Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 29.04.2025
DOI : 10.15672/hujms.1605434

Research Article

Well-posedness of a class generalized split quasi-inverse tensor variational inequalities

Qing Nie[1], Vo Minh Tam[2], Boling Chen*[3]

[1] *College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, Peoples Republic of China*
[2] *Department of Mathematics, Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam*
[3] *Center for Applied Mathematics of Guangxi, Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin, 537000, Guangxi, Peoples Republic of China*

Abstract

This paper aims to study a generalized split quasi-inverse tensor variational inequality (GSQITVI) in tensor spaces. Building on the concept of well-posedness, we establish several metric-based features that provide necessary and sufficient conditions for the well-posedness of the GSQITVI. By utilizing the measure of non-compactness and the correlation theorem, we also derive results concerning the well-posedness of the problem. These findings emphasize the key properties of the GSQITVI and offer an analysis of the convergence of its solutions.

Mathematics Subject Classification (2020). 47H04, 49J40, 90C30

Keywords. Generalized split quasi-inverse tensor variational inequality, well-posedness, measure of non-compactness, convergence analysis

1. Introduction

Let $\mathbb{C}^{[V_i,h_i]}$ be the set of all tensors of orders V_i and dimensions h_i, for all $i = 1, 2$. Consider two set-valued mappings, $J : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_1,h_1]}$ and $M : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_2,h_2]}$, with values that are nonempty, closed, and convex. Let $j : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_1,h_1]}$ and $m : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_2,h_2]}$ be tensor mappings, and $h : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$, $\theta : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$, and $\tau : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ be given functions. Using these assumptions, in this paper, we study a generalized split quasi-inverse tensor variational inequality (GSQITVI) as follows:

*Corresponding Author.
Email addresses: 1925948806@qq.com (Q. Nie), vmtam@dthu.edu.vn (V.M. Tam), bolinchenylnu@163.com (B. Chen)
Received: 22.12.2024; Accepted: 23.02.2025

Problem 1.1. Find $A^* \in \mathbb{C}^{[V_1,h_1]}$ and $B^* \in \mathbb{C}^{[V_2,h_2]}$ such that

$$\begin{cases} j\left(A^*,B^*\right) \in J\left(A^*,B^*\right), \\ m\left(A^*,B^*\right) \in M\left(A^*,B^*\right), \\ h\left(A^*,B^*\right) \leq 0, \\ \langle j'-j\left(A^*,B^*\right),A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A^*,B^*\right)\right) \leq 0, \forall j' \in J\left(A^*,B^*\right), \\ \langle m'-m\left(A^*,B^*\right),B^*\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A^*,B^*\right)\right) \leq 0, \forall m' \in M\left(A^*,B^*\right). \end{cases}$$

In particular, if J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$. Then, the GSQITVI become to the following generalized split inverse tensor variational inequality problem (for brevity, GSITVI):

Problem 1.2. Find $A^* \in J$, $B^* \in M$, such that

$$\begin{cases} j\left(A^*,B^*\right) \in J, \\ m\left(A^*,B^*\right) \in M, \\ h\left(A^*,B^*\right) \leq 0, \\ \langle j'-j\left(A^*,B^*\right),A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A^*,B^*\right)\right) \leq 0, \forall j' \in J, \\ \langle m'-m\left(A^*,B^*\right),B^*\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A^*,B^*\right)\right) \leq 0, \forall m' \in M. \end{cases}$$

The study of variational inequality theory in finite-dimensional spaces began in 1980. Since then, the theory has experienced rapid development. For an overview of recent advancements in variational inequality problems and methods, as well as related applications, readers are referred to [12]. Notably, Noor [16] introduced an iterative algorithm and a projection technique for obtaining approximate solutions to general variational inequalities. Let H be a real Hilbert space, and let g and T be two continuous mappings where $g : H \to H$ and $T : H \to H$. Also, let K be a nonempty, closed, and convex subset of H. The problem is to find $u \in K$ such that

$$\langle g(u) - g(v), T(u)\rangle \leq 0, \quad \forall g(u), g(v) \in K, \tag{1.1}$$

which is referred to as the general nonlinear variational inequality problem. If $g(u) = u$ and $g(v) = v$, the classical variational inequality problem (for brevity, VIP) is defined as finding $u \in K$ such that

$$\langle u - v, T(u)\rangle \leq 0, \quad \forall v \in K. \tag{1.2}$$

Noor [17, 18] introduced the basic concepts of variational inequalities and utilizes various methods to present the main results related to generalized variational inequalities.

At the same time, variational inequalities have found diverse applications, and their rapid development has led to increasingly comprehensive research into various aspects. While we know that in variational inequalities the constraint set is independent of the variable, in quasi-variational inequalities the constraint set depends on the variable. Therefore, quasi-variational inequalities be said generalize variational inequalities. Let $T : H \to H$ be a mapping and H a real Hilbert space. Let C be a subset of H and $K : C \rightrightarrows C$ a set-valued mapping. The quasi-VI problem is then to find $u \in K(u)$ such that

$$\langle u - v, T(u)\rangle \leq 0, \quad \forall v \in K(u). \tag{1.3}$$

In real-life scenarios, most optimization problems are typically constrained by multiple variables. Kanzow-Steck [11] analyzed such optimization problems by focusing on two key properties: the pseudo-monotonicity of the variational operator and the Moscow-type continuity of the feasible-set mapping. Their paper proposes that these assumptions can be used to establish the existence of a solution and its computability through appropriate approximation techniques.

The concepts of well-posedness to split variational inequalities, inverse variational inequalities and tensor variational inequality are significant and intriguing subjects in the exploration of variational problems. In [14, 15], Lignola-Morgan introduced specific notions regarding the well-posedness of quasi-VI with unique solutions, providing equivalent characterizations of these concepts and discussing the well-posedness of quasi-VI. Censor-Gibali-Reich [5] developed an iterative algorithm to study split variational inequalities under suitable conditions, which includes a prototype split inverse problem and a newly introduced variational problem. He-Ling-Xu [6] explored split-variable inequalities within the framework of product spaces. By leveraging the split structure of these inequalities, they proposed a projection method that is straightforward to implement and ensures global convergence. We can also find the study on generalized split variational inequalities in [9, 10]. Hu-Fang [7] broadened the concept of well-posedness to include split inverse variational inequality problems, establishing well-posedness characterizations in the style of Furi-Vignoli. Crespi-Guerraggio-Rocca [4] introduced well-posedness concepts for vector optimization problems and differential vector variational inequalities, further examining the well-posedness of vector optimization problems within the framework of c-quasiconvexity. Wang-Huang-Qi [20] investigated the global uniqueness and solvability of tensor variational inequalities under appropriate assumptions. Specifically, a tensor variational inequality is defined by the following problem: Let $\mathbb{C}$ be a tensor space and H a subset of $\mathbb{C}^{[V,h]}$, with $J : H \longrightarrow \mathbb{C}^{[V,h]}$. The goal is to find $A \in H$ such that

