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Global Boundedness and Mass Persistence of Solutions to A Chemotaxis-Competition System with Logistic Source

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Abstract: This article examines the population dynamics of solutions such as global existence, global boundedness, and mass persistence, to a parabolic elliptic type of chemotaxis-competition system including logistics kinetics under no-flux boundary conditions in a smoothly bounded domain. Tello and Winkler were the first to investigate the global existence and global boundedness of the system mentioned above. Then Tao and Winkler examined qualitative properties of the given system such as the mass persistence of solutions. This study improves some known results and reveals that under some suitable conditions, there exists a classical solution to the system described above that is globally bounded. In addition, it is shown that the population as a whole is never extinct. In other words, any globally defined positive solution eventually persists in mass from below and is bounded in pointwise above by certain positive constants. We wanted to highlight that the methods and techniques utilized in this article are completely different from the approach used in the previous research.

Bir Lojistik Kemotaksis-Rekabet Sisteminin Çözümlerinin Küresel Sınırlılığı ve Kitlesel Kalıcılığı

Anahtar Kelimeler
Lineer hassasiyet,
Lojistik kaynak,
Global varlık,
Global sınırlılık,
Kitlesel kalıcılık

Öz: Bu makale, düzgün sınırlı bir alanda akısız sınır koşulları altında lojistik kinetik içeren parabolik eliptik tipte bir kemotaksi-rekabet sistemine ait global varlık, global sınırlılık ve kütlesel kalıcılığı gibi çözümlerin popülasyon dinamiklerini incelemektedir. Tello ve Winkler'in ilk olarak yukarıda belirtilen sistemin küresel varlığını ve sınırlılığını incelemiştir. Daha sonra Tao ve Winkler, verilen sistemin çözümlerin kütlesel kalıcılığı gibi dinamik özelliklerini araştırmışlardır. Bu çalışmada, bilinen bazı sonuçlar geliştirilmiştir ve uygun koşullar altında sistemin küresel olarak var ve sınırlı olan tek bir klasik çözüme sahip olduğunu göstermiştir. Buna ilave olarak, popülasyonun bir bütün olarak asla yok olmadığı gösterilmiştir. Başka bir deyişle, küresel olarak tanımlanmış herhangi bir pozitif çözüm er yada geç belirli birer pozitif sabitlerle noktasal olrak üstten ve kütlesel olarak alttan sınırlandırılır. Bu makalede kullanılan yöntem ve tekniklerin, daha önceki araştırmalarda kullanılan yaklaşımdan tamamen farklıdır.

1. Introduction

The term chemotaxis depicts the motion of living organisms or mobile cells in return for a chemical gradient. Keller and Segel in [1, 2] presented this phenomenon in the 1970's and it plays an important role in many biological circumstances, for instance, immune system response, gravitational collapse, tumor growth, population dynamics, and immune cell migration, etc. In these models, the density of cells or creatures correlates with the concentration of the chemical, leading them to migrate towards areas of elevated chemical concentrations or away from areas of heightened chemical concentrations. Keller-Segel type chemotaxis models are typically expressed by partial differential equations (PDEs), which delineate the spatiotemporal evolution of cell density and chemical concentration. Through the interplay of processes such as diffusion and chemotaxis, these equations provide a framework for comprehending the intricate dynamics of chemotactic systems.

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Core challenges in chemotaxis models determines whether solutions blows-up in finite-time or exist globally. The following question is whether these solutions are bounded provided that solutions are exist. Furthermore, if they are ultimately bounded, it is crucial to examine the long-term dynamics of bounded solutions, including persistence, stability, etc. In this context, a ton of research have been performed on the dynamical properties of many chemotaxis systems, for example the investigation of local existence, uniqueness, global existence, global boundedness, a finite time blow-up, persistence and asymptotic stability, etc. We refer the reader to explore the articles [3, 4, 5] for more details.

The following logistic chemotaxis model will be investigated in this paper:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + hu - ku^2, \\ 0 = \Delta v - av + bu, \end{cases} \tag{1}$$

for $x\in\Omega$, $t>0$, along with the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

and the initial conditions $u_0(x) := u(0,x;u_0)$ that satisfies

$$u_0 \in C^0(\bar{\Omega}) \quad and \quad u_0 \geq 0, \tag{3}$$

as well as $\Omega\subset\mathbb{R}^N$ with $N\geq 1$ is a smooth domain and $\chi, a, b, h, k > 0$ are positive numbers.

