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CROSSED CORNERS OF LIE ALGEBRAS

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ABSTRACT

In this work, we explore the concept of crossed corners in lie algebras and establish a connection between the category of crossed corners of lie algebras and the category of reduced simplicial lie algebras with Moore complex has length 2.

Keywords: Lie algebra, Crossed modules, Crossed corner.

1 INTRODUCTION

The concept of crossed modules of groups was first introduced by Whitehead in [1]. The lie algebra analog of crossed modules was studied by Kassel and Loday in [2]. Ellis, in [3], defined a Lie algebra version of crossed squares. In [4] the equivalence between simplicial lie algebras and these crossed structures was established. Later, quadratic modules of Lie algebra were presented in [5]. The study of quasi quadratic modules over Lie algebras can be found in [6]. For additional research, braided crossed modules of Lie algebras can also be given in [7,8,9].

Alp introduced the concept of crossed corners of groups, which are closely related to crossed squares, and explored the connections between them [10,11,12]. Crossed corner and related structures are given for commutative algebras [13,14].

The primary goal of this paper is to define crossed corners over lie algebras. We examine the close relationship between the categories of crossed corners of lie algebras and reduced simplicial lie algebras with a Moore complex of length 2, using Peiffer pairings in the Moore complex.

2 MATERIAL AND METHOD

As recalled in Kassel and Loday [2], a crossed module of lie algebras is a homomorphism of lie-algebras $\gamma: L_1 \to L_2$, with a Lie algebra action of L_1 on L_2, satisfying the following axioms:

LieCM1. $\gamma(\iota_2 \cdot \iota_1) = [\iota_2, \gamma(\iota_1)]$

LieCM2. $\gamma(\iota_1) \cdot \iota_1{}^* = [\iota_1, \iota_1{}^*]$

for all $\iota_1, \iota_1{}^* \in L_1,\ \iota_2 \in L_2$. Morever, a crossed module is isomorphic to a simplicial lie algebra whose Moore complex has length 1.

Definition: Let L_1, L_2 and L_3 are Lie algebras. Suppose given a diagram of Lie algebras

$$\begin{tikzcd} L_1 \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] & L_2 \\ L_3 & \end{tikzcd}$$

is a crossed corner with Lie actions of L_2 on L_1 and L_3 on L_1, and a function $h: L_2 \times L_3 \to L_1$. Then

LieCC1. γ and γ' are crossed modules of Lie algebras.

LieCC2. $h([\iota_2, \iota_2'], \iota_3) = \iota_2 \cdot h(\iota_2', \iota_3) - \iota_2' \cdot h(\iota_2, \iota_3)$

$h(\iota_2, [\iota_3, \iota_3']) = \iota_3 \cdot h(\iota_2, \iota_3') - \iota_3' \cdot h(\iota_2, \iota_3)$

LieCC3. $h(\gamma(\iota_2), \iota_3) = \iota_3 \cdot \iota_1$

$h(\iota_2, \gamma'(\iota_1)) = \iota_2 \cdot \iota_1$

LieCC4. $(\iota_2 \cdot \iota_3) \cdot \iota_1 = [\iota_2, \iota_3] \cdot \iota_1$

$$(\iota_3 \cdot \iota_2) \cdot \iota_1 = -[\iota_2, \iota_3] \cdot \iota_1$$

In which there are Lie actions L_2 on L_3 and L_3 on L_2,

$$\iota_3 \cdot \iota_2 = \gamma' h(\iota_2, \iota_3)$$

$$\iota_2 \cdot \iota_3 = \gamma h(\iota_2, \iota_3)$$

for all $\iota_1 \in L_1,\ \iota_2, \iota_2' \in L_2, \iota_3, \iota_3' \in L_3$.

These actions are well defined as shown in the first authors masters thesis [15]. Also here, the crossed corner category $\boldsymbol{LieCC}$ for Lie algebras is created.

