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Rupture Degree of Some Wheel Related Graphs

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Abstract: It is desirable that computer or communication networks continue to operate without interruption in the case of an attack or partial disruption. If a network modeled by a graph, then there are various graph theoretical parameters used to express the stability and vulnerability of communication networks. One of them is the concept of rupture degree. In this paper, we determine exact values for the rupture degree of wheel related graphs.

Tekerlekle İlgili Bazı Çizgelerin Kopukluk Derecesi

Anahtar Kelimeler
Ağ hassasiyeti,
Kopukluk derecesi,
Tekerlekle ilgili çizgeler

Öz: Bilgisayar ya da iletişim ağlarının bir saldırı ya da kısmi bozulma durumunda kesintiye uğramadan çalışmaya devam etmesi arzu edilen bir durumdur. Bir ağ bir çizge ile modellenirse, ağın kararlılığını ve hassasiyetini ifade etmek için kullanılan çeşitli teorik çizge parametreleri mevcuttur. Bunlardan biri de kopukluk derecesi kavramıdır. Bu makalede tekerlekle ilgili çizgelerin kopukluk derecesi için kesin değerleri belirledik.

1. INTRODUCTION

In this paper, we consider simple, finite, undirected graphs. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. The number of vertices and the number of edges of the graph G are denoted by $|V(G)|, |E(G)|$ respectively [1]. We denote the number of components of a graph G by $\omega(G)$ and the order of the largest component of G by $m(G)$. We use $\lfloor x \rfloor$ for the largest integer not larger than x and $\lceil x \rceil$ for the smallest integer not smaller than x. Two vertices are said to cover each other in a graph G if they are incident in G. A vertex cover in G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the vertex covering number of G and is denoted by $\alpha(G)$. An independent set of vertices of a graph G is a set of vertices of G whose elements are pairwise nonadjacent. The independence number $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G [2, 3]. If G is a graph of order n, then $\alpha(G) + \beta(G) = n$. The *connectivity* of G, denoted by $K(G)$, is the minimal size of a vertex set S such that $G - S$ is disconnected or has only one vertex [4]. Terminology and notation not defined in this paper can be found in [4]. We denote the minimum vertex degree of a graph G by $\delta(G)$. A set $S \subseteq V(G)$ is a vertex cut of G, if either $G - S$ is disconnected or $G - S$ has only one vertex. The graph $W_n = K_1 + C_n$ is called a *wheel graph*. In wheel graph, the vertex c of degree n is called the *central vertex* while the vertices on the cycle C_n are called *rim vertices*.

A communication network is composed of processors and communication links. Network designers attach importance the reliability and stability of a network. If the network begins losing communication then there is a loss in its effectiveness. This event is called as the vulnerability of communication networks. In a communication network, vulnerability measures the resistance of the network after a breakdown of some of its processors or communication links [5]. If we think of a graph as modeling a network, then we have some graph parameters to measure the vulnerability. In a communication network, the vulnerability measures are essential to guide the designers in choosing an appropriate topology. They have an impact on solving difficult optimization problems for networks and they can also be investigated in different metric spaces by looking at the algebraic relationship with the topologies of the graph structures we consider [6, 7, 8].

The vulnerability of communication networks measures the resistance of a network to a disruption in operation after the failure of certain processors and communication links. Network designers require greater degree of stability and reliability or less vulnerability in communication networks. Many graph theoretical parameters have been used to describe the stability and reliability of communication networks. These parameters deal with some or all of the following three questions:

(1) What is the number of elements that are not functioning?
(2) What is the number of remaining connected subnetworks?
(3) What is the size of a largest remaining group within which mutual communication can still occur?

The *tenacity* of an incomplete connected graph G is defined as

$$T(G) = \left\{ \frac{|S| + m(G-S)}{w(G-S)} : S \subset V(G) \text{ and } \omega(G-S) > 1 \right\}$$

and the tenacity of K_n is defined as n. It is natural to consider the additive dual of tenacity. We call this parameter *the rupture degree* of graphs. Formally, the rupture degree of an incomplete connected graph G is defined by

$$r(G) = max\{\omega(G-S) - |S| - m(G-S) : S \subset V(G), \omega(G-S) \geq 2\}$$

and the rupture degree of K_n is defined as $1-n$ [9, 10].