The biological interpretation of model (1) describes the mechanism of cellular movement, where mobile cells migrate towards regions with a higher concentration of chemical substance. The chemotaxis term $-\chi\nabla\cdot(u\nabla v)$ depicts the impact of chemotactic migration, and the parameter $h>0$ denotes the growth rate of cells; $k>0$ indicates self-limitation of the cells; $a>0$ reflects the rate at which degradation occurs of the chemical substance; and $b>0$ represents the pace at which production occurs of the mobile cells.

We remark that model (1) has been analyzed in many research works up to now, and many results have been obtained in the existing literature. When $N\geq 2$ and $h=k=0$, it was shown that a finite-time blows-up occurs in solutions of (1) under some restriction on the initial condition, look at [6, 7, 8, 9] for more details. When $a=b=1$ and $h,k>0$, Tello and Winkler in [10] demonstrated that for any suitable nonnegative initial function, (1) possesses a global bounded solution if

$$N \leq 2 \ \text{ or } N \geq 3 \text{ whenever } \ \chi < \frac{kN}{N-2}. \tag{4}$$

The long-term behaviors of solutions has also been examined in many research paper, for example, the readers are referred to the research papers [11, 12, 13, 14, 15, 16, 17, 18] for the other studies including weak solutions, boundedness, and stability of positive constant solution, etc.

For reader intersrts, we present the following parabolic-elliptic chemotaxis system with a logistics source and singular sensitivity:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot\left(\frac{u}{v^m}\nabla v\right) + hu - ku^2, & x\in\Omega, \\ \quad -\Delta v + av = bu, & x\in\Omega. \end{cases}$$

When $n=2$ and $m=1$, Fujue, Winkler and Yokota in [19] showed that any finite-time blow-up was not observed in the above system and all global solutions are globally bounded provided that

$$h > \begin{cases} \frac{a\chi^2}{4}, & \text{for } \ 0<\chi\leq 2 \\ a(\chi-1), & \text{for } \ \chi>2. \end{cases}$$

Then, this result was generalized by the authors Kurt and Shen in [20] to the all space dimensions, and they proved that any positive classical solutions exists globally and stays bounded under some parameter conditions between a,χ and the initial function u_0.

When $n=2$ and $m\in(0,1)$, Zhao in [21] proved that mentioned system admits a globally bounded classical solution with k being large enough. Recently, Kurt in [22] demostrated that there is a global classical solution for sufficiently large k. Moreover, the global boundedness are given under the additional restriction

$$m < \frac{1}{2} + \frac{1}{n} \quad \text{with} \quad n\geq 2.$$

For additional variants of the aforementioned model, concerning the long-term properties of nonnegative solutions including the problems such as uniform boundedness, persistence, stability, entire and periodic solutions and more, we recommend the reader to explore the research articles [23, 24, 25, 26, 27, 28, 29] for further reading.

Our motivation to investigate the problems discussed in this paper is outlined as follows. It is well known that [10, Lemma 2.4] proves that if (4) holds, then model (1) admits a classical solution that is globally bounded, i.e.,

$$\sup \| u(t,\cdot) \|_{L^\infty(\Omega)} < \infty, \qquad \forall t>0. \tag{5}$$

This result was improved by Hu and Tao in [12, Theorem 1.1], and they obtained the same result even at the critical point. Furthermore, Tao and Winkler in [30, Theorem 1.1] studied the mass persistence of solutions and they proved that when $\Omega\subset\mathbb{R}^N$ with $N\geq 1$ is convex, all positive solutions to model (1) always persists as a whole, in other words, for every given nonnegative global classical solution (u,v) of model (1), if Ω is a convex domain and (4) is valid, then for some $k_0>0$, we have

$$\int_{\Omega} u(t,x)dx \geq k_0 \quad \text{for all} \quad t > 0. \tag{6}$$

However, it still remains open whether (6) is true when the case $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is not convex. As far as our knowledge, there has not been carried out any research on this problem yet. This papers aims to answer to this question.

1.1 Fundamental results

We provide our results in this subsection. The generic constant C is not dependent on solutions, and may vary in value at different places. Moreover, we assume that

$$N \leq 2 \text{ or } \chi \leq \frac{k}{b} \cdot \frac{N}{N-2} \text{ with } N \geq 3. \tag{7}$$

The first result is on the global L^p-norm of u.