Example: Let I be a Lie algebra and $I_1,\ I_2$ be any two ideals of I.

$$\begin{tikzcd}
I_1 \cap I_2 \arrow[r, "\eta"] \arrow[d, "\eta'"'] & I_2 \\
I_1 &
\end{tikzcd}$$

The diagram of inclusions together with the actions of $I_1,\ I_2$ on $I_1 \cap I_2$ given by Lie bracket and the function

$$h: I_1 \otimes I_2 \to I_1 \cap I_2$$

$$(\iota_1, \iota_2) \mapsto [\iota_1, \iota_2]$$

gives a crossed corner.

3 RESULTS AND DISCUSSION

Theorem: The category of reduced simplicial Lie algebras of dimension two and smaller L of the Moore complex and the category of crossed corner defined on Lie algebras are equivalent to each other.

Proof: For the reduced simplicial Lie algebra L whose Moore complex is NL, let $NL_1 = Çekd_0^1$ and $NL_1' = Çekd_1^1$. We obtain the following diagram of Lie algebras

$$\begin{tikzcd}
NL_2/\gamma_3(NL_3 \cap I_3) \arrow[r, "\gamma_2"] \arrow[d, "\gamma_2^*"'] & NL_1 \\
NL_1' &
\end{tikzcd}$$

and where the h morphism

$$h: NL_1 \times NL_1{}' \longrightarrow {}^{NL_2}/_{\gamma_3(NL_3 \cap I_3)}$$

$$(\iota, \iota') \longmapsto [s_1(\iota), s_1(\iota') - s_0(\iota')]$$

for all $\iota \in NL_1$, $\iota' \in NL_1{}'$. Then, we will demonstrate that all the axioms of a crossed corner are satisfied.

$LieCC1$. γ_2 ,$\gamma_2{}'$ are crossed modules for lie algebras.

$LieCC2$. We will show that

$$h([\iota_1, \iota_2], \iota') = \iota_1 \cdot h(\iota_2, \iota') - \iota_2 \cdot h(\iota_1, \iota')$$

by taking $\iota_1, \iota_2 \in NL_1, \iota' \in NL_1{}'$.

$$\begin{aligned}
h([\iota_1, \iota_2], \iota') &= h(\iota_1\iota_2 - \iota_2\iota_1, \iota') \\
&= [s_0(\iota_1\iota_2 - \iota_2\iota_1), s_1(\iota'), -s_0(\iota')] \\
&= [s_0(\iota_1\iota_2) - s_0(\iota_2\iota_1), s_1(\iota') - s_0(\iota')] \\
&= [s_0(\iota_1)s_0(\iota_2) - s_0(\iota_2)s_0(\iota_1), s_1(\iota') - s_0(\iota')] \\
&= [s_0(\iota_1)s_0(\iota_2), s_1(\iota') - s_0(\iota')] - [s_0(\iota_2)s_0(\iota_1), s_1(\iota') - s_0(\iota')] \\
&= \iota_1 \cdot [s_1(\iota_2) - s_1(\iota'), s_0(\iota')] - \iota_2 \cdot [s_0(\iota_1) - s_1(\iota'), s_0(\iota')] \\
&= \iota_1 \cdot h(\iota_2, \iota') - \iota_2 \cdot h(\iota_1, \iota')
\end{aligned}$$

Akça and Arvasi [4] have introduced the functions $M_{\alpha,\beta}$ in the Moore complex for a simplicial Lie algebra L. We will use these functions in the following axioms

$LieCC3$. We will show that

$$h(\gamma_2(\iota^*), \iota') = \iota' \cdot \iota^*$$

by taking $\iota^* \in {}^{NL_2}/_{\gamma_3(NL_3 \cap I_3)}$, $\iota' \in NL_1{}'$

$$\begin{aligned}
h(\gamma_2(\iota^*), \iota') &= [s_1 d_2(\iota^*) - s_0 d_2(\iota^*), s_1(\iota')] \\
&= [s_1 d_2(\iota^*), s_1(\iota')] - [s_0 d_2(\iota^*), s_1(\iota')]
\end{aligned}$$