$$\langle A - B, J(A)\rangle \leq 0, \quad \forall B \in H. \tag{1.4}$$

Additionally, Barbagallo-Mauro [2] investigated the behavior of control policies in oligopolistic market equilibrium problems through the use of inverse tensor variational inequalities. Let K be a subset of the tensor space $\mathbb{C}^{[V,h]}$ and $j : \mathbb{C}^{[V,h]} \longrightarrow \mathbb{C}^{[V,h]}$ be a tensor mapping. Anceschi-Barbagallo-Bianco [1] introduced a class of inverse variational inequalities of the tensor type, which consists of finding $u^* \in K$ such that

$$j\left(u^*\right) \in K, \quad \langle u^*, J - j(u^*)\rangle \leq 0, \quad \forall J \in K. \tag{1.5}$$

Some results of the well-posedness analysis are also established in [1]. With the advancement of quasi-variational inequalities, Cao-Kong-Zeng [3] introduced the concept of a generalized split inverse variational inequality in the context of Hilbert space, also defining relevant notions of well-posedness and generalized well-posedness. The well-posedness of the generalized split quasi-variational inequality is then examined using the theory of noncompactness measures and the generalized Cantor theorem. Some results on well-posedness and generalized well-posedness for various kinds of quasi-variational inequalities have been developed recently, see, e.g., [8, 19] and the references therein.

Motivated by previous articles, the innovation of this article lies in considering a new class of split quasi-inverse tensor variational inequalities, Problem 1.1, that involves two independent variables, where the constraint set simultaneously depends on both of these variables. Aims to explore some results on the well-posedness of the GSQITVI under specific assumptions. The structure of the paper is as follows: Section 2 presents some essential concepts that are vital for the subsequent proofs. In Section 3, we will outline the key findings related to the well-posedness of the GSQITVI. Also, we will focus on the generalized well-posedness of the GSQITVI and derive relevant metric characterizations. Finally, some remarks and conclusions are given in Section 4.

2. Preliminaries

In general, we will denote by $\mathbb{C}^{[V,h]}$ the set of all V-order h-dimensional tensors. We assume that V and h are integers with $V, h > 2$, unless stated otherwise; in this context, we will only consider real tensors. Occasionally, we will refer to $h_1 \times h_2 \times h_3 \times \ldots \times h_V$ as the size of $\mathbb{C}^{[V,h]}$. This section focuses on revisiting some fundamental concepts, properties, and important results related to tensor spaces $\mathbb{C}^{[V,h]}$.

Definition 2.1 ([12]). For tensors $A, B \in \mathbb{C}^{[V,h]}$, we specify the inner product between $\mathbb{C}^{[V,h]} \times \mathbb{C}^{[V,h]}$ and $\mathbb{R}$ as:

$$\langle A, B\rangle = \sum_{i_1=1}^{h} \dots \sum_{i_V=1}^{h} a_{i_1,\dots,i_V} b_{i_1,\dots,i_V}.$$

The norm that arises from this inner product is denoted as follows:

$$\|A\| = \sqrt{\sum_{i_1=1}^{h} \cdots \sum_{i_V=1}^{h} |a_{i_1,\dots,i_V}|^2},$$

which is termed the Frobenius norm. Additionally, within the category of $\mathbb{C}^{[V,h]}$ the distance between A and B is stipulated as $d_{\mathbb{C}^{[V,h]}}(A, B) := \|A - B\|$.

We now introduce several core concepts to facilitate proving well-posedness results for the GSQITVI.

Definition 2.2 ([12]). Let $\mathcal{W}$ be a nonempty subset of $\mathbb{C}^{[V,h]}$ and let $A \in \mathbb{C}^{[V,h]}$. Then

(a) the diameter of $\mathcal{W}$ is defined by

$$\operatorname{diam} \mathcal{W} = \sup\{\|A - B\| : A, B \in \mathcal{W}\};$$

(b) the distance between the tensor A and the set $\mathcal{W}$ is defined by

$$d(A, \mathcal{W}) = \inf\{\|A - B\| : B \in \mathcal{W}\}.$$

We recall the definitions of the measure of noncompactness and the Hausdorff distance as stated in tensor spaces

Definition 2.3 ([12]). Let H be a non-empty subset of $\mathbb{C}^{[V,h]}$. The non-compactness measure μ for the set H is stipulated as

$$\mu(H) := \inf\left\{\epsilon > 0 : H = \bigcup_{i=1}^{n} H_i,\ \operatorname{diam}(H_i) < \epsilon,\ i = 1, 2, \dots, n\right\},$$

where every H_i $(i = 1, \dots, n)$ is a finite covering of the set H.

Next, we introduce the Hausdorff distance for use in well-posedness analysis.

Definition 2.4 ([12]). Let Ω_1 and Ω_2 be two non-empty subsets of the tensor space $\mathbb{C}^{[V,h]}$. The Hausdorff metric between Ω_1 and Ω_2 is defined in the following way

$$\mathcal{H}(\Omega_1, \Omega_2) = \max\{e(\Omega_1, \Omega_2), e(\Omega_2, \Omega_1)\},$$

where $e(\Omega_1, \Omega_2)$ is given by

$$e(\Omega_1, \Omega_2) = \sup_{A\in\Omega_1} \inf_{B\in\Omega_2} \|A - B\|.$$

The following definitions generalize the concepts of topological and closedness, lower semicontinuity, and upper semicontinuity properties to set-valued tensor mappings.

Definition 2.5. A set-valued mapping $\Xi : \mathbb{C}^{[V,h]} \rightrightarrows \mathbb{S}^{[M,n]}$ is said to be:

(i) closed if, for arbitrary sequence $A_n \to A$ in $\mathbb{C}^{[V,h]}$ and $B_n \to B$ in $\mathbb{S}^{[M,n]}$ where $B_n \in \Xi(A_n)$, we have $B \in \Xi(A)$.

(ii) lower semicontinuous if, for any sequence $A_n \to A$ in $\mathbb{C}^{[V,h]}$ and any $B \in \Xi(A)$, there exists a sequence $\{B_n\}$ such that $B_n \in \Xi(A_n)$ and $B_n \to B$ in $\mathbb{S}^{[M,n]}$.

(iii) upper semicontinuous if, for every sequence $\{A_n\}$ converging in $\mathbb{C}^{[V,h]}$ and any sequence $\{B_n\} \subseteq \mathbb{S}^{[M,n]}$ with $B_n \in \Xi(A_n)$ besides $B_{nk} \to B^*$ in $\mathbb{S}^{[M,n]}$.

