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$\tilde{\rho}$ Katlı Bessel $\mathfrak{T}_2$-Δ_θ^n-Asimptotik İstatistiksel Denk Çift Dizilerin Bazı Özellikleri

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Anahtar Kelimeler
Asimptotik denklik,
Bessel istatistiksel yakınsaklık,
Lacunary dizi

Öz: Dizi uzayları ve toplanabilirlik teorisi çerçevesinde, klasik yakınsama ve sınırlılık kavramlarının teorik temelleri üzerine önemli ilerlemeler kaydedilmiş olmakla birlikte, yenilikçi yakınsama yöntemlerinin araştırılması hala kritik bir alan olarak öne çıkmaktadır. Literatürde, özellikle Bessel fonksiyonlarının bu teorik yapılarla entegrasyonu yönünde belirgin bir eksiklik bulunmaktadır. Bu çalışma, çift diziler için $\tilde{\rho}$ katlı Bessel $\mathfrak{T}_2$-Δ_θ^n-asimptotik istatistiksel denklik ve güçlü Bessel $\mathfrak{T}_2$-Δ_θ^n-asimptotik denklik kavramlarını tanıtarak, bu boşluğu doldurmayı ve söz konusu kavramlar arasındaki kapsama ilişkilerini kapsamlı bir biçimde incelemeyi amaçlamaktadır. Bessel fonksiyonlarının bu bağlamda teorilere dahil edilmesi, yalnızca matematiksel analiz alanındaki mevcut teorik çerçeveleri genişletmekle kalmayıp, aynı zamanda alanın ileri düzeydeki araştırmalarına ve uygulamalı problemlerin çözümüne yönelik sağlam bir temel teşkil etmektedir. Bu yenilikçi kavramlar, Bessel fonksiyonlarıyla ilişkili dizi davranışlarının daha derin bir şekilde anlaşılmasına katkıda bulunmakta ve bu anlayışın matematiksel uygulamalarda kullanılabilirliğini artırmaktadır.

Some Properties of Bessel $\mathfrak{T}_2$-Δ_θ^n-Asymptotically Statistical Equivalent Double Sequence of Order $\tilde{\rho}$

Keywords
Asymptotical equivalence,
Bessel statistical convergence,
Lacunary sequence

Abstract: In the context of sequence spaces and summability theory, significant progress has been made in the theoretical foundations of classical concepts such as convergence and boundedness. However, the exploration of innovative convergence methods continues to emerge as a critical area of research. A notable gap in the literature is the integration of Bessel functions into these theoretical frameworks. This study aims to fill this gap by introducing the concepts of Bessel $\mathfrak{T}_2$-Δ_θ^n-asymptotically statistical equivalence of order $\tilde{\rho}$ and strong Bessel $\mathfrak{T}_2$-Δ_θ^n-asymptotically equivalence of order $\tilde{\rho}$ for double sequences, examining the comprehensive interrelations of these concepts. The inclusion of Bessel functions in these theories not only extends the existing theoretical frameworks within mathematical analysis but also provides a solid foundation for further research and the solution of applied problems in the field. These innovative concepts contribute to a deeper understanding of sequence behavior related to Bessel functions and enhance their applicability in mathematical applications.

1. Introduction

Statistical convergence, first introduced by Fast [1] in 1951, represents a noteworthy generalization of the classical idea of convergence, enhancing its relevance across diverse mathematical domains. In 1993, Fridy [2] introduced lacunary statistical convergence as a refined extension to the established notion of statistical convergence. Subsequently, Fridy and Orhan [2] established a relationship between these two forms of convergence. Later, Çakan and Altay [3] extended these findings to multidimensional settings, building on the work of Fridy and Orhan. For a deeper understanding of lacunary convergence and its generalizations, additional insights can be found in the study by Freedman et al. [4] along with various other sources.

The concept of statistical convergence was further generalized by Kostyrko et al. [5], who introduced ideal convergence. These types of convergence are based on an ideal which is nonempty and closed under finite unions and subsets. Das et al. [6] expanded this framework to include ideal

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convergence of double sequences within metric spaces. Tripathy [7] extended the discussion by introducing lacunary ideal convergence for real sequences. Additionally, Das et al. [8] developed the notions of ideal statistical convergence and ideal lacunary statistical convergence by integrating the idea of ideals. Belen et al. [9] further explored ideal statistical convergence for double sequences, presenting it as a new generalization of both statistical and conventional convergence. Kumar et al. [10] contributed by defining ideal lacunary statistical convergence for double sequences. Studies on statistical convergence and ideal convergence can be reviewed in references [11-26].

Marouf [27] provided the definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [28] expanded on these concepts by formulating an asymptotically statistical equivalent counterpart and establishing natural regularity conditions for nonnegative summability matrices. Subsequently, Patterson and Savaş [29] applied the definitions from [28] to lacunary sequences, thereby extending their applicability. Additionally, Savaş [30] examined ideal asymptotically statistical equivalent sequences and ideal asymptotically lacunary statistical equivalent sequences.

Et and Çolak [31], Et et al. [32-34] expanded the framework of difference sequence spaces through their generalizations.