When we compare the tenacity number and rupture degree parameters, which deal with all three questions above, Zhang et al. shows that there exist graphs G_1 and G_2 such that $T(G_1) = T(G_2)$ and $r(G_1) \neq r(G_2)$. That is, the rupture degree and tenacity differ in showing the vulnerability of networks [9]. Consequently the rupture degree is a better parameter to measure the vulnerability of some networks. Zhang et al. obtained several results on the rupture degree of a graph. Li, F and Li, X [10] showed the relationship between tenacity and rupture degree as $r(G) \leq \alpha(G)\big(1 - T(G)\big)$.

Kırlangıç et al. calculated the rupture degree of complete $k-ary\ tree$, graph operations and gear graphs [11, 12]. Aytaç et al. calculated the rupture degree of thorn graphs and E_p^t consisting of t legs such that each leg is a path graph of order p and they gave the formulas for the rupture degree of the corona operation of some special graphs and they calculated the rupture degree for composite graphs [13, 14, 15]. Agnes et al. determined the exact values of the rupture degree of the wheel graph and also that of cartesian product of graphs [16].

Zhang et al. presented some results about rupture degree of join graphs and Nordhaus−Gaddum type results. They also gave some results on bounds for rupture degree and following theorems.

Theorem 1.1. [9] The rupture degree of the comet $C_{t,r}$ is

$$r(C_{t,r}) = \begin{cases} r-1, & if\ t\ is\ even \\ r-2, & if\ t\ is\ odd. \end{cases}$$

Theorem 1.2. [9] The rupture degree of the path P_n ($n \geq 3$) is

$$r(P_n) = \begin{cases} -1, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd. \end{cases}$$

Theorem 1.3. [9] The rupture degree of the cycle C_n is

$$r(C_n) = \begin{cases} -1, & if\ n\ is\ even \\ -2, & if\ n\ is\ odd. \end{cases}$$

Theorem 1.4. [9] The rupture degree of the star $K_{1,n-1}(n \geq 3)$ is $n-3$.

Theorem 1.5. [9] Let G be a non−complete connected graph of order n. Then

(a) $r(G) \leq n - 2\delta - 1$,
(b) $3 - n \leq r(G) \leq n - 3$,
(c) $2\beta(G) - n - 1 \leq r(G) \leq \frac{(\beta(G)^2 - K(G))(\beta(G)-1)-n}{\beta(G)}$.

Theorem 1.6. [16] The rupture degree of the wheel W_n ($n \geq 5$) is

$$r(W_n) = \begin{cases} -3, & if\ n\ is\ even \\ -2, & if\ n\ is\ odd. \end{cases}$$

In this paper, we obtain rupture degree of some wheel related graphs such as the helm graph H_n, the closed helm CH_n, the flower graph Fl_n, the sunflower graph SF_n and the web graph $W(t,n)$.

2. RUPTURE DEGREE OF SOME WHEEL RELATED GRAPHS

In this section, firstly the graphs mentioned in the paper are introduced with their figures, and then the rupture degree of these graphs are calculated.

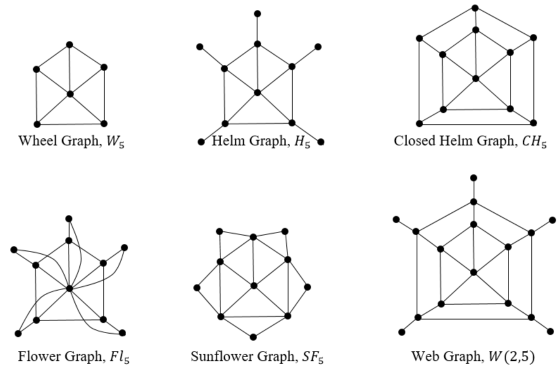

Figure 1. Some wheel related graphs

Definition 2.1. [17] The *helm* H_n is a graph obtained from a wheel W_n with central vertex c, by attaching a pendant edge to each rim vertex of W_n.