3. Main results

In this section, we will provide the results of well-posedness for the GSQITVI by establishing metric characterizations and specific conditions. We also study the well-posedness in the generalized sense of the GSQITVI which gives a metric characterization employing the Hausdorff metric between the approximating solution set and the solution set of the GSQITVI. To begin, we will introduce the concept of the approximating sequence of the GSQITVI.

Definition 3.1. A sequence $\{(A_n, B_n)\}$ in $\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$ is said to be an approximating sequence of the GSQITVI, if there is a positive sequence $\{\epsilon_n\}$ where $\epsilon_n \to 0$ as n approaches infinity such that

$$\begin{cases} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n,B_n), J(A_n,B_n)) \leq \epsilon_n, \\ d_{\mathbb{C}^{[V_2,h_2]}}(m(A_n,B_n), M(A_n,B_n)) \leq \epsilon_n, \\ h\left(A_n, B_n\right) \leq \epsilon_n, \\ \langle j' - j\left(A_n, B_n\right), A_n \rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A_n, B_n\right)\right) \leq \epsilon_n, \quad \forall j' \in J\left(A_n, B_n\right), \\ \langle m' - m\left(A_n, B_n\right), B_n \rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A_n, B_n\right)\right) \leq \epsilon_n, \quad \forall m' \in M\left(A_n, B_n\right). \end{cases}$$

Remark 3.2. If $J(A,B) = J$ and $M(A,B) = M$ for each $(A,B) \in \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$, where J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$, then the approximating sequence $\{(A_n, B_n)\}$ of GSITVI is given by

$$\begin{cases} j\left(A_n, B_n\right) \in J, \\ m\left(A_n, B_n\right) \in M, \\ h\left(A_n, B_n\right) \leq \epsilon_n, \\ \langle j' - j\left(A_n, B_n\right), A_n \rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A_n, B_n\right)\right) \leq \epsilon_n, \quad \forall j' \in J, \\ \langle m' - m\left(A_n, B_n\right), B_n \rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A_n, B_n\right)\right) \leq \epsilon_n, \quad \forall m' \in M, \end{cases}$$

where $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n \to 0$ as $n \to \infty$.

Let S denote the solution set of the GSQITVI. We now introduce the concept of well-posedness and generalized well-posedness to the GSQITVI.

Definition 3.3. (i) The GSQITVI is said to be well-posed if $S = \{(A^*, B^*)\}$ is one-point, and each approximating sequence $\{(A_n, B_n)\}$ for the GSQITVI converges to (A^*, B^*);

(ii) The GSQITVI is said to be generalized well-posed if $S \neq \emptyset$ and each approximating sequence of the GSQITVI has a subsequence that converges to an element within S.

Let us consider the definition of the approximating solution set of the GSQITVI.

Definition 3.4. For every $\epsilon > 0$, the approximating solution set $T(\epsilon)$ of the GSQITVI defined by

$$T(\epsilon) = \left\{ \begin{array}{l} (A,B) \in \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}: \\ \qquad d_{\mathbb{C}^{[V_1,h_1]}}(j(A,B), J(A,B)) \leq \epsilon, \\ \qquad d_{\mathbb{C}^{[V_2,h_2]}}(m(A,B), M(A,B)) \leq \epsilon, \\ \qquad h(A,B) \leq \epsilon, \\ \qquad \langle j' - j\left(A, B\right), A \rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A, B\right)\right) \leq \epsilon, \forall j' \in J(A,B), \\ \qquad \langle m' - m\left(A, B\right), B \rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A, B\right)\right) \leq \epsilon, \forall m' \in M(A,B) \end{array} \right\}.$$

Remark 3.5. If $\epsilon_1 > \epsilon_2$, then $T(\epsilon_1) - T(\epsilon_2) > 0$, so $T(\epsilon)$ is monotonous of the GSQITVI. Since $\epsilon \to 0$, we can get $T(\epsilon) \to T(0)$, that means $T(0) = S$.

Remark 3.6. If $J(A,B) = J$ and $M(A,B) = M$ for each $(A,B) \in \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$, where J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$, then the approximating solution set $T^*(\epsilon)$ of GSITVI is given by

$$T^*(\epsilon) = \left\{ \begin{array}{c} (A,B) \in \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} : j\,(A,B) \in J,\, m\,(A,B) \in M,\, h(A,B) \leq \epsilon, \\ \langle j' - j\,(A,B)\,, A\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\,(j') - \theta\,(j\,(A,B)) \leq \epsilon,\, \forall j' \in J, \\ \langle m' - m\,(A,B)\,, B\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\,(m') - \tau\,(m\,(A,B)) \leq \epsilon,\, \forall m' \in M \end{array} \right\}.$$

3.1. The descriptions of well-posedness features for the GSQITVI

We now provide the first main result on the equivalence between the well-posedness and the existence of solutions for the GSQITVI under suitable conditions.

Theorem 3.7. *Let $J : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_1,h_1]}$ and $M : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_2,h_2]}$ be two set-valued mappings. Then* GSQITVI *is well-posed, precisely if and only if the solution set of* GSQITVI *is nonempty, and*

$$\lim_{\epsilon \to 0} \operatorname{diam}(T(\epsilon)) = 0.$$

Proof. **Necessity:** Assume that GSQITVI is well-posed, which suggests that there exists a singular solution $(A^*, B^*) \in \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$. Consequently, we have $S = \{(A^*, B^*)\}$, meaning S is nonempty. Now, suppose for contradiction that $\operatorname{diam}(T(\epsilon)) \neq 0$ as $\epsilon \to 0$. There exists a constant $\beta > 0$ and a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that there are points $(A_n^1, B_n^1)\,, (A_n^2, B_n^2) \in T\,(\epsilon_n)$ satisfying

$$\left\|\left(A_n^1, B_n^1\right) - \left(A_n^2, B_n^2\right)\right\|_{\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}} > \beta > 0, \quad \forall n \in \mathbb{N}.$$

Since $(A_n^1, B_n^1)\,, (A_n^2, B_n^2) \in T\,(\epsilon_n)$, we get that

$$\begin{cases} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n^i, B_n^i), J(A_n^i, B_n^i)) \leq \epsilon_n, \\ d_{\mathbb{C}^{[V_2,h_2]}}(m(A_n^i, B_n^i), M(A_n^i, B_n^i)) \leq \epsilon_n, \\ h\,(A_n^i, B_n^i) \leq \epsilon_n, \\ \langle j' - j\,(A_n^i, B_n^i)\,, A_n\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\,(j') - \theta\,(j\,(A_n^i, B_n^i)) \leq \epsilon_n, \quad \forall j' \in J\,(A_n^i, B_n^i)\,, \\ \langle m' - m\,(A_n^i, B_n^i)\,, B_n\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\,(m') - \tau\,(m\,(A_n^i, B_n^i)) \leq \epsilon_n, \quad \forall m' \in M\,(A_n^i, B_n^i) \end{cases}$$

fo all $i = 1, 2$. This implies that $\{(A_n^1, B_n^1)\}$ and $\{(A_n^2, B_n^2)\}$ are two approximating sequences for the GSQITVI. Based on the definition of the well-posedness (where every approximating sequence converges to the exclusive point), we have

$$\lim_{n \longrightarrow \infty} (A_n^1, B_n^1) = \lim_{n \longrightarrow \infty} (A_n^2, B_n^2) = (A^*, B^*).$$