Mathematics provides an effective approach for tackling and resolving issues associated with circular and cylindrical shapes, which are modeled using Bessel functions. Originally introduced by the German mathematician Friedrich Bessel [35], these functions have become essential tools across various domains of mathematics, science, and engineering due to their relevance and adaptability. The exploration of Bessel functions involves an extensive array of mathematical investigations, such as their properties, asymptotic characteristics, integral forms, and special cases. Beyond their theoretical significance, Bessel functions are also widely applied in practical fields, including physics, engineering, and other scientific areas. The Bessel function of the first kind, denoted as $\mathcal{J}_t$, can be expressed through the following infinite series representation:

$$\mathcal{J}_t(\varrho) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+t+1)s!}\left(\frac{\varrho}{2}\right)^{2s+t},$$

where t represents the order of the Bessel function, and, and Γ denotes the gamma function. The gamma function, also known as Euler's integral, is given for $\varrho > 0$ by

$$\Gamma(\varrho) = \int_0^{\infty} e^{-y} y^{\varrho-1} dy.$$

Ibrahim et al. [36] explored and defined concepts such as Bessel-based convergence, Bessel-type boundedness, statistical Bessel Cauchy sequences, and statistical convergence within the framework of Bessel functions.

The main results of this study are presented in the following section. We introduce new concepts, including Bessel $\mathfrak{I}_2$ - Δ_θ^n -asymptotically statistical equivalence of order $\tilde{\rho}$ and strong Bessel $\mathfrak{I}_2$ - Δ_θ^n - asymptotically equivalence of order $\tilde{\rho}$ for double sequences.

2. Material and Method

The sequence space $\mathcal{J}_t^{\mu}(\mathcal{X})$ is characterized as follows:

$$\mathcal{J}_t^{\mu}(\mathcal{X}) = \{\overline{\mathcal{W}} = (\overline{\mathcal{W}}_\sigma) : \mathcal{J}_t^{\mu}(\overline{\mathcal{W}}_\sigma) \in \mathcal{X}\}$$

where $\mathcal{X}$ represents any sequence space, $\mu \in \mathbb{N}$, t is a real number and

$$\mathcal{J}_t^{\mu}(\overline{\mathcal{W}}_\sigma) = \sum_{z=0}^{\mu} \frac{(-1)^z}{\Gamma(z+1+t)z!}\left(\frac{\overline{\mathcal{W}}_{\sigma+z}}{2}\right)^{2z+t}.$$

Since $\mathcal{X}$ is a linear space, $\mathcal{J}_t^{\mu}(\mathcal{X})$ will also be a linear space.

Definition 2.1 A sequence $(\overline{\mathcal{W}}_\sigma)$ is considered statistically convergent to the value $\mathcal{W}^*$ provided that for all $\tau > 0$,

$$\lim_{z\to\infty} \frac{1}{z}\left|\{\sigma \le z : |\overline{\mathcal{W}}_\sigma - \mathcal{W}^*| > \tau\}\right| = 0.$$

This convergence can be expressed as $\mathrm{St} - \lim \overline{\mathcal{W}}_\sigma = \mathcal{W}^*$.

Definition 2.2 A sequence $(\overline{\mathcal{W}}_\sigma)$ is described as Bessel bounded (or $\mathcal{J}_t^{\mu}$-bounded) if there exists a positive value $P \in \mathbb{R}^+$ such that

$$\left|\mathcal{J}_t^{\mu}(\overline{\mathcal{W}}_\sigma)\right| \le P \text{ for each } \sigma \in \mathbb{N}.$$

The collection of all $\mathcal{J}_t^{\mu}$ -bounded sequences is demonstrated by $\ell_\infty[\mathcal{J}_t^{\mu}]$.

Throughout the paper, we assume that $c, d, t_1, t_2 \in (0,1]$. We use $\tilde{\rho}$ to represent (c, d) and $\tilde{\lambda}$ to represent (t_1, t_2). We establish the following relations:

$$\tilde{\rho} \preccurlyeq \tilde{\lambda} \Leftrightarrow c \le t_1 \text{ and } d \le t_2,$$
$$\tilde{\rho} \prec \tilde{\lambda} \Leftrightarrow c < t_1 \text{ and } d < t_2,$$
$$\tilde{\rho} \cong \tilde{\lambda} \Leftrightarrow c = t_1 \text{ and } d = t_2,$$
$$\tilde{\rho} \in (0,1] \Leftrightarrow c, d \in (0,1]$$
$$\tilde{\lambda} \in (0,1] \Leftrightarrow t_1, t_2 \in (0,1],$$
$$\tilde{\rho} \cong 1 \text{ when } c = d = 1,$$
$$\tilde{\lambda} \cong 1 \text{ when } t_1 = t_2 = 1,$$
$$\tilde{\rho} \succ 1 \text{ when } c > 1 \text{ and } d > 1.$$