Theorem 2.1. Let H_n be a Helm graph of order $2n+1$. Then

$$r(H_n) = \begin{cases} \frac{n-6}{2} & , if\ n\ is\ even \\ \frac{n-7}{2} & , if\ n\ is\ odd \end{cases}$$

Proof. **Case 1.** n is even.

Let S be an arbitrary vertex cut of H_n and $|S| = k$. If $k \leq \frac{n}{2} + 1$, then $\omega(H_n - S) \leq n = 2(k-1)$. Therefore we have $m(H_n - S) \geq \left\lceil \frac{2n+1-k}{2(k-1)} \right\rceil$. Hence

$$\begin{aligned}\omega(H_n - S) - |S| - m(H_n - S) &\leq 2k - 2 - k - \left\lceil \frac{2n+1-k}{2(k-1)} \right\rceil \\ &\leq k - \left\lceil \frac{2n+1-k}{2(k-1)} \right\rceil - 2 \\ &\leq \frac{n}{2} + 1 - \left\lceil \frac{2n+1-k}{2(k-1)} \right\rceil - 2 \\ &\leq \frac{n}{2} - 3.\end{aligned}$$

If $|S| = k > \frac{n}{2} + 1$, then $\omega(H_n - S) < n$ and $m(H_n - S) \geq 2$. Hence

$$\begin{aligned}\omega(H_n - S) - |S| - m(H_n - S) &\leq n - \frac{n}{2} - 1 - 2 \\ &= \frac{n}{2} - 3.\end{aligned}$$

$= \frac{n}{2} - 3.$

From the choice of S, we obtain $r(H_n) \leq \frac{n}{2} - 3$. It is easy to see that there is a vertex cut S^* of H_n such that $|S^*| = \frac{n+2}{2}$, $\omega(H_n - S^*) = n$ and $m(H_n - S^*) = 2$. From the definition of rupture degree , we have

$$\begin{aligned}r(H_n) &\geq \omega(H_n - S^*) - |S^*| - m(H_n - S^*) \\ &\geq n - \frac{n+2}{2} - 2 \\ &= \frac{n-6}{2}.\end{aligned}$$

This implies that $r(H_n) = \frac{n-6}{2}$.

Case 2. n is odd.

Let S be an arbitrary vertex cut of H_n and $|S| = k$. If $k \leq \frac{n+3}{2}$, then $\omega(H_n - S) \leq n = 2k - 3$. Therefore we have $m(H_n - S) \geq \left\lceil \frac{2n+1-k}{2k-6} \right\rceil$. Hence

$$\begin{aligned}\omega(H_n - S) - |S| - m(H_n - S) &\leq 2k - 3 - k - \left\lceil \frac{2n+1-k}{2k-6} \right\rceil \\ &\leq k - 3 - \left\lceil \frac{2n+1-k}{2k-6} \right\rceil \\ &\leq \frac{n+3}{2} - 3 - \left\lceil \frac{2n+1-k}{2k-6} \right\rceil \\ &\leq \frac{n+3}{2} - 5 = \frac{n-7}{2}.\end{aligned}$$

If $|S| = k > \frac{n+3}{2}$, then $\omega(H_n - S) < n$ and $m(H_n - S) \geq 2$. Hence

$$\begin{aligned}\omega(H_n - S) - |S| - m(H_n - S) &\leq n - \frac{n+3}{2} - 2 \\ &= \frac{n-7}{2}.\end{aligned}$$

From the choice of S, we obtain $r(H_n) \leq \frac{n-7}{2}$. It is easy to see that there is a vertex cut S^* of H_n such that $|S^*| = \frac{n+3}{2}$, $\omega(H_n - S^*) = n$ and $m(H_n - S^*) = 2$. From the definition of rupture degree , we have

$$\begin{aligned}r(H_n) &\geq \omega(H_n - S^*) - |S^*| - m(H_n - S^*) \\ &\geq n - \frac{n+3}{2} - 2 \\ &= \frac{n-7}{2}.\end{aligned}$$

This implies that $r(H_n) = \frac{n-7}{2}$.