From this, we deduce:

$$\begin{aligned} 0 < \beta &< \left\|\left(A_n^1, B_n^1\right) - \left(A_n^2, B_n^2\right)\right\|_{\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}} \\ &\leq \left\|\left(A_n^1, B_n^1\right) - (A^*, B^*)\right\|_{\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}} + \left\|\left(A_n^2, B_n^2\right) - (A^*, B^*)\right\|_{\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}} \\ &\to 0. \end{aligned}$$

This contradiction indicates that $\lim_{\epsilon \to 0} \operatorname{diam}(T(\epsilon)) = 0$ must hold.

Sufficiency: Now assume $S \neq \emptyset$ and that $\lim_{\epsilon \to 0} \operatorname{diam}(T(\epsilon)) = 0$ for every $\epsilon > 0$. From the definition, we know $S \subset T(\epsilon)$ for all $\epsilon > 0$, so S is a singleton. Let $\{(A_n, B_n)\}$ be an approximating sequence of the GSQITVI. There exists a strictly positive sequence $\{\epsilon_n\}$ with the property that ϵ_n tends to zero as n goes to infinity such that

$$\begin{cases} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n, B_n), J(A_n, B_n)) \leq \epsilon_n, \\ d_{\mathbb{C}^{[V_2,h_2]}}(m(A_n, B_n), M(A_n, B_n)) \leq \epsilon_n, \\ h\,(A_n, B_n) \leq \epsilon_n, \\ \langle j' - j\,(A_n, B_n)\,, A_n\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\,(j') - \theta\,(j\,(A_n, B_n)) \leq \epsilon_n, \quad \forall j' \in J\,(A_n, B_n)\,, \\ \langle m' - m\,(A_n, B_n)\,, B_n\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\,(m') - \tau\,(m\,(A_n, B_n)) \leq \epsilon_n, \quad \forall m' \in M\,(A_n, B_n) \end{cases}$$

which implies that

$$(A_n, B_n) \in T(\epsilon_n) \quad \forall n \in \mathbb{N}.$$

Let (A^*, B^*) be the unique solution of the GSQITVI. By the approximating set's definition, we know $(A^*, B^*) \in T(\epsilon_n)$. Thus, we have

$$\|(A_n, B_n) - (A^*, B^*)\|_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}} \le \operatorname{diam}(T(\epsilon_n)) \to 0.$$

This means that $\{(A_n, B_n)\}$ converges to the unique solution (A^*, B^*) as $n \to \infty$. This proves that GSQITVI is well-posed. □

The following result gives the equivalence between the well-posedness and the existence of solutions for GSITVI, which is derived directly from Theorem 3.7.

Corollary 3.8. *Let J and M be two non-empty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$. Then GSITVI is well-posed, precisely if and only if the solution set S^* of GSITVI is nonempty and*

$$\lim_{\epsilon\to 0} \operatorname{diam}(T^*(\epsilon)) = 0.$$

The next result examines the well-posedness of the GSQITVI without using the condition that the solution set $S \neq \emptyset$ in Theorem 3.7. To do this, we need to introduce the following relevant assumptions:

$H(j,m,h)$**:** The functions $j : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{V}^{[V_1,h_1]}, m : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_2,h_2]}$, and $h : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ are continuous.

$H(J,M)$**:** The set-valued mappings $J : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_1,h_1]}$ and $M : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_2,h_2]}$ have closed, lower semicontinuous, and upper semicontinuous convex values.

$H(\theta,\tau)$**:** The functions $\theta : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ and $\tau : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ are continuous.

Theorem 3.9. *Under the conditions $H(j,m,h), H(J,M)$, and $H(\theta,\tau)$, the GSQITVI is well-posed if and only if the*

$$\lim_{\epsilon\to 0} \operatorname{diam}(T(\epsilon)) = 0, \text{ and } \quad T(\epsilon) \neq \emptyset, \epsilon > 0. \tag{3.1}$$

Proof. The necessity follows directly from Theorem 3.7. Now, we will verify the sufficiency.

Suppose that (3.1) holds. Let $\{(A_n, B_n)\} \subset \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$ be an approximating sequence for the GSQITVI. By definition, there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n > 0$ for every $n \in N$ and $\epsilon_n \to 0$ as $n \to +\infty$ such that

$$\begin{cases} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n,B_n), J(A_n,B_n)) \le \epsilon_n, \\ d_{\mathbb{C}^{[V_2,h_2]}}(m(A_n,B_n), M(A_n,B_n)) \le \epsilon_n, \\ h(A_n,B_n) \le \epsilon_n, \\ \langle j' - j(A_n,B_n), A_n\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j') - \theta(j(A_n,B_n)) \le \epsilon_n, \quad \forall j' \in J(A_n,B_n), \\ \langle m' - m(A_n,B_n), B_n\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau(m') - \tau(m(A_n,B_n)) \le \epsilon_n, \quad \forall m' \in M(A_n,B_n). \end{cases}$$

This implies that

$$(A_n, B_n) \in T(\epsilon_n) \quad \forall n \in \mathbb{N}.$$

Since $\lim_{\epsilon\to 0} \operatorname{diam}(T(\epsilon_n)) = 0$ as $n \to +\infty$, we can conclude that $\{(A_n, B_n)\}$ is a Cauchy sequence. Therefore, in accordance with the definition of a Cauchy sequence, $(A_n, B_n) \to (A^*, B^*)$ in $\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$ as $n \to +\infty$. Now, we will show that (A^*, B^*) is the unique tensor solution of the GSQITVI.

• First, we show that $j(A^*, B^*) \in J(A^*, B^*), m(A^*, B^*) \in M(A^*, B^*)$, and $h(A^*, B^*) \le 0$. Indeed, for each $n \in N$, this approximating sequence satisfies the inequality

$$h(A_n, B_n) \le \epsilon_n,$$

and since $(A_n, B_n) \to (A^*, B^*) \in \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$ and $\epsilon_n \to 0$, by the continuity of h, we obtain

$$h(A^*, B^*) = \lim_{n\to\infty} h(A_n, B_n) \leq \lim_{n\to\infty} \epsilon_n = 0.$$

Therefore, $h(A^*, B^*) \leq 0$.