The sequence $\theta_{\mathrm{q},\mathrm{y}} = \{(u_{\mathrm{q}}, v_{\mathrm{y}})\}$ is referred to as a double lacunary sequence if there exist two strictly increasing integer sequences (u_{q}) and (v_{y}) satisfying the following conditions:

$$u_0 = 0, \mathrm{h}_{\mathrm{q}} = u_{\mathrm{q}} - u_{\mathrm{q}-1} \to \infty \text{ as } \mathrm{q} \to \infty,$$
$$v_0 = 0, \bar{\mathrm{h}}_{\mathrm{y}} = v_{\mathrm{y}} - v_{\mathrm{y}-1} \to \infty \text{ as } \mathrm{y} \to \infty.$$

The intervals determined by $\theta_{\mathrm{q},\mathrm{y}}$ will be denoted by

$$\mathrm{I}_{\mathrm{q}} = \{(u): u_{\mathrm{q}-1} < u \le u_{\mathrm{q}}\},$$
$$\mathrm{I}_{\mathrm{y}} = \{(v): v_{\mathrm{y}-1} < v \le v_{\mathrm{y}}\}$$
$$\mathrm{I}_{\mathrm{q},\mathrm{y}} = \{(u, v): u_{\mathrm{q}-1} < u \le u_{\mathrm{q}} \text{ and } v_{\mathrm{y}-1} < v \le v_{\mathrm{y}}\},$$

$\mathrm{q}_{\mathrm{q}} = \frac{u_{\mathrm{q}}}{u_{\mathrm{q}-1}}, \bar{\mathrm{q}}_{\mathrm{y}} = \frac{v_{\mathrm{y}}}{v_{\mathrm{y}-1}}$, and $\mathrm{q}_{\mathrm{q},\mathrm{y}} = \mathrm{q}_{\mathrm{q}}\bar{\mathrm{q}}_{\mathrm{y}}$, where $\mathrm{k}_{\mathrm{q},\mathrm{y}} = u_{\mathrm{q}} v_{\mathrm{y}}$ and $\mathrm{h}_{\mathrm{q},\mathrm{y}} = \mathrm{h}_{\mathrm{q}}\bar{\mathrm{h}}_{\mathrm{y}}$.

The symbols for difference double sequences, originally defined by Tripathy and Sarma [37, 38], are expressed as follows:

$$\Delta^n x_{\sigma\omega} = \sum_{u_1=0}^{n} \sum_{u_2=0}^{n} (-1)^{u_1+u_2} \binom{n}{u_1} \binom{n}{u_2} x_{\sigma+u_1,\omega+u_2}$$

where the first-order difference is defined as:

$\mathrm{x}_{\sigma\omega} = \mathrm{x}_{\sigma,\omega} - \mathrm{x}_{\sigma,\omega+1} - \mathrm{x}_{\sigma+1,\omega} + \mathrm{x}_{\sigma+1,\omega+1}$ for all $\sigma, \omega \in \mathbb{N}$.

3. Results

Definition 3.1 Consider $\overline{\mathcal{W}} = (\overline{\mathcal{W}}_{\sigma\omega})$ and $\overline{\mathcal{Z}} = (\overline{\mathcal{Z}}_{\sigma\omega})$ as two real-valued double sequences such that $\Delta^n \overline{\mathcal{W}}_{\sigma\omega}$ and $\Delta^n \overline{\mathcal{Z}}_{\sigma\omega}$ are nonnegative. $\overline{\mathcal{W}}$ and $\overline{\mathcal{Z}}$ are referred to as Bessel $\mathfrak{T}_2$-Δ^n-asymptotically equivalent of order $\tilde{\rho}$ if, for any $\tau > 0$,

$$\left\{(\sigma, \omega) \in \mathbb{N}^2 : \left| \frac{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma \right| \ge \tau \right\} \in \mathfrak{T}_2.$$

This relationship can be expressed as $\overline{\mathcal{W}} \overset{\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho}}{\sim} \overline{\mathcal{Z}}$.

Definition 3.2 For $\tilde{\rho} \in (0,1]$, the sequences $(\overline{\mathcal{W}}_{\sigma\omega})$ and $(\overline{\mathcal{Z}}_{\sigma\omega})$ are considered Bessel $\mathfrak{T}_2$ - Δ^n - asymptotically statistical equivalent of order $\tilde{\rho}$ if, for all $\tau, \gamma > 0$, the set

$$\left\{(\mathrm{q}, \mathrm{y}) \in \mathbb{N}^2 : \frac{1}{u_{\mathrm{q}}^c v_{\mathrm{y}}^d} \left| \left\{(\sigma, \omega) : \sigma \le u_{\mathrm{q}} \text{ and } \omega \le v_{\mathrm{y}}, \left| \frac{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma \right| \ge \tau \right\} \right| \ge \gamma \right\} \in \mathfrak{T}_2.$$

This concept is represented as $\overline{\mathcal{W}} \overset{\mathrm{S}^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

