma x+\gamma y+xy)^\beta=[(\gamma+x)(\gamma+y)]^\beta=(\gamma+x)^\beta(\gamma+y)^\beta=\psi(x)\psi(y).$$

Finally, we note that $\phi(0)=0^\alpha(\gamma+0)^\beta=0$.
As a direct application of Theorem 3.1, adopting the same notation, for any $a\in(0,+\infty)$ and $p\in(0,+\infty)$, we have

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx\leq\frac{1}{p}\int_0^a x^{1-p}\frac{\phi\left[f(x)\right]}{\phi(x)}dx,$$

that is

$$\int_0^a x^{1-p}\frac{F^\alpha(x)\left[\gamma+F(x)\right]^\beta}{x^{2\alpha}(\gamma+x)^{2\beta}}dx\leq\frac{1}{p}\int_0^a x^{1-p}\frac{f^\alpha(x)\left[\gamma+f(x)\right]^\beta}{x^\alpha(\gamma+x)^\beta}dx$$

or, equivalently,

$$\int_0^a \frac{F^\alpha(x)\left[\gamma+F(x)\right]^\beta}{x^{2\alpha+p-1}(\gamma+x)^{2\beta}}dx\leq\frac{1}{p}\int_0^a \frac{f^\alpha(x)\left[\gamma+f(x)\right]^\beta}{x^{\alpha+p-1}(\gamma+x)^\beta}dx.$$

To the best of our knowledge, it is a new Hardy-type integral inequality.

Example 2: For any $\alpha \in [1,+\infty)$ and $\beta \in [e,+\infty)$, the function $\phi(x) = x^{\alpha}\log(\beta+x)$ is twice differentiable, convex because we have

$$\phi''(x) = x^{\alpha-1}\frac{2\alpha(\beta+x)-x}{(\beta+x)^2} + \alpha(\alpha-1)x^{\alpha-2}\log(\beta+x) \geq x^{\alpha-1}\frac{2\alpha\beta+(2\alpha-1)x}{(\beta+x)^2} \geq x^{\alpha-1}\frac{2\alpha\beta+x}{(\beta+x)^2} \geq 0,$$

and submultiplicative because of the product of two submultiplicative functions, i.e., x^{α} and $\log(\beta+x)$, as demonstrated in [14, Theorem 1] for this logarithmic function. Finally, we note that $\phi(0) = 0^{\alpha}\log(\beta+0) = 0$.
As a direct application of Theorem 3.1, adopting the same notation, for any $a \in (0,+\infty)$ and $p \in (0,+\infty)$, we have

$$\int_0^a x^{1-p}\frac{\phi[F(x)]}{\phi^2(x)}dx \leq \frac{1}{p}\int_0^a x^{1-p}\frac{\phi[f(x)]}{\phi(x)}dx,$$

that is

$$\int_0^a x^{1-p}\frac{F^{\alpha}(x)\log[\beta+F(x)]}{x^{2\alpha}[\log(\beta+x)]^2}dx \leq \frac{1}{p}\int_0^a x^{1-p}\frac{f^{\alpha}(x)\log[\beta+f(x)]}{x^{\alpha}\log(\beta+x)}dx$$

or, equivalently,

$$\int_0^a \frac{F^{\alpha}(x)\log[\beta+F(x)]}{x^{2\alpha+p-1}[\log(\beta+x)]^2}dx \leq \frac{1}{p}\int_0^a \frac{f^{\alpha}(x)\log[\beta+f(x)]}{x^{\alpha+p-1}\log(\beta+x)}dx.$$

Again, to the best of our knowledge, it is a new Hardy-type integral inequality.

5. Conclusion

In this article, we have revised an existing result on a special Hardy-type integral inequality of the literature, which has the property of combining convexity and submultiplicativity assumptions. This revision was done in four steps: the determination of a counterexample, the correction of the statement in the form of a theorem, the corresponding detailed proof, and the discussion of the new result with comparison to the known result. In addition, two examples of new integral inequalities are given, both of the Hardy type. The perspectives of this article include the extensions of the obtained inequalities to multiple dimensions, their applications in functional analysis, and the deeper study of the notion of submultiplicativity in other settings of integral inequalities. The practical aspect of our theory also needs to be examined, particularly in the context of its applications to differential equations, operator theory, and mathematical physics. We leave these aspects for future research.

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