Next, by assuming condition $H(j, m)$ holds, we have $j(A_n, B_n) \to j(A^*, B^*)$ and $m(A_n, B_n) \to m(A^*, B^*)$ as $n \to \infty$. We now prove

$$d_{\mathbb{C}^{[V_1,h_1]}}(j(A^*, B^*), J(A^*, B^*)) \leq \liminf_{n\to\infty} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n, B_n), J(A_n, B_n)) \leq \lim_{n\to\infty} \epsilon_n = 0. \tag{3.2}$$

Assuming the left inequality of (3.2) fails, there exists a constant r greater than zero such that

$$\liminf_{n\to\infty} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n, B_n), J(A_n, B_n)) < r < d_{\mathbb{C}^{[V_1,h_1]}}(j(A^*, B^*), J(A^*, B^*)).$$

This implies that there exists a subsequence $\{(A_{n_k}, B_{n_k})\}$ of $\{(A_n, B_n)\}$ converging to a point (A^*, B^*) and a sequence $\{w_{n_k}\}$ with $w_{n_k} \in J(A_{n_k} B_{n_k})$ such that

$$\|(j(A_{n_k}, B_{n_k}) - w_{n_k}\|_{\mathbb{C}^{[V_1,h_1]}} \leq r - \frac{1}{n_k} \quad \forall k \in N.$$

Since J is upper semicontinuous, assume $w_{n_k} \to w^*$ in $\mathbb{C}^{[V_1,h_1]}$; by the closedness of J, we have $w^* \in J(A^*, B^*)$. By the continuity of j, we get $j(A_{n_k}, B_{n_k}) \to j(A^*, B^*)$ as $k \to \infty$. Then, we obtain

$$\begin{aligned} r < d_{\mathbb{C}^{[V_1,h_1]}}(j(A^*, B^*), J(A^*, B^*)) &\leq \|j(A^*, B^*) - w^*\|_{\mathbb{C}^{[V_1,h_1]}} \\ &\leq \liminf_{k\to\infty} \|j(A_{n_k}, B_{n_k}) - w_{n_k}\|_{\mathbb{C}^{[V_1,h_1]}} \\ &\leq r. \end{aligned}$$

This contradiction confirms that $j(A^*, B^*) \in J(A^*, B^*)$. By similar reasoning, we also get $m(A^*, B^*) \in M(A^*, B^*)$.

- Secondly, we show the following:

$$\begin{aligned} &\langle j' - j(A^*, B^*), A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j') - \theta(j(A^*, B^*)) \leq 0, \quad \forall j' \in J(A^*, B^*), \\ &\langle m' - m(A^*, B^*), B^*\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau(m') - \tau(m(A^*, B^*)) \leq 0, \quad \forall m' \in M(A^*, B^*). \end{aligned}$$

For any $j' \in J(A^*, B^*)$, since J is lower semicontinuous, there is a sequence $\{j'_n\}$ with $j'_n \in J(A_n, B_n)$ such that $j'_n \to j'$ in $\mathbb{C}^{[V_1,h_1]}$. Using the continuity of θ and j, we conclude:

$$\begin{aligned} &\langle j' - j(A^*, B^*), A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j') - \theta(j(A^*, B^*)) \\ &\leq \liminf_{n\to\infty} \left[\langle j'_n - j(A_n, B_n), A_n\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j'_n) - \theta(j(A_n, B_n))\right] \\ &\leq \liminf_{n\to\infty} \epsilon_n = 0, \end{aligned}$$

and so

$$\langle j' - j(A^*, B^*), A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j') - \theta(j(A^*, B^*)) \leq 0.$$

Similarly, we also obtain

$$\langle m' - m(A^*, B^*), B^*\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau(m') - \tau(m(A^*, B^*)) \leq 0.$$

Thus, we conclude that $(A^*, B^*) \in S$.

To accomplish the proof, we prove that S is a singleton. If S has two distinct solutions (A_1^*, B_1^*) and (A_2^*, B_2^*), then $(A_1^*, B_1^*), (A_2^*, B_2^*) \in T(\epsilon)$ for all $\epsilon > 0$. We have

$$0 < \|(A_1^*, B_1^*) - (A_2^*, B_2^*)\|_{\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}} \leq \lim_{\epsilon\to 0} \operatorname{diam}(T(\epsilon)) = 0,$$

which leads to a contradiction. Therefore, GSQITVI is well-posed. □

Next, we also derive the well-posedness for GSITVI by assuming $T^*(\epsilon) \neq \emptyset$, which is a consequence of Theorem 3.9.

Corollary 3.10. *Let J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$, j and m are two continuous tensor mappings, $j : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_1,h_1]}$ and $m : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_2,h_2]}$, the h, θ and τ are given continuous functions, $h : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$, $\theta : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ and $\tau : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$. Then GSITVI is well-posed if and only if*

$$\lim_{\epsilon \to 0} \operatorname{diam}(T^*(\epsilon)) = 0, \textit{and} \quad T^*(\epsilon) \neq \emptyset, \epsilon > 0.$$

3.2. The characterizations of well-posed in the generalized sense of the GSQITVI

Next, we investigate the well-posedness in the generalized sense of the GSQITVI. The following result sets up a metric characterization by means of the employment of the Hausdorff metric between the approximating solution set and the solution set of the GSQITVI. To begin with, we establish the following lemma.

Lemma 3.11. *The conditions $H(j, m, h), H(J, M)$, and $H(\theta, \tau)$ are satisfied. For each $\epsilon > 0$, the approximating solution set $T(\epsilon)$ for the GSQITVI is closed.*

Proof. For a fixed $\epsilon > 0$, let $\{(A_n, B_n)\} \subset T(\epsilon)$ such that $(A_n, B_n) \to (A^*, B^*)$ as $n \to \infty$. Now, we aim to prove that $(A^*, B^*) \in T(\epsilon)$. Indeed, since $(A_n, B_n) \in T(\epsilon)$, we have

$$\begin{cases} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n, B_n), J(A_n, B_n)) \leq \epsilon, \\ d_{\mathbb{C}^{[V_2,h_2]}}(m(A_n, B_n), M(A_n, B_n)) \leq \epsilon, \\ h\left(A_n, B_n\right) \leq \epsilon, \\ \left\langle j' - j\left(A_n, B_n\right), A_n\right\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A_n, B_n\right)\right) \leq \epsilon, \quad \forall j' \in J\left(A_n, B_n\right), \\ \left\langle m' - m\left(A_n, B_n\right), B_n\right\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A_n, B_n\right)\right) \leq \epsilon, \quad \forall m' \in M\left(A_n, B_n\right). \end{cases}$$