For $\alpha = (0)$, $\beta = (2,1)$, image of $M_{\alpha,\beta}$

$$\begin{aligned}
d_3\left(M_{(2,1),(0)}(\iota, \iota^*)\right) &= d_3([s_2 s_1(\iota), s_0(\iota^*) - s_1(\iota^*) + s_2(\iota^*)]) \\
&= [d_3 s_2 s_1(\iota), d_3 s_0(\iota^*) - d_3 s_1(\iota^*) + d_3 s_2(\iota^*)]
\end{aligned}$$

$$= [s_1(\imath), \imath^*, s_0 d_2(\imath^*) - s_1 d_2(\imath^*) + \imath^*]$$

$$= [s_1(\imath), s_0 d_2(\imath^*)] - [s_1(\imath), s_1 d_2(\imath^*)] + [s_1(\imath), \imath^*] \in \gamma_3(NL_3 \cap I_3)$$

Thus,

$$h(\imath, \gamma_2{}'(\imath^*),) \equiv [s_1(\imath), \imath^*] \qquad mod\ \gamma(NL_3 \cap I_3)$$

$$= \imath \cdot \imath^*$$

Similarly, we show that

$$h(\gamma_2(\imath^*), \imath') = \imath' \cdot \imath^*$$

for $\imath^* \in {}^{NL_2}/_{\gamma_3(NL_3 \cap I_3)}$, $\imath' \in NL_1{}'$

$$h(\gamma_2(\imath^*), \imath') = [s_1 d_2(\imath^*), s_1(\imath') - s_0(\imath')]$$

$$= [s_1 d_2(\imath^*), s_1(\imath')] - [s_1 d_2(\imath^*), s_0(\imath')]$$

for $\alpha = (1), \beta = (2,0)$, image of $M_{\alpha,\beta}$

$$d_3\left(M_{(1),(2,0),}(\imath^*, \imath')\right) = d_3[s_2 s_0(\imath') - s_2 s_1(\imath'), s_1(\imath^*) - s_2(\imath^*)]$$

$$= [d_3 s_2 s_0(\imath') - d_3 s_2 s_1(\imath'), d_3 s_1(\imath^*) - d_3 s_2(\imath^*)]$$

$$= [s_0(\imath') - s_1(\imath'), s_1 d_2(\imath^*) - \imath^*]$$

$$= [s_0(\imath'), s_1 d_2(\imath^*)] - [s_1(\imath'), s_1 d_2(\imath^*)] - [s_0(\imath'), \imath^*] + [s_1(\imath'), \imath^*]$$

Thus,

$$h(\gamma_2(\imath^*), \imath') \equiv [s_0(\imath'), \imath^*] - [s_1(\imath'), \imath^*] \qquad mod\ \gamma(NL_3 \cap I_3)$$

$$= [s_1(\imath'), \imath^*]$$

$$= \imath' \cdot \imath^* \qquad (\text{since}, s_0 \imath' = s_1 s_0 d_1 \imath' = 0)$$

$LieCC4$. We will show that

$$(\imath' \cdot \imath'') \cdot \imath^* = [\imath', \imath''] \cdot \imath^*$$

for all $\imath' \in NL_1$, $\imath'' \in NL_1{}'$, $\imath^* \in {}^{NL_2}/_{\gamma_3(NL_3 \cap I_3)}$

$$(\imath' \cdot \imath'') \cdot \imath^* = \left(\gamma_2 h(\imath' \cdot \imath'')\right) \cdot \imath^*$$

$$= d_2[s_1(\imath'), s_1(\imath'') - s_0(\imath'')] \cdot \imath^*$$

$$= [d_2 s_1(\imath'), d_2 s_1(\imath'') - d_2 s_0(\imath'')] \cdot \imath^*$$

$$= [\iota', \iota'' - s_0 d_1(\iota'')] \cdot \iota^*$$

$$= [\iota', \iota''] \cdot \iota^* \qquad\qquad (\text{since}, \gamma_1(\iota'') = 0)$$

Similarly the axiom $(\iota'' \cdot \iota') \cdot \iota^*$ is satisfied.