Definition 2.2. [17] The *closed helm* CH_n is the graph with central vertex c, obtained from a helm by joining each pendant vertex to form a cycle.

Theorem 2.2. Let CH_n be a closed Helm graph of order $2n + 1$. Then

$$r(CH_n) = \begin{cases} -2 & , if\ n\ is\ even \\ -4 & , if\ n\ is\ odd \end{cases}$$

Proof. **Case 1.** n is even.

Let S be an arbitrary vertex cut of CH_n and $|S| = k$. If $k \leq n + 1$, then $\omega(CH_n - S) \leq n = k - 1$. Therefore we have $m(CH_n - S) \geq \left\lceil \frac{2n+1-k}{k-1} \right\rceil$. Hence

$$\begin{aligned}\omega(CH_n - S) - |S| - m(CH_n - S) &\leq k - 1 - k - \left\lceil \frac{2n+1-k}{k-1} \right\rceil \\ &\leq \left\lceil \frac{2n}{k-1} - 1 \right\rceil - 1 \\ &\leq -2.\end{aligned}$$

If $|S| = k > n + 1$, then $\omega(CH_n - S) < n$ and $m(CH_n - S) \geq 1$. Hence

$$\begin{aligned}\omega(CH_n - S) - |S| - m(CH_n - S) &\leq n - n - 1 - 1 \\ &= -2\end{aligned}$$

$= -2.$

From the choice of S, we obtain $r(CH_n) \leq -2$. It is easy to see that there is a vertex cut S^* of CH_n such that $|S^*| = n + 1$, $\omega(CH_n - S^*) = n$ and $m(CH_n - S^*) = 1$. From the definition of rupture degree , we have

$$\begin{aligned}r(CH_n) &\geq \omega(CH_n - S^*) - |S^*| - m(CH_n - S^*) \\ &\geq n - n - 1 - 1 \\ &= -2.\end{aligned}$$

This implies that $r(CH_n) = -2$.

Case 2. n is odd.

Let S be an arbitrary vertex cut of CH_n and $|S| = k$. If $k \leq n + 2$, then $\omega(CH_n - S) \leq n - 1 = k - 3$. Therefore we have $m(CH_n - S) \geq \left\lceil \frac{2n+1-k}{k-3} \right\rceil$. Hence

$$\begin{aligned}\omega(CH_n - S) - |S| - m(CH_n - S) &\leq k - 3 - k - \left\lceil \frac{2n+1-k}{k-3} \right\rceil \\ &\leq -4\end{aligned}$$

If $|S| = k > n + 2$, then $\omega(CH_n - S) < n - 1$ and $m(CH_n - S) \geq 1$. Hence

$$\begin{aligned}\omega(CH_n - S) - |S| - m(CH_n - S) &\leq n - 1 - n - 2 - 1 \\ &= -4.\end{aligned}$$

From the choice of S, we obtain $r(CH_n) \leq -4$. It is easy to see that there is a vertex cut S^* of CH_n such that $|S^*| = n+1$, $\omega(CH_n - S^*) = n-1$ and $m(CH_n - S^*) = 2$. From the definition of rupture degree , we have

$$r(CH_n) \geq \omega(CH_n - S^*) - |S^*| - m(CH_n - S^*) \\ \geq n-1-n-1-2 \\ = -4.$$

This implies that $r(CH_n) = -4$.

Definition 2.3. [17]The *flower graph* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex c of the helm.