Given any pair (A, B), the sets $J(A, B)$ and $M(A, B)$ are both closed, convex subsets of $\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$. This implies that there exists an element $j_n \in J\left(A_n, B_n\right)$ such that $\|j\left(A_n, B_n\right) - j_n\| \leq \epsilon$, as indicated by $d_{\mathbb{C}^{[V_1,h_1]}}\left(j\left(A_n, B_n\right), J\left(A_n, B_n\right)\right) \leq \epsilon$. Due to the upper semicontinuity and closedness of J, there is a subsequence $\{j_{n_k}\}$ of $\{j_n\}$ such that $j_{n_k} \to j^* \in J\left(A^*, B^*\right)$. Additionally, since $j\left(A_n, B_n\right) \to j\left(A^*, B^*\right)$ by continuity of j, we get

$$\|j\left(A_{n_k}, B_{n_k}\right) - j_{n_k}\| \leq \epsilon.$$

Taking the liminf on both sides, we obtain

$$\|j\left(A^*, B^*\right) - j^*\| \leq \liminf_{n \to \infty} \|j\left(A_{n_k}, B_{n_k}\right) - j_{n_k}\| \leq \epsilon$$

This implies

$$d_{\mathbb{C}^{[V_1,h_1]}}\left(j\left(A^*, B^*\right), J\left(A^*, B^*\right)\right) \leq \epsilon.$$

Similarly, we can show

$$d_{\mathbb{C}^{[V_2,h_2]}}\left(m\left(A^*, B^*\right), M\left(A^*, B^*\right)\right) \leq \epsilon.$$

By the continuity of h, we have

$$\left|h\left(A^*, B^*\right)\right| \leq \left|h\left(A_n, B_n\right)\right| + \left|h\left(A_n, B_n\right) - h\left(A^*, B^*\right)\right| \to \epsilon \quad \text{as } n \to \infty,$$

thus, $h\left(A^*, B^*\right) \leq \epsilon$.

Next, for any $n \in \mathbb{N}$, we have

$$\left\langle j' - j\left(A_n, B_n\right), A_n\right\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A_n, B_n\right)\right) \leq \epsilon, \quad \forall j' \in J\left(A_n, B_n\right).$$

Since J is lower semicontinuous, for any $j^* \in J(A^*, B^*)$, a sequence $\{j_n\}$ exists with $j_n \in J(A_n, B_n)$ such that $j_n \to j^*$. Then, thanks to the continuity of j and θ, we obtain

$$\begin{aligned}&\langle j^* - j(A^*, B^*), A^* \rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j^*) - \theta(j(A^*, B^*)) \\ &= \lim_{n\to\infty} \left[\langle j_n - j(A_n, B_n), A_n \rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j_n) - \theta(j(A_n, B_n)) \right] \leq \epsilon, \quad \forall j_n \in J(A_n, B_n).\end{aligned}$$

This implies that

$$\langle j^* - j(A^*, B^*), A^* \rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta(j^*) - \theta(j(A^*, B^*)) \leq \epsilon, \quad \forall j^* \in J(A^*, B^*).$$

By the same argument above, we find that

$$\langle m^* - m(A^*, B^*), B^* \rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau(m^*) - \tau(m(A^*, B^*)) \leq \epsilon, \quad \forall m^* \in M(A^*, B^*).$$

Thus, we conclude that $(A^*, B^*) \in T(\epsilon)$. This confirms that $T(\epsilon)$ is closed for each $\epsilon > 0$. □

Remark 3.12. Let J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$, j and m are two continuous tensor mappings, $j : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_1,h_1]}$ and $m : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_2,h_2]}$, the h, θ and τ are given continuous functions, $h : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$, $\theta : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ and $\tau : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$. Then, $T^*(\epsilon)$ is closed by using the same method in Lemma 3.11.

Theorem 3.13. *Let $J : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_1,h_1]}$ and $M : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \rightrightarrows \mathbb{C}^{[V_2,h_2]}$ be two set-valued mappings. Then GSQITVI is well-posed in the generalized sense if and only if the solution set S of the GSQITVI is nonempty compact and*

$$\lim_{\epsilon\to 0^+} e(T(\epsilon), S) = 0. \tag{3.3}$$

Proof. **Necessity:** Suppose initially that GSQITVI is well-posed in the generalized sense, which implies that the solution set S of the GSQITVI is nonempty (see Definition 3.3(ii)) and $S \subseteq T(\epsilon) \neq \emptyset$, for all $\epsilon > 0$. To verify that S is compact, consider any sequence $\{(A_n, B_n)\} \subset S$, we have $(A_n, B_n) \in T(\epsilon)$ for each $n \in \mathbb{N}$ and all $\epsilon > 0$, i.e., the sequence $\{(A_n, B_n)\}$ serves as an approximating sequence for the GSQITVI. In light of Definition 3.3(ii), there exists a subsequence $\{(A_{n_k}, B_{n_k})\}$ of $\{(A_n, B_n)\}$ that converges to some element within S. This confirms that S is compact.

Next, we aim to prove (3.3). Assuming that $e(T(\epsilon), S)$ does not converge to zero as $\epsilon \to 0^+$, then for every positive sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$ there exist $\beta > 0$ and a corresponding sequence $(A'_n, B'_n) \in T(\epsilon_n)$ such that

$$d_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}}\left((A'_n, B'_n), S\right) > \beta \quad \forall n \in \mathbb{N}.$$

Since $\{(A'_n, B'_n)\}$ is an approximating sequence of the GSQITVI, and due to the generalized well-posedness of the GSQITVI (see Definition 3.3(ii)), there exists a subsequence $\left\{\left(A'_{n_k}, B'_{n_k}\right)\right\}$ of $\{(A'_n, B'_n)\}$ such that $\left(A'_{n_k}, B'_{n_k}\right)$ converging to some point of S, yielding

$$0 < \beta < d_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}}\left(\left(A'_{n_k}, B'_{n_k}\right), S\right) \to 0 \quad \text{as } k \to 0.$$

This contradiction verifies that (3.3) holds.