We can define the construction of a reduced simplicial Lie algebra with a Moore complex of length ≤ 2 from a crossed corner of lie algebras. Then

$$\begin{tikzcd}
L_1 \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] & L_2 \\
L_3 &
\end{tikzcd}$$

by using the equivalence of the reduced crossed square of Lie algebras with crossed corners,

$$\begin{tikzcd}
L_1 \arrow[r, "\gamma"] \arrow[d, "\gamma'"'] & L_2 \arrow[d, "\mu"] \\
L_3 \arrow[r, "\mu'"'] & \{0\}
\end{tikzcd}$$

we can obtain a bisimplicial Lie algebra with this square. Where γ, γ', μ, μ' are crossed modules for Lie algebras. Since $\mu: L_2 \to \{0\}$ is the zero Lie algebra morphism

$$\begin{tikzcd}
\cdots L_2 \ltimes (L_2 \ltimes \{0\}) \arrow[r, shift left=3] \arrow[r, shift left=1.5] \arrow[r] & L_2 \ltimes \{0\} \arrow[l, shift left=-1.5] \arrow[l, shift left=-3] \arrow[r, shift left=1.5] \arrow[r] & \{0\} \arrow[l, shift left=-1.5]
\end{tikzcd}$$

this is reduced simplicial Lie algebra. From $L_2 \times \{0\} \cong L_2$ and $\mu': L_3 \to \{0\}$ respectively

$$\begin{tikzcd}
\cdots L_2 \ltimes L_2 \arrow[r, shift left=3] \arrow[r, shift left=1.5] \arrow[r] & L_2 \arrow[l, shift left=-1.5] \arrow[l, shift left=-3] \arrow[r, shift left=1.5] \arrow[r] & \{0\} \arrow[l, shift left=-1.5]
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots L_3 \ltimes L_3 \arrow[r, shift left=3] \arrow[r, shift left=1.5] \arrow[r] & L_3 \arrow[l, shift left=-1.5] \arrow[l, shift left=-3] \arrow[r, shift left=1.5] \arrow[r] & \{0\} \arrow[l, shift left=-1.5]
\end{tikzcd}$$

are reduced simplicial Lie algebras.

We will use the functor defined by Artin and Mazur [16] to transform bisimplicial lie algebras into simplicial lie algebras

$$\begin{array}{ccccc}
\cdots & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & (L_1 \ltimes L_3) \ltimes ((L_1 \ltimes L_3) \ltimes L_2) & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & L_2 \ltimes L_2 \\
\upuparrows\downarrow\downarrow\downarrow & & \upuparrows\downarrow\downarrow\downarrow & & \upuparrows\downarrow\downarrow\downarrow \\
((L_1 \ltimes L_3) \ltimes L_2) \ltimes (L_2 \ltimes L_2) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & (L_1 \ltimes L_3) \ltimes L_2 & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & L_2 \\
\uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow & & \uparrow\downarrow\downarrow \\
L_3 \ltimes L_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow} & L_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow} & \{0\}
\end{array}$$

The subset of the lie algebra $L_{1,0} \times L_{0,1} = (L_2 \times \{0\}) \times (L_3 \times \{0\})$ is

$$L_1 = \{((\iota_2, 0), (\iota_3, 0)) | d_1^h(\iota_3, 0) = d_0^v(\iota_2, 0) = 0\}$$

where $\{0\} \cong L_0$.

$$\varphi: L_1 \to L_3 \times L_2 \times \{0\}$$

$$((\iota_2, 0), (\iota_3, 0)) \mapsto (\iota_3, \iota_2, 0)$$

is a isomorphism. Then

$$d_0(\iota_3, \iota_2, 0) = 0$$

$$d_1(\iota_3, \iota_2, 0) = 0$$

Hence, we obtain $\{L_1, L_0\}$ as a reduced 1-truncated simplicial lie algebra together with $d_{0,1}$ zero lie algebra homomorphisms.