Theorem 1 (L^p-boundedness). Suppose that (3) and (7) are valid. Then

$$\int_{\Omega} u^p(t,x;u_0)dx \leq C,$$

for all $t \in (0, T_{\max})$.

The second result is on the global existence & boundedness of solutions.

Theorem 2. Suppose that (3) and (7) hold.

- (Global existence) The solution (u, v) is global, which means,

$$T_{\max}(u_0) = \infty.$$

- (Global boundedness) There is $C_\infty > 0$ such that

$$\| u \|_{L^\infty(\Omega)} \leq C_\infty \quad \forall\, t > 0.$$

The last result is on the mass persistence of solutions.

Theorem 3 (Mass persistence). Suppose that (3) and (7) are valid. Then

$$\int_{\Omega} u(t,x;u_0)dx \geq \delta_* \quad \forall\, \mathrm{t} > t_0 > 0,$$

for some constant $\delta_* > 0$.

We organized the remain part of this paper as below. Section 2 is dedicated to present some key estimates and some well known formulas and estimates. Section 3 is devoted to the analysis of the L^p-norm of u, the global existence and global boundedness of solutions to (1). Additionally, we will investigate mass persistence of globally bounded solutions of (1). Then we discuss and compare our results to previous results obtained in the previous research papers in the literature in Section 4.

2. Material and Method

This section is dedicated to present some elementary lemmas.

Lemma 1. Assume that $\mathrm{u_0}$ satisfies (3). Then there is $\mathrm{T_{max}(u_0)} \in (0, \infty]$ such that (1) admits a unique classical solution, which is symbolized by $(\mathrm{u(t,x;u_0)}, \mathrm{v(t,x;u_0)})$, on $(0, \mathrm{T_{max}(u_0)})$ with $\mathrm{u(0,x;u_0) = u_0(x)}$ and $u \in C^{2,1}((0,T_{\max}) \times \bar{\Omega}) \cap \mathrm{C}((0,T_{\max}) \times \bar{\Omega})$, and $v \in C^{2,0}\big((0,T_{\max}) \times \bar{\Omega}\big)$. Moreover, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} \|u(t,\cdot)\|_{C^0(\bar{\Omega})} = \infty.$$

Proof. The proof can be obtained from the similar arguments of [10, Theorem 2.1].

Second, suppose $1 < p < \infty$ and let $X_p = L^p(\Omega)$ and $-\Delta + aI : D(-\Delta + aI) \subset L^p(\Omega) \to L^p(\Omega)$ with $D(-\Delta + aI) = \left\{u \in W^{2,p}(\Omega) \mid \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega\right\}$. Assume that $(aI - \Delta)^\beta$ is the fractional power operator of $(aI - \Delta)$ and let $X_p^\beta = \mathcal{D}((aI-\Delta)^\beta)$ with graph norm

$$\| u \|_{X_p^\beta} = \| (aI - \Delta)^\beta u \|_{L^p(\Omega)}$$

for $\beta \geq 0$ and $u \in X_p^\beta$. See [31, Definitions 1.4.1 and 1.4.7] for more details.

Lemma 2. Assume that $1 < p < \infty$.

- If $2\beta - \frac{N}{p} > \theta \geq 0$, then $X_p^\beta \hookrightarrow C^\theta(\Omega)$.

- Assume $\varphi \in C^1(\bar{\Omega})$ satisfying $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$. Then there are constants $\zeta, C_p > 0$ such that

$$\| e^{-(aI-\Delta)t} \nabla \cdot \varphi \|_{L^p(\Omega)} \leq C_p \left(1 + t^{-\frac{1}{2}}\right) e^{-\zeta t} \| \varphi \|_{L^p(\Omega)}$$

for every $t > 0$.

Proof. (1) might be obtained from [6, Theorem 1.6.1]. (2) follows from [32, Lemma 1.3].

Lemma 3. It holds that

$$\int_{\Omega} u \leq m_0 := max\left\{\frac{h}{k}|\Omega|, \int_{\Omega} u_0\right\}$$

for every $t \in (0, T_{\max})$, where $|\Omega|$ is the Lebesgue measure of Ω.

Proof. Integrating (1) and employing Hölder's inequality entails

$$\frac{d}{dt}\int_\Omega u = h\int_\Omega u - k\int_\Omega u^2 \le h\int_\Omega u - \frac{k}{|\Omega|}\left(\int_\Omega u\right)^2$$

for every $t \in (0, T_{\max})$. Then ODE's comparison principle completes the proof.

3. Results

We provide our fundamental results throughout this section.