Definition 3.3 Let $\theta = \{(u_{\mathrm{q}}, v_{\mathrm{y}})\}$ denote a double lacunary sequence, and assume $\tilde{\rho} \in (0,1]$. The sequences $(\overline{\mathcal{W}}_{\sigma\omega})$ and $(\overline{\mathcal{Z}}_{\sigma\omega})$ are referred to as Bessel $\mathfrak{T}_2$-Δ_θ^n-asymptotically statistical equivalent of order $\tilde{\rho}$ if, for any $\tau, \gamma > 0$,

$$\left\{(\mathrm{q}, \mathrm{y}) \in \mathbb{N}^2 : \frac{1}{\mathfrak{h}_{\mathrm{q}}^c \bar{\mathfrak{h}}_{\mathrm{y}}^d} \left| \left\{(\sigma, \omega) \in I_{\mathrm{q},\mathrm{y}} : \left| \frac{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma \right| \ge \tau \right\} \right| \ge \gamma \right\} \in \mathfrak{T}_2,$$

where $I_{\mathrm{q},\mathrm{y}} = \{(u, v): u_{\mathrm{q}-1} < u \le u_{\mathrm{q}} \textit{ and } v_{\mathrm{y}-1} < v \le v_{\mathrm{y}}\}$ and $\mathfrak{h}_{\mathrm{q},\mathrm{y}}^{\tilde{\rho}}$ represents the $\tilde{\rho}$th power $(\mathfrak{h}_{\mathrm{q},\mathrm{y}})^{\tilde{\rho}}$ of $\mathfrak{h}_{\mathrm{q},\mathrm{y}}$.

This is denoted as $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

Definition 3.4 Let $\theta = \{(u_{\mathrm{q}}, v_{\mathrm{y}})\}$ represent a lacunary sequence, and assume $\tilde{\rho} \in (0,1]$. The sequences $(\overline{\mathcal{W}}_{\sigma\omega})$ and $(\overline{\mathcal{Z}}_{\sigma\omega})$ are defined as strong Bessel $\mathfrak{T}_2$-Δ_θ^n-asymptotically equivalent of order $\tilde{\rho}$ if, for all $\tau > 0$,

$$\left\{(\mathrm{q}, \mathrm{y}) \in \mathbb{N}^2 : \frac{1}{\mathfrak{h}_{\mathrm{q}}^c \bar{\mathfrak{h}}_{\mathrm{y}}^d} \sum_{(\sigma,\omega) \in I_{\mathrm{q},\mathrm{y}}} \left| \frac{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathrm{H}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma \right| \ge \tau \right\} \in \mathfrak{T}_2.$$

This concept is expressed as $\overline{\mathcal{W}} \overset{N_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

Theorem 3.1 Let $\tilde{\rho}, \tilde{\lambda} \in (0,1]$ satisfy $\tilde{\rho} \preccurlyeq \tilde{\lambda}$. Under this condition, if $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho})}{\sim} \overline{\mathcal{Z}}$, it follows that $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\lambda})}{\sim} \overline{\mathcal{Z}}$.

Proof. Omitted.

Theorem 3.2 Let $\theta = \{(u_{\mathrm{q}}, v_{\mathrm{y}})\}$ represent a double lacunary sequence, and let $\tilde{\rho} \in (0,1]$. If

$$\liminf \mathrm{q}_{\mathrm{q},\mathrm{y}}^{\tilde{\rho}} > 1 (\liminf \mathrm{q}_{\mathrm{q}}^c > 1 \text{ and } \liminf \bar{\mathrm{q}}_{\mathrm{y}}^d > 1),$$

then $\overline{\mathcal{W}} \overset{\mathrm{S}^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho})}{\sim} \overline{\mathcal{Z}}$ implies $\overline{\mathcal{W}} \overset{\mathrm{S}_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{H}}), \tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

Proof. Suppose that $\liminf \mathrm{q}_{\mathrm{q}}^c > 1$ and $\liminf \bar{\mathrm{q}}_{\mathrm{y}}^d > 1$, say $\liminf \mathrm{q}_{\mathrm{q}}^c = c_1$ and $\liminf \mathrm{q}_{\mathrm{y}}^d = d_1$ and define $c_2 = \frac{(c_1 - 1)}{2}$ and $d_2 = \frac{(d_1 - 1)}{2}$. At that time, there are positive integers q_0 and y_0 such that $\mathrm{q}_{\mathrm{q}}^c > 1 + c_2$ for $\mathrm{q} \ge \mathrm{q}_0$, and $\mathrm{q}_{\mathrm{y}}^d > 1 + d_2$ for $\mathrm{y} \ge \mathrm{y}_0$. Therefore, for $\mathrm{q} \ge \mathrm{q}_0$ and $\mathrm{y} \ge \mathrm{y}_0$, the following supplies:

$$\begin{aligned} \mathfrak{h}_{\mathrm{q}}^c \bar{\mathfrak{h}}_{\mathrm{y}}^d \frac{1}{u_{\mathrm{q}}^c v_{\mathrm{y}}^d} &\ge \left(1 - \left(\frac{u_{\mathrm{q}-1}^c}{u_{\mathrm{q}}^c}\right)\right) \times \left(1 - \left(\frac{v_{\mathrm{y}-1}^d}{v_{\mathrm{y}}^d}\right)\right) \\ &= \left(1 - \frac{1}{q_{\mathrm{q}}^c}\right) \times \left(1 - \frac{1}{\bar{q}_{\mathrm{y}}^d}\right) \\ &\ge \left(1 - \frac{1}{1 + c_2}\right) \times \left(1 - \frac{1}{1 + d_2}\right) \\ &= \frac{c_2}{1 + c_2} \times \frac{d_2}{1 + d_2}. \end{aligned}$$