Sufficiency: Assume that S is nonempty and compact, and $\lim_{\epsilon\to 0^+} e(T(\epsilon), S) = 0$ holds. Let $\{(A_n, B_n)\}$ be an approximating sequence for the GSQITVI. Then there exists a positive sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that $(A_n, B_n) \in T(\epsilon_n)$ for all $n \in \mathbb{N}$. By (3.3), we can find a sequence $\{(A^*_n, B^*_n)\} \subset S$ such that

$$\|(A_n, B_n) - (A^*_n, B^*_n)\|_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}} \to 0 \text{ as } n \to \infty.$$

Due to the compactness of S, there exists a subsequence $\left\{\left(A^*_{n_k}, B^*_{n_k}\right)\right\}$ of $\{(A^*_n, B^*_n)\}$ that converges to some element $(A^*, B^*) \in S$. Thus, we obtain

$$\begin{aligned}
&\|(A_{n_k}, B_{n_k}) - (A^*, B^*)\|_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}} \\
\leq &\left\|(A_{n_k}, B_{n_k}) - \left(A^*_{n_k}, B^*_{n_k}\right)\right\|_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}} \\
&+ \left\|\left(A^*_{n_k}, B^*_{n_k}\right) - (A^*, B^*)\right\|_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}} \to 0,
\end{aligned}$$

as $k \to 0$, which implies that the GSQITVI is well-posed in the generalized sense. □

Corollary 3.14. *Let J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$, j and m are two continuous tensor mappings, $j : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_1,h_1]}$ and $m : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_2,h_2]}$, the h, ψ and θ are given functions, $h : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$, $\theta : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ and $\tau : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$. Then GSITVI is well-posed in the generalized sense if and only if the solution set S^* of GSITVI is nonempty compact and*

$$\lim_{\epsilon\to 0^+} e(T^*(\epsilon), S^*) = 0.$$

According to the proof of Theorem 3.13, we can see that the compactness of S is a crucial factor. In the next theorem, we can formulate a metric description of well-posedness in the generalized sense by utilizing the measurability of the non-compactness of the approximating solution set to relax the compactness of S.

Theorem 3.15. *Suppose that conditions $H(j, m, h)$, $H(J, M)$, and $H(\theta, \tau)$ are satisfied. Then GSQITVI is well-posed in the generalized sense if and only if*

$$T(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \text{ and } \quad \lim_{\epsilon\to 0} \mu(T(\epsilon)) = 0. \tag{3.4}$$

Proof. **Necessity:** Assume that GSQITVI is well-posed in the generalized sense. By Definition 3.3(ii), the solution set S of the GSQITVI is nonempty, and so $S \subset T(\epsilon) \neq \emptyset$ for all $\epsilon > 0$. Hence, it gives

$$\mathcal{H}(T(\epsilon), S) = \max\{e(T(\epsilon), S), e(S, T(\epsilon))\} = e(T(\epsilon), S), \tag{3.5}$$

for all $\epsilon > 0$. It follows from Theorem 3.13 that S is compact, and so

$$\mu(S) = 0. \tag{3.6}$$

Combining (3.5) and (3.6) leads to

$$\mu(T(\epsilon)) \leq 2\mathcal{H}(T(\epsilon), S) + \mu(S) = 2e(T(\epsilon), S).$$

Moreover, from Theorem 3.13, we have

$$\lim_{\epsilon\to 0^+} e(T(\epsilon), S) = 0.$$

Hence, we conclude

$$\lim_{\epsilon\to 0} \mu(T(\epsilon)) = 0.$$

This implies that conditions (3.4) are satisfied.

Sufficiency: Assume that (3.4) holds. Thanks to Lemma 3.11, for each $\epsilon > 0$, $T(\epsilon)$ is closed. We define $S' = \bigcap_{\epsilon>0} T(\epsilon)$. By the generalized Cantor theorem (see [13]), we have $\lim_{\epsilon\to 0} \mathcal{H}(T(\epsilon), S') = 0$ and S' is nonempty and compact.

In the following, we prove that $S' = S$. It is easy to see that $S \subseteq S'$, so we only need to verify that $S' \subseteq S$. For any $(A^*, B^*) \in S'$ and $\epsilon > 0$ fixed, it holds

$$d_{\mathbb{C}^{[V_1,h_1]}\times\mathbb{C}^{[V_2,h_2]}}\left((A^*, B^*), T(\epsilon)\right) = 0.$$

Then, for each $n \in \mathbb{N}$, there exists a sequence $\{(A_n, B_n)\} \subset T(\epsilon_n)$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\|(A_n, B_n) - (A^*, B^*)\| \leq \epsilon_n.$$

Hence, $(A_n, B_n) \to (A^*, B^*)$ in $\mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]}$ as $n \to \infty$. Since $(A_n, B_n) \in T(\epsilon_n)$, it gives

$$\begin{cases} d_{\mathbb{C}^{[V_1,h_1]}}(j(A_n, B_n), J(A_n, B_n)) \leq \epsilon, \\ d_{\mathbb{C}^{[V_2,h_2]}}(m(A_n, B_n), M(A_n, B_n)) \leq \epsilon, \\ h\left(A_n, B_n\right) \leq \epsilon, \\ \langle j' - j\left(A_n, B_n\right), A_n\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j'\right) - \theta\left(j\left(A_n, B_n\right)\right) \leq \epsilon, \quad \forall j' \in J\left(A_n, B_n\right), \\ \langle m' - m\left(A_n, B_n\right), B_n\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m'\right) - \tau\left(m\left(A_n, B_n\right)\right) \leq \epsilon, \quad \forall m' \in M\left(A_n, B_n\right). \end{cases}$$

By the continuity of j, m and h, we have

$$j\left(A_n, B_n\right) \to j\left(A^*, B^*\right), m\left(A_n, B_n\right) \to m\left(A^*, B^*\right), \text{ and } h\left(A_n, B_n\right) \to h\left(A^*, B^*\right).$$

Employing the same arguments as presented in Theorem 3.9, it follows that

$$\begin{aligned} d_{\mathbb{C}^{[V_1,h_1]}}\left(j\left(A^*, B^*\right), J\left(A^*, B^*\right)\right) &\leq \liminf_{n\to\infty} d_{\mathbb{C}^{[V_1,h_1]}}\left(j\left(A_n, B_n\right), J\left(A_n, B_n\right)\right) \leq \lim_{n\to\infty} \epsilon_n = 0, \\ d_{\mathbb{C}^{[V_2,h_2]}}\left(m\left(A^*, B^*\right), M\left(A^*, B^*\right)\right) &\leq \liminf_{n\to\infty} d_{\mathbb{C}^{[V_2,h_2]}}\left(m\left(A_n, B_n\right), M\left(A_n, B_n\right)\right) \\ &\leq \lim_{n\to\infty} \epsilon_n = 0 \end{aligned}$$

and

$$h\left(A^*, B^*\right) = \lim_{n\to\infty} h\left(A_n, B_n\right) \leq \lim_{n\to\infty} \epsilon_n = 0.$$

This implies that $j\left(A^*, B^*\right) \in J\left(A^*, B^*\right)$, $m\left(A^*, B^*\right) \in M\left(A^*, B^*\right)$ and $h\left(A^*, B^*\right) \leq 0$.