$L_{2,0} \times L_{1,1} \times L_{0,2} = (L_2 \times (L_2 \times \{0\})) \times ((L_1 \times L_3) \times (L_2 \times \{0\})) \times (L_3 \times (L_3 \times \{0\}))$ subset of lie algebra of $((\iota_2', \iota_2'', 0), (\iota'_1, \iota_3), (\iota_2, 0), (\iota_3', \iota_3'', 0))$ contains elements. And, maps are

$$d_0^v(\iota_2', \iota_2'', 0) = d_1^h(\iota'_1, \iota_3, \iota_2, 0)$$

$$d_1^v(\iota'_1, \iota_3, \iota_2, 0) = d_2^h(\iota_3', \iota_3'', 0)$$

Therefore, it can be expressed in terms of elements of L_2

$$((\iota_2', \iota_2'', 0), (\iota'_1, (\gamma'(\iota'_1))\iota_3'), (\iota_2, 0)), (\gamma'(\iota'_1)\iota_3, \iota_3'', 0))$$

where $\iota_2'' = \iota_2$ ve $\gamma'(\iota'_1)\iota_3 = \iota_3'$.

We can say that φ' is isomorphism defined by

$$\varphi'\left((\iota_2', \iota_2'', 0), ((\iota'_1, \iota_3), (\iota_2'', 0)), (\gamma'(\iota'_1)\iota_3, \iota_3'', 0)\right) = ((\iota'_1, (\iota_3, \iota_2'')), (\iota_3'', (\iota_2, 0))$$

As a result, by applying the Artin Mazur [16] functor, we obtain the corresponding reduced simplicial Lie algebra.

$$L : (L_1 \ltimes (L_3 \ltimes L_2)) \ltimes (L_3 \ltimes (L_2 \ltimes \{0\})) \quad (L_3 \ltimes (L_2 \ltimes \{0\})) \quad \{0\}$$

where, maps are

$$d_0^1(\iota_3, \iota_2, 0) = 0$$

$$d_1^1(\iota_3, \iota_2, 0) = 0$$

$$s_0^0(0) = (0, 0, 0)$$

$$d_0^2\Big(\big(\iota'_1, (\iota'_3, \iota''_2)\big), \big(\iota''_3, (\iota'_2, 0)\big)\Big) = (\iota''_3, \gamma(\iota'_1)\iota''_2, 0)$$

$$d_1^2\Big(\big(\iota'_1, (\iota'_3, \iota''_2)\big), \big(\iota''_3, (\iota'_2, 0)\big)\Big) = (\iota''_3, \gamma'(\iota'_1)\iota'_3, \iota''_2\iota'_2, 0)$$

$$d_2^2\Big(\big(\iota'_1, (\iota'_3, \iota''_2)\big), \big(\iota''_3, (\iota'_2, 0)\big)\Big) = (\iota'_3, \iota'_2, 0)$$

$$s_0^1(\iota'_3, \iota'_2, 0) = \Big(\big(0, (0, \iota'_2)\big), \big(\iota'_3, (0, 0)\big)\Big)$$

$$s_1^1(\iota'_3, \iota'_2, 0) = \Big(\big(0, (\iota'_3, 0)\big), \big(0, (\iota'_2, 0)\big)\Big)$$

We can obtain the 2-truncated reduced simplicial lie algebra.

Equivalences between reduced simplicial Lie algebras, quadratic modules, and braided crossed modules are given for [5, 7], for the group case of this construction see also [17]. Using this theorem, equivalences of these structures can also be provided with crossed corners. Additionally, it can be adapted within Leibniz algebra, Lie-Rinehart algebra, and Leibniz-Rinehart algebra [18, 19, 20, 21].

4 CONCLUSION AND SUGGESTIONS

In this study, we introduce the lie algebra of crossed corners and show that the category of reduced simplicial lie algebras with a Moore complex of length 2 is equivalent to the category of crossed corners of lie algebras. For future research, this framework could also be extended to other algebraic structures.

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Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

Işıl Zekiye Kurtuluş contributed to the preparation of the article.

Özgün Gürmen Alansal contributed to the interpretation of the data and management of the study.

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