Since $\overline{\mathcal{W}} \overset{S^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathrm{\natural}}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$, we can express the following:

$$\frac{1}{u_q^c v_\eta^d}\left|\left\{(\sigma,\omega):\sigma\le u_q \text{ and } \omega\le v_\eta, \left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|$$
$$\ge \frac{1}{u_q^c v_\eta^d}\left|\left\{(\sigma,\omega)\in I_{q,\eta}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|$$
$$=\mathfrak{h}_q^c\bar{\mathfrak{h}}_\eta^d\frac{1}{u_q^c v_\eta^d}\frac{1}{\mathfrak{h}_q^c\bar{\mathfrak{h}}_\eta^d}\left|\left\{(\sigma,\omega)\in I_{q,\eta}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|$$
$$\ge\left(\frac{c_2}{1+c_2}\times\frac{d_2}{1+d_2}\right)\frac{1}{\mathfrak{h}_q^c\bar{\mathfrak{h}}_\eta^d}\left|\left\{(\sigma,\omega)\in I_{q,\eta}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|.$$

Additionally, we derive

$$\left\{(q,\eta)\in\mathbb{N}^2:\frac{1}{\mathfrak{h}_q^c\bar{\mathfrak{h}}_\eta^d}\left|\left\{(\sigma,\omega)\in I_{q,\eta}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|\ge\gamma\right\}$$
$$\subseteq\left\{(q,\eta)\in\mathbb{N}^2:\frac{1}{u_q^c v_\eta^d}\left|\left\{(\sigma,\omega):\sigma\le u_q \text{ and } \omega\le v_\eta:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|\ge\gamma\left(\frac{c_2}{1+c_2}\times\frac{d_2}{1+d_2}\right)\right\}\in\mathfrak{T}_2,$$

for all $\tau,\gamma>0$. Thus, we conclude $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

The following results are presented without including their proofs.

Theorem 3.3 Let $\theta=\{(u_q,v_\eta)\}$ be a double lacunary sequence, with $\tilde{\rho}\in(0,1]$ and

$$\limsup q_{q,\eta}^{\tilde{\rho}} \prec \infty \quad \left(\limsup q_q^c <\infty \text{ and } \limsup \bar{q}_\eta^d<\infty\right).$$

Then, if $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$, it follows that $\overline{\mathcal{W}} \overset{S^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

Theorem 3.4. If $\overline{\mathcal{W}} \overset{N_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$, then $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$.

Definition 3.5 ([3]) A double index sequence $\theta'=\{(u'_q,v'_\eta)\}$ is referred to as a double lacunary refinement of the sequence $\theta=\{(u_q,v_\eta)\}$ when $\{(u_q,v_\eta)\}\subset\{(u'_q,v'_\eta)\}$.

Theorem 3.5 Assume that θ' is a double lacunary refinement of the double lacunary sequence θ and let $I_{q,\eta}\subset I'_{q,\eta}$. If there exists a constant $\tau>0$ such that for every $I'_{\rho,\lambda}\subseteq I_{q,\eta}$ and $\tilde{\rho},\tilde{\lambda}\in(0,1]$,

$$\frac{\left|I'_{\rho,\lambda}\right|^{\tilde{\lambda}}}{\left|I_{q,\eta}\right|^{\tilde{\rho}}}\ge\tau$$

holds, then if $\overline{\mathcal{W}} \overset{S_\theta^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\rho})}{\sim} \overline{\mathcal{Z}}$ it follows that $\overline{\mathcal{W}} \overset{S_{\theta'}^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\natural}),\tilde{\lambda})}{\sim} \overline{\mathcal{Z}}$.

Proof. For all $\tau>0$ and each $I'_{\rho,\lambda}$, there is an $I_{q,\eta}$ such that $I'_{\rho,\lambda}\subseteq I_{q,\eta}$, In this case, we have the following:

$$\frac{1}{\left|I'_{\rho,\lambda}\right|^{\tilde{\lambda}}}\left|\left\{(\sigma,\omega)\in I'_{\rho,\lambda}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|$$
$$=\left(\frac{\left|I_{q,\eta}\right|^{\tilde{\rho}}}{\left|I'_{\rho,\lambda}\right|^{\tilde{\lambda}}}\right)\times\left(\frac{1}{\left|I_{q,\eta}\right|^{\tilde{\rho}}}\right)\times\left|\left\{(\sigma,\omega)\in I'_{\rho,\lambda}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|$$
$$\le\left(\frac{\left|I_{q,\eta}\right|^{\tilde{\rho}}}{\left|I'_{\rho,\lambda}\right|^{\tilde{\lambda}}}\right)\times\left(\frac{1}{\left|I_{q,\eta}\right|^{\tilde{\rho}}}\right)\times\left|\left\{(\sigma,\omega)\in I_{q,\eta}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|$$
$$\le\left(\frac{1}{\tau}\right)\times\left(\frac{1}{\left|I_{q,\eta}\right|^{\tilde{\rho}}}\right)\times\left|\left\{(\sigma,\omega)\in I_{q,\eta}:\left|\frac{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\natural}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\ge\tau\right\}\right|,$$