Theorem 2.3. Let Fl_n be a Flower graph of order $2n+1$. Then

$$r(Fl_n) = \begin{cases} \frac{n-6}{2} & ,if\ n\ is\ even \\ \frac{n-7}{2} & ,if\ n\ is\ odd \end{cases}$$

Proof. **Case 1.** n is even

Let S be an arbitrary vertex cut of Fl_n and $|S| = k$. If $k \leq \frac{n}{2}+1$, then $\omega(Fl_n - S) \leq 2k-2$ and $m(Fl_n) - S \geq 2$. Hence, $r(Fl_n) \leq \max_k \{2k-2-k-2\} = \max_k\{k-4\}$. The function $f(k) = k-4$ is an increasing function and takes its maximum value at $k = \frac{n}{2}+1$. Then

$$r(Fl_n) \leq \frac{n}{2}+1-4 = \frac{n}{2}-3$$

From the choice of S, we obtain $r(Fl_n) \leq \frac{n}{2}-3$. On the other hand there is a vertex cut $S^*$of Fl_n such that $|S^*| = \frac{n+2}{2}$, $\omega(Fl_n - S^*) = n$ and $m(Fl_n - S^*) = 2$. From the definition of rupture degree, we have

$$r(Fl_n) \geq \omega(Fl_n - S^*) - |S^*| - m(Fl_n - S^*) \\ \geq n - \frac{n+2}{2} - 2 \\ = \frac{n-6}{2}$$

This implies that $r(Fl_n) = \frac{n-6}{2}$.

Case 2. n is odd

Let S be an arbitrary vertex cut of Fl_n and $|S| = k$. If $k \leq \frac{n+3}{2}$, then $\omega(Fl_n - S) \leq 2k-3$ and $m(Fl_n) - S \geq 2$. Hence, $r(Fl_n) \leq \max_k \{2k-3-k-2\} = \max_k\{k-5\}$. The function $f(k) = k-5$ is an increasing function and takes its maximum value at $k = \frac{n+3}{2}$. Then

$$r(Fl_n) \leq \frac{n+3}{2} - 5 = \frac{n-7}{2}$$

From the choice of S, we obtain $r(Fl_n) \leq \frac{n-7}{2}$. On the other hand there is a vertex cut $S^*$of Fl_n such that $|S^*| = \frac{n+3}{2}$, $\omega(Fl_n - S^*) = n$ and $m(Fl_n - S^*) = 2$. From the definition of rupture degree, we have

$$r(Fl_n) \geq \omega(Fl_n - S^*) - |S^*| - m(Fl_n - S^*) \\ \geq n - \frac{n+3}{2} - 2 \\ = \frac{n-7}{2}$$

This implies that $r(Fl_n) = \frac{n-7}{2}$.

Definition 2.4. [17]The *sunflower graph* SF_n is a graph obtained from a wheel with central vertex c, $n-cycle$ $v_0, v_1, \dots, v_{n-1}$ and additional n vertices $w_0, w_1, \dots, w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, \dots, n-1$ where $i+1$ is taken modulo n.

Theorem 2.4. Let SF_n be a sunflower graph of order $2n+1$. Then $r(SF_n) = 0$.

Proof. Let S be an arbitrary vertex cut of SF_n and $|S| = k$. If $k \leq n$, then $\omega(Fl_n - S) \leq n+1 = k+1$. Thefore $m(SF_n) - S \geq \left\lceil \frac{2n+1-(t+1)}{t+1} \right\rceil$. Hence,

$$\omega(SF_n - S) - |S| - m(SF_n - S) \leq k+1-k-\left\lceil \frac{2n-k}{k+1} \right\rceil \\ \leq 0$$

If $|S| = k > n$, then $\omega(SF_n - S) < n+1$ and $m(CH_n - S) \geq 1$. Hence

$$\omega(SF_n - S) - |S| - m(SF_n - S) \leq n+1-n-1 \\ \leq 0$$

≤ 0.