Since J is lower semicontinuous, for any $j^* \in J\left(A^*, B^*\right)$, there exists a sequence $\{j_n'\} \subset J\left(A_n, B_n\right)$ such that $j_n' \to j^*$ as $n \to \infty$. Given that θ and τ are continuous, we can write

$$\begin{aligned} &\langle j^* - j\left(A^*, B^*\right), A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j^*\right) - \theta\left(j(A^*, B^*)\right) \\ &= \lim_{n\to\infty} \left[\langle j_n' - j\left(A_n, B_n\right), A_n\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j_n'\right) - \theta\left(j\left(A_n, B_n\right)\right)\right] \leq \lim_{n\to\infty} \epsilon_n = 0. \end{aligned}$$

Hence, we obtain

$$\langle j^* - j\left(A^*, B^*\right), A^*\rangle_{\mathbb{C}^{[V_1,h_1]}} + \theta\left(j^*\right) - \theta\left(j(A^*, B^*)\right) \leq 0 \quad \forall j^* \in J\left(A^*, B^*\right).$$

Similarly, we obtain

$$\langle m^* - m\left(A^*, B^*\right), B^*\rangle_{\mathbb{C}^{[V_2,h_2]}} + \tau\left(m^*\right) - \tau\left(m(A^*, B^*)\right) \leq 0 \quad \forall m^* \in M\left(A^*, B^*\right).$$

Thus, we arrive at the conclusion that $(A^*, B^*) \in S$, implying $S' = S$. Hence, $\lim_{\epsilon\to 0} H(T(\epsilon), S) = 0$ and $\lim_{\epsilon\to 0} e(T(\epsilon), S) = 0$. Considering the compactness of S and applying Theorem 3.13, it can be concluded that GSQITVI is well-posed in the generalized sense. □

Following a similar approach to the proof of Theorem 3.15, we establish the well-posedness of GSITVI in the generalized sense, which is a specific case of the GSQITVI.

Corollary 3.16. *Let J and M are two nonempty, closed, and convex subsets of $\mathbb{C}^{[V,h]}$, j and m are two continuous tensor mappings, $j : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_1,h_1]}$ and $m : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{C}^{[V_2,h_2]}$, the h, θ and τ are given continuous functions, $h : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$, $\theta : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$ and $\tau : \mathbb{C}^{[V_1,h_1]} \times \mathbb{C}^{[V_2,h_2]} \to \mathbb{R}$. Then GSITVI is well-posed in the generalized sense if and only if the*

$$T^*(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \textit{ and } \quad \lim_{\epsilon\to 0} \mu(T^*(\epsilon)) = 0.$$

4. Conclusions

This work provides a comprehensive study of the generalized split quasi-inverse tensor variational inequality (GSQITVI) in tensor spaces. Then by building upon the concept of well-posedness, we have established key metric-based features that offer necessary and sufficient conditions for the well-posedness of the GSQITVI. Also, we have further derived important results regarding the well-posedness of the GSQITVI by using the measure of non-compactness. The primary innovation of this work lies in the consideration of a new class of split quasi-inverse tensor variational inequalities that involve two independent variables, where the constraint set simultaneously depends on both variables. These findings provide valuable insights into the convergence properties of solutions of the GSQITVI.

Acknowledgements

The authors are grateful to the anonymous referees for their insightful comments and constructive suggestions to improve the presentation of this paper.

References

[1] F. Anceschi, A. Barbagallo and S. G. Lo Bianco, *Inverse tensor variational inequalities and applications*, J. Optim. Theory Appl. **196**, 570–589, 2023.

[2] A. Barbagallo and P. Mauro, *Inverse variational inequality approach and applications*, Numer. Funct. Anal. Optim. **35**, 851–867, 2014.

[3] L. Cao, H. Kong and S. D. Zeng, *On the well-posedness of the generalized split quasi-inverse variational inequalities*, J. Nonlinear Sci. Appl. **9**, 5497–5509, 2016.

[4] G. P. Crespi, A. Guerraggio and M. Rocca, *Well posedness in vector optimization problems and vector variational inequalities*, J. Optim. Theory Appl. **132**, 213–226, 2007.

[5] H. He, C. Ling and H. K. Xu, *A relaxed projection method for split variational inequalities*, J. Optim. Theory Appl. **166**, 213–233, 2015.

[6] R. Hu and Y. P. Fang, *Levitin–Polyak well-posedness by perturbations of inverse variational inequalities*, Optim. Lett. **7**, 343–359, 2013.

[7] R. Hu and Y. P. Fang, *Well-posedness of the split inverse variational inequality problem*, Bull. Malays. Math. Sci. Soc. **40**, 1733–1744, 2017.

[8] N. V. Hung, *Generalized Levitin–Polyak well-posedness for controlled systems of FMQHI-fuzzy mixed quasi-hemivariational inequalities of Minty type*, J. Comput. Appl. Math. **386**, 113263, 2021.

[9] N. V. Hung, V. M. Tam and D. Baleanu, *Regularized gap functions and error bounds for split mixed vector quasivariational inequality problems*, Math. Methods Appl. Sci. **43**, 4614–4626, 2020.

[10] N. V. Hung, V. M. Tam and D. O'Regan, *Error bound analysis for split weak vector mixed quasi-variational inequality problems in fuzzy environment*, Appl. Anal. **102**, 1874–1888, 2023.

[11] C. Kanzow and D. Steck, *Quasi-variational inequalities in Banach spaces: theory and augmented Lagrangian methods*, SIAM J. Optim. **29**, 3174–3200, 2019.

[12] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, SIAM, 2000.

[13] K. Kuratowski, *Topology*, New edition, revised and augmented, Academic Press, 1966.

[14] M. B. Lignola, *Well-posedness and L-well-posedness for quasivariational inequalities*, J. Optim. Theory Appl. **128**, 119–138, 2006.

[15] M. B. Lignola and J. Morgan, *Vector quasi-variational inequalities: approximate solutions and well-posedness*, J. Convex Anal. **13**, 373, 2006.

[16] M. A. Noor, *General variational inequalities*, Appl. Math. Lett. **1**, 119–122, 1988.

[17] M. A. Noor, *Some developments in general variational inequalities*, Appl. Math. Comput. **152**, 199–277, 2004.

[18] M. A. Noor, *Differentiable non-convex functions and general variational inequalities*, Appl. Math. Comput. **199**, 623–630, 2008.

[19] V. M. Tam, N. V. Hung, Z. Liu and J. C. Yao, *Levitin–Polyak well-posedness by perturbations for the split hemivariational inequality problem on Hadamard manifolds*, J. Optim. Theory Appl. **195**, 684–706, 2022.

[20] Y. Wang, Z. H. Huang and L. Qi, *Global uniqueness and solvability of tensor variational inequalities*, J. Optim. Theory Appl. **177**, 137–152, 2018.