and so

$$\left\{(\rho,\lambda)\in\mathbb{N}^2:\frac{1}{|I'_{\rho,\lambda}|^{\tilde{\lambda}}}\left|\left\{(\sigma,\omega)\in I'_{\rho,\lambda}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\right|\geq\gamma\right\}\subseteq\left\{(\mathfrak{q},\mathfrak{y})\in\mathbb{N}^2:\frac{1}{|I_{\mathfrak{q},\mathfrak{y}}|^{\tilde{\rho}}}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\right|\geq\tau.\gamma\right\}\in\mathfrak{T}_2.$$

Therefore, we obtain the desired result.

Theorem 3.6 Consider double lacunary sequences θ and θ'. Define $I_{\mathfrak{q},\mathfrak{y},\rho,\lambda}=I_{\mathfrak{q},\mathfrak{y}}\cap I'_{\rho,\lambda}(\mathfrak{q},\mathfrak{y},\rho,\lambda=1,2,3,\dots)$. If there is a $\tau>0$ such that

$$\frac{|I_{\mathfrak{q},\mathfrak{y},\rho,\lambda}|^{\tilde{\lambda}}}{|I_{\mathfrak{q},\mathfrak{y}}|^{\tilde{\rho}}}\geq\tau \text{ for all } \tilde{\rho},\tilde{\lambda}\in(0,1] \text{ provided } I_{\mathfrak{q},\mathfrak{y},\rho,\lambda}\neq\emptyset,$$

then the relation $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$ leads to $\overline{\mathcal{W}}\overset{S_{\theta'}^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$.

Theorem 3.7. Let $\tilde{\rho},\tilde{\lambda}\in(0,1]$ be given such that $\tilde{\rho}\preccurlyeq\tilde{\lambda}$ and $\theta=\{(u_{\mathfrak{q}},v_{\mathfrak{y}})\}$ and $\theta'=\{(u'_{\mathfrak{q}},v'_{\mathfrak{y}})\}$ be two double lacunary sequences such that $I_{\mathfrak{q},\mathfrak{y}}\subset I'_{\mathfrak{q},\mathfrak{y}}$

(i) If

$$\liminf_{\mathfrak{q},\mathfrak{y}}\frac{h_{\mathfrak{q}}^c\bar{h}_{\mathfrak{y}}^d}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}>0, \tag{1}$$

then $\overline{\mathcal{W}}\overset{S_{\theta'}^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$ implies $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$,

(ii) If

$$\lim_{\mathfrak{q},\mathfrak{y}}\frac{h'_{\mathfrak{q}}\bar{h}'_{\mathfrak{y}}}{h_{\mathfrak{q}}^{t_1}\bar{h}_{\mathfrak{y}}^{t_2}}=0, \tag{2}$$

then $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$ implies $\overline{\mathcal{W}}\overset{S_{\theta'}^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$.

Proof.

(i) Suppose that $I_{\mathfrak{q},\mathfrak{y}}\subset I'_{\mathfrak{q},\mathfrak{y}}$ for each $\mathfrak{q},\mathfrak{y}\in\mathbb{N}$ and that condition (1) holds. For a given $\tau>0$, we obtain

$$\left\{(\sigma,\omega)\in I'_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\supseteq\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}$$

and so

$$\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I'_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\right|\geq\frac{h_{\mathfrak{q}}^c\bar{h}_{\mathfrak{y}}^d}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\frac{1}{h_{\mathfrak{q}}^c\bar{h}_{\mathfrak{y}}^d}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\right|,$$

for all $\mathfrak{q},\mathfrak{y}\in\mathbb{N}$. Then, we can write

$$\left\{(\mathfrak{q},\mathfrak{y})\in\mathbb{N}^2:\frac{1}{\mathfrak{h}_{\mathfrak{q}}^c\bar{\mathfrak{h}}_{\mathfrak{y}}^d}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\right|\geq\gamma\right\}\subseteq\left\{(\mathfrak{q},\mathfrak{y})\in\mathbb{N}^2:\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I'_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}_{\sigma\omega}})}{\Delta^n\mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}_{\sigma\omega}})}-\varsigma\right|\geq\tau\right\}\right|\geq\gamma\frac{h_{\mathfrak{q}}^c\bar{h}_{\mathfrak{y}}^d}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\right\}\in\mathfrak{T}_2,$$

for every $\tau,\gamma>0$. Hence, $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$.