From the choice of S, we obtain $r(SF_n) \leq 0$. On the other hand there is a vertex cut $S^*$of SF_n such that $|S^*| = n$, $\omega(SF_n - S^*) = n+1$ and $m(SF_n - S^*) = 1$. From the definition of rupture degree, we have

$$r(SF_n) \geq \omega(SF_n - S^*) - |S^*| - m(SF_n - S^*) \\ \geq n+1-n-1 \\ \geq 0$$

This implies that $r(SF_n) = 0$.

Definition 2.5. [17] A *web graph* is the graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t,n)$ is the generalized web with t cycles each of order n.

Theorem 2.5. Let $W(t,n)$ be a web graph of order $nt+n+1$. Then

$$r(W(t,n)) = \begin{cases} \frac{n-6}{2} & ,if\ n\ is\ even \\ \frac{n-t-6}{2} & ,if\ n\ is\ odd\ and\ t\ is\ odd \\ \frac{n-t-5}{2} & ,if\ n\ is\ odd\ and\ t\ is\ even \end{cases}$$

Proof. **Case 1.** n is even

Let S be an arbitrary vertex cut of $W(t,n)$ and $|S| = k$. If $k \leq \frac{nt}{2} + 1$, then $\omega(W(t,n) - S) \leq \frac{n}{2} + k - 1$. Therefore we have $m(W(t,n) - S) \geq \left\lceil \frac{nt+n+1-k}{\frac{nt+n}{2}} \right\rceil$.

Hence

$$\omega(W(t,n)-S) - |S| - m(W(t,n)-S) \leq \frac{n}{2} + k - 1 - k - \left\lceil \frac{nt+n+1-k}{\frac{nt+n}{2}} \right\rceil \leq \frac{n}{2} - 1 - 2\left\lceil 1 + \frac{1-k}{nt+n} \right\rceil \leq \frac{n}{2} - 3.$$

If $|S| = k > \frac{nt}{2} + 1$, then $\omega(W(t,n) - S) < \frac{n}{2}(t+1)$ and $m(W(t,n) - S) \geq \left\lceil \frac{nt+n+1-k}{\frac{nt+n}{2}} \right\rceil$. Hence

$$\omega(W(t,n)-S) - |S| - m(W(t,n)-S) \leq \frac{nt}{2} + \frac{n}{2} - \frac{nt}{2} - 1 - 2\left\lceil \frac{nt+n+1-k}{nt+n} \right\rceil \leq \frac{n}{2} - 1 - 2\left\lceil 1 + \frac{1-k}{nt+n} \right\rceil \leq \frac{n}{2} - 3$$

From the choice of S, we obtain $r(W(t,n) \leq \frac{n-6}{2}$. It is easy to see that there is a vertex cut S^* of $W(t,n)$ such that $|S^*| = \frac{nt+2}{2}$, $\omega(W(t,n) - S^*) = \frac{n}{2}(t+1)$ and $m(W(t,n) - S^*) = 2$. From the definition of rupture degree, we have

$$r(W(t,n)) \geq \omega(W(t,n)-S^*) - |S^*| - m(W(t,n)-S^*) \geq \frac{nt}{2} + \frac{n}{2} - \frac{nt}{2} - 1 - 2 = \frac{n-6}{2}$$

This implies that $r(W(t,n)) = \frac{n-6}{2}$.

Case 2. n is odd.

Subcase 2.1. t is odd

Let S be an arbitrary vertex cut of $W(t,n)$ and $|S| = k$. If $k \leq \frac{nt+3}{2}$, then $\omega(W(t,n) - S) \leq \frac{n-1}{2}(t+1) + 1 = k + 1 + \frac{n-t}{2}$. Therefore we have $m(W(t,n) - S) \geq \left\lceil \frac{nt+n+1-k}{\frac{(n-1)(t+1)}{2}+1} \right\rceil$. Hence

$$\omega(W(t,n)-S) - |S| - m(W(t,n)-S) \leq \frac{n-t}{2} + 1 + k - k - \left\lceil \frac{nt+n+1-k}{\frac{nt+n}{2}} \right\rceil \leq \frac{n-t}{2} + 1 - 2\left\lceil \frac{nt+n+1-k}{nt+n-t+1} \right\rceil \leq \frac{n-t}{2} + 1 - 4 \leq \frac{n-t-6}{2}.$$