(ii) Let $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\mathfrak{n}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$ and condition (2) holds. Since $I_{\mathfrak{q},\mathfrak{y}}\subset I'_{\mathfrak{q},\mathfrak{y}}$, for $\tau>0$, we can express:

$$
\begin{aligned}
&\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I'_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&=\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{u'_{\mathfrak{q}-1}<\sigma\leq u_{\mathfrak{q}-1}\wedge v'_{\mathfrak{y}-1}<\omega\leq v_{\mathfrak{y}-1}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&+\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{u_{\mathfrak{q}}<\sigma\leq u'_{\mathfrak{q}}\wedge v_{\mathfrak{y}}<\omega\leq v'_{\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&+\frac{1}{h'^{t_1}_{r}\bar{h}'^{t_2}_{s}}\left|\left\{u_{\mathfrak{q}-1}<\sigma\leq u_{\mathfrak{q}}\wedge v_{\mathfrak{y}-1}<\omega\leq v_{\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&\leq\frac{(u_{\mathfrak{q}-1}-u'_{\mathfrak{q}-1})(v_{\mathfrak{y}-1}-v'_{\mathfrak{y}-1})}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}+\frac{(u'_{\mathfrak{q}}-u_{\mathfrak{q}})(v'_{\mathfrak{y}}-v_{\mathfrak{y}})}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}} \\
&+\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&\leq\frac{(u'_{\mathfrak{q}}-u'_{\mathfrak{q}-1})(v'_{\mathfrak{y}}-v'_{\mathfrak{y}-1})}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}+\frac{(u'_{\mathfrak{q}}-u'_{\mathfrak{q}-1})(v'_{\mathfrak{y}}-v'_{\mathfrak{y}-1})}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}} \\
&+\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&=\frac{2h'_{\mathfrak{q}}\bar{h}'_{\mathfrak{y}}}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}+\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right| \\
&\leq\frac{2h'_{\mathfrak{q}}\bar{h}'_{\mathfrak{y}}}{h^{t_1}_{\mathfrak{q}}\bar{h}^{t_2}_{\mathfrak{y}}}+\frac{1}{h^{c}_{\mathfrak{q}}\bar{h}^{d}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right|,
\end{aligned}
$$

for each $\tau\in\mathbb{N}$. Then, we can express

$$
\begin{aligned}
&\left\{(\mathfrak{q},\mathfrak{y})\in\mathbb{N}^2:\frac{1}{h'^{t_1}_{\mathfrak{q}}\bar{h}'^{t_2}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I'_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right|\geq\gamma\right\} \\
&\subseteq\left\{(\mathfrak{q},\mathfrak{y})\in\mathbb{N}^2:\frac{1}{\mathfrak{h}^{c}_{\mathfrak{q}}\mathfrak{h}^{d}_{\mathfrak{y}}}\left|\left\{(\sigma,\omega)\in I_{\mathfrak{q},\mathfrak{y}}:\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\varsigma\right|\geq\tau\right\}\right|\geq\gamma\right\}\in\mathfrak{T}_2.
\end{aligned}
$$

At that time, $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$ leads to $\overline{\mathcal{W}}\overset{S_{\theta'}^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$.

Theorem 3.8. Let $\tilde{\rho},\tilde{\lambda}\in(0,1]$ be given such that $\tilde{\rho}\preccurlyeq\tilde{\lambda}$ and $\theta=\{(u_{\mathfrak{q}},v_{\mathfrak{y}})\}$ and $\theta'=\{(u'_{\mathfrak{q}},v'_{\mathfrak{y}})\}$ be two double lacunary sequences such that $I_{\mathfrak{q},\mathfrak{y}}\subset I'_{\mathfrak{q},\mathfrak{y}}$.

(i) If (1) holds, then $\overline{\mathcal{W}}\overset{N_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$ implies $\overline{\mathcal{W}}\overset{N_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$,

(ii) If (2) holds and $\overline{\mathcal{W}},\overline{\mathcal{Z}}\in\ell_\infty^2(\Delta^n\mathcal{J}_t^{\text{ɥ}})$, then $\overline{\mathcal{W}}\overset{N_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$ implies $\overline{\mathcal{W}}\overset{N_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$, where

$$\ell_\infty^2(\Delta^n\mathcal{J}_t^{\text{ɥ}})=\left\{\overline{\mathcal{W}}=(\overline{\mathcal{W}}_{\sigma\omega}):\sup_{\sigma,\omega}\left|\Delta^n\overline{\mathcal{W}}_{\sigma\omega}\right|<\infty\right\}.$$

Proof. Omitted.

Theorem 3.9. Let $\tilde{\rho},\tilde{\lambda}\in(0,1]$ be given such that $\tilde{\rho}\preccurlyeq\tilde{\lambda}$ and $\theta=\{(u_{\mathfrak{q}},v_{\mathfrak{y}})\}$ and $\theta'=\{(u'_{\mathfrak{q}},v'_{\mathfrak{y}})\}$ be two double lacunary sequences such that $I_{\mathfrak{q},\mathfrak{y}}\subset I'_{\mathfrak{q},\mathfrak{y}}$.

(i) Let (1) hold, if $\overline{\mathcal{W}}\overset{N_{\theta'}^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$ then $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$,

(ii) Let (2) supply and $\overline{\mathcal{W}}\in\ell_\infty^2(\Delta^n\mathcal{J}_t^{\text{ɥ}})$, if $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$, then $\overline{\mathcal{W}}\overset{N_{\theta'}^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\lambda})}{\sim}\overline{\mathcal{Z}}$.

Proof.

(i) Omitted.

(ii) Suppose that $\overline{\mathcal{W}}\overset{S_\theta^\varsigma(\mathfrak{T}_2(\Delta^n\mathcal{J}_t^{\text{ɥ}}),\tilde{\rho})}{\sim}\overline{\mathcal{Z}}$ and $\overline{\mathcal{W}},\overline{\mathcal{Z}}\in\ell_\infty^2(\Delta^n\mathcal{J}_t^{\text{ɥ}})$. Then, there exists a constant $P>0$ such that $\left|\frac{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n\mathcal{J}_t^{\text{ɥ}}(\overline{\mathcal{Z}}_{\sigma\omega})}-\right.$

$\varsigma\Big| \le P$ for all $\sigma, \omega \in \mathbb{N}$. Consequently, for any $\tau > 0$, we obtain

$$\frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I'_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right|$$
$$= \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I'_{q,\mathfrak{y}}-I_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| + \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right|$$
$$\le \left(\frac{\left(h'_{q} - h_{q}\right)\left(\bar{h}'_{\mathfrak{y}} - \bar{h}_{\mathfrak{y}}\right)}{h'^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}}\right) P + \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_w^{m}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_w^{m}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right|$$
$$= \left(\frac{h'_{q}\bar{h}'_{\mathfrak{y}} + h_{q}\bar{h}_{\mathfrak{y}} - h'_{q}\bar{h}_{\mathfrak{y}} - h_{q}\bar{h}'_{\mathfrak{y}}}{h'^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}}\right) P + \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right|$$
$$\le \left(\frac{2h'_{q}\bar{h}'_{\mathfrak{y}}}{h'^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}} + \frac{2h^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}}{h^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}}\right) P + \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right|$$
$$\le \left(\frac{h'_{q}\bar{h}'_{\mathfrak{y}}}{h'^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}} + 1\right) 2P + \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{\substack{(\sigma,\omega)\in I_{q,\mathfrak{y}} \\ \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| \ge \tau}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| + \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{\substack{(\sigma,\omega)\in I_{q,\mathfrak{y}} \\ \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| < \tau}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right|$$
$$\le \left(\frac{h'_{q}\bar{h}'_{\mathfrak{y}}}{h'^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}} + 1\right) 2P + \frac{P}{h^{c}_{q}\bar{h}^{d}_{\mathfrak{y}}} \left|\left\{(\sigma,\omega) \in I_{q,\mathfrak{y}} : \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| \ge \tau\right\}\right| + \frac{h'_{q}\bar{h}'_{\mathfrak{y}}}{h'^{t_1}_{q}\bar{h}^{t_2}_{\mathfrak{y}}}\tau,$$

and so

$$\left\{(q,\mathfrak{y}) \in \mathbb{N}^2 : \frac{1}{h'^{t_1}_{q}\bar{h}'^{t_2}_{\mathfrak{y}}} \sum_{(\sigma,\omega)\in I'_{q,\mathfrak{y}}} \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| \ge \tau\right\} \subseteq \left\{(q,\mathfrak{y}) \in \mathbb{N}^2 : \frac{1}{\mathfrak{h}^{c}_{q}\bar{\mathfrak{h}}^{d}_{\mathfrak{y}}} \left|\left\{(\sigma,\omega) \in I_{q,\mathfrak{y}} : \left|\frac{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{W}}_{\sigma\omega})}{\Delta^n \mathcal{J}_t^{\mathfrak{n}}(\overline{\mathcal{Z}}_{\sigma\omega})} - \varsigma\right| \ge \tau\right\}\right| \ge \frac{\gamma}{P}\right\} \in \mathfrak{T}_2,$$

for every $\tau, \gamma > 0$. Hence, we have $\overline{\mathcal{W}} \overset{N_{\theta'}^{\varsigma}(\mathfrak{T}_2(\Delta^n \mathcal{J}_t^{\mathfrak{n}}),\tilde{\lambda})}{\sim} \overline{\mathcal{Z}}$.

4. Discussion and Conclusion

This study makes a significant contribution to the field of sequence spaces and summability theory by introducing the concepts of Bessel $\mathfrak{T}_2$ - Δ_θ^n -asymptotically statistical equivalence of order $\tilde{\rho}$ and strong Bessel $\mathfrak{T}_2$-Δ_θ^n-asymptotically equivalence of order $\tilde{\rho}$ for double sequences. The integration of Bessel functions into these frameworks provides a fresh perspective on the behavior of sequences and addresses a notable gap in the existing literature. The results obtained here contribute to the advancement of summability theory and open up new opportunities for further research and applications in mathematical analysis.

Declaration of Ethical Code

In this study, we hereby commit that all necessary rules outlined in the "Regulation on Scientific Research and Publication Ethics in Higher Education Institutions" have been followed, and none of the actions listed under the section "Actions Contrary to Scientific Research and Publication Ethics" in the mentioned regulation have been carried out.

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