If $|S| = k > \frac{nt+3}{2}$, then $\omega(W(t,n) - S) < \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2}$ and $m(W(t,n) - S) \geq 2$. Hence

$$\omega(W(t,n)-S) - |S| - m(W(t,n)-S) \leq \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2} - \frac{nt}{2} - \frac{3}{2} - 2 \leq \frac{n-t}{2} - 3 \leq \frac{n-t-6}{2}.$$

From the choice of S, we obtain $r(W(t,n) \leq \frac{n-t-6}{2}$. It is easy to see that there is a vertex cut S^* of $W(t,n)$ such that $|S^*| = \frac{nt+3}{2}$, $\omega(W(t,n) - S^*) = \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2}$ and $m(W(t,n) - S^*) = 2$. From the definition of rupture degree, we have

$$r(W(t,n)) \geq \omega(W(t,n)-S^*) - |S^*| - m(W(t,n)-S^*) \geq \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2} - \frac{nt}{2} - \frac{3}{2} - 2 \geq \frac{n-t-6}{2}.$$

This implies that $r(W(t,n)) = \frac{n-t-6}{2}$.

Subcase 2.2. t is even.

Let S be an arbitrary vertex cut of $W(t,n)$ and $|S| = k$. If $k \leq \frac{nt+2}{2}$, then $\omega(W(t,n) - S) \leq \frac{n-1}{2}(t+1) + 1 = \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2} = k - \frac{1}{2} + \frac{n-t}{2}$. Therefore we have $m(W(t,n) - S) \geq \left\lceil \frac{nt+n+1-k}{\frac{2k+n-t-1}{2}} \right\rceil$. Hence

$$\omega(W(t,n)-S) - |S| - m(W(t,n)-S) \leq \frac{n-t}{2} - \frac{1}{2} + k - k - 2\left\lceil \frac{nt+n+1-k}{2k+n-t-1} \right\rceil \leq \frac{n-t-5}{2}.$$

If $|S| = k > \frac{nt+2}{2}$, then $\omega(W(t,n) - S) < \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2}$ and $m(W(t,n) - S) \geq 2$. Hence

$$\omega(W(t,n)-S) - |S| - m(W(t,n)-S) \leq \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2} - \frac{nt}{2} - 1 - 2 \leq \frac{n-t}{2} - \frac{5}{2} \leq \frac{n-t-5}{2}.$$

From the choice of S, we obtain $r(W(t,n)) \leq \frac{n-t-5}{2}$. It is easy to see that there is a vertex cut S^* of $W(t,n)$ such that $|S^*| = \frac{nt+2}{2}$, $\omega(W(t,n) - S^*) = \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2}$ and $m(W(t,n) - S^*) = 2$. From the definition of rupture degree, we have

$$r(W(t,n)) \geq \omega(W(t,n)-S^*) - |S^*| - m(W(t,n)-S^*) \geq \frac{nt}{2} - \frac{t}{2} + \frac{n}{2} + \frac{1}{2} - \frac{nt}{2} - 1 - 2 \geq \frac{n-t-5}{2}.$$

This implies that $r(W(t,n)) = \frac{n-t-5}{2}$.

4. DISCUSSION AND CONCLUSION

When we make a comparison by looking at the results between the graphs with the same order examined in this study, we see that there is an inequality $r(CH_n) < r(SF_n) < r(Fl_n) = r(H_n)$. Therefore, we can say that the stability of the flower graph and the helm graph is more powerful than the stability of the sunflower graph and the stability of the closed helm graph is the least powerful. In the web graph $W(t,n)$, we see that the rupture degree decreases as the value of t increases in the case of n is odd. This reduces the stability and robustness of the graph. The results obtained show the effect on the graphs considered in case of an external attack. Hence, designers for choosing the appropriate topology can use these results.

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