3.1. L^p-boundedness

This section examines the L^p-norm of u. We first give some estimates.

Lemma 4. For all $m \in \mathbb{R}$, we have

$$m\int_\Omega u^{m-1}\nabla u\cdot\nabla v + a\int_\Omega vu^m = b\int_\Omega u^{m+1},$$

for every $t \in (0, T_{\max})$.

Proof. Let us multiply the equation (1) by u^{m-1} and integrating by parts over Ω. Then

$$\int_\Omega u^{m-1}\cdot(\Delta v - av + bu) = 0$$

for all $t \in (0, T_{\max})$. The proof thus holds.

Lemma 5. For given $q \in \left(1, \frac{\chi b}{(\chi b - k)_+}\right)$, there exists $C_q > 0$ such that

$$\int_\Omega u^q(t,x;u_0)dx \le C_q, \tag{8}$$

for all $t \in (0, T_{\max})$.

Proof. First of all, we remark that if $k \ge \chi b$, then (8) naturally follows. Therefore, we assume that $k < \chi b$. For every given $\chi, b, k > 0$, let us define $q > 1$ such that

$$q := 1 + \frac{k}{\chi b - k}. \tag{9}$$

Next, let us multiply (1) by u^{q-1} and integrating over Ω. Then

$$\frac{1}{q}\cdot\frac{d}{dt}\int_\Omega u^q = \int_\Omega u^{q-1}\Delta u - \chi\int_\Omega u^{q-1}\nabla\cdot(u\nabla v) + h\int_\Omega u^q - k\int_\Omega u^{q+1}$$

$$= -(q-1)\int_\Omega u^{q-2}|\nabla u|^2$$

$$+(q-1)\chi\int_\Omega u^{q-1}\nabla u\cdot\nabla v + h\int_\Omega u^q - k\int_\Omega u^{q+1}, \tag{10}$$

for $t \in (0, T_{\max})$. Note that letting $m = q$ in Lemma 4 together with (9) yields that

$$(q-1)\chi\int_\Omega u^{q-1}\nabla u\cdot\nabla v \le \frac{(q-1)\chi b}{q}\int_\Omega u^{q+1}$$

$$= k\int_\Omega u^{q+1}. \tag{11}$$

Note also that, by the Erhling type lemma, for given $\varepsilon > 0$, we find $C(\varepsilon) > 0$ such that

$$\int_\Omega u^q \le \frac{4\varepsilon}{q^2}\int_\Omega |\nabla u^{\frac{q}{2}}|^2 + C(\varepsilon,q)(\int_\Omega u)^q$$

$$\le \varepsilon\int_\Omega u^{q-2}|\nabla u|^2 + C(\varepsilon,q),$$

which, by Lemma 3, implies

$$-(q-1)\int_\Omega u^{q-2}|\nabla u|^2 \le -\left(\frac{1}{q}+h\right)\int_\Omega u^q + C(h,q)(m_0)^q, \tag{12}$$

for all $t \in (0, T_{\max})$. Collecting from (10) to (12), we obtain

$$\frac{d}{dt}\int_\Omega u^q \le -\int_\Omega u^q + C(h,q)(m_0)^q$$

for all $t \in (0, T_{\max})$. The ODE's comparison principle yields

$$\int_\Omega u^q(t,x;u_0)dx \le \max\left\{\int_\Omega u_0^q(x)dx, 2C(h,q)(m_0)^q\right\}$$

for all $t \in (0, T_{\max})$. The lemma thus follows.

Proof of Theorem 1. First, since $\chi \le \frac{k}{b}\cdot\frac{N}{N-2}$, Lemma 5 guarantees that there are some $q > \frac{N}{2}$ such that

$$\int_\Omega u^q(t,x)dx \le K_q \quad \text{for all } t \in (0, T_{\max}).$$

Next, in view of the Gagliardo-Nirenberg embedding theorem, Young's inequality, we obtain for all given $\varepsilon > 0$, we have

$$\int_\Omega u^{p+1} = \| u^{\frac{p}{2}} \|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}}$$

$$\leq C \parallel \nabla u^{\frac{p}{2}} \parallel_{L^2(\Omega)}^{\frac{2(p+1)\theta}{p}} \parallel u^{\frac{p}{2}} \parallel_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2(p+1)(1-\theta)}{p}}$$

$$+C \parallel u^{\frac{p}{2}} \parallel_{L^{\frac{2q}{p}}(\Omega)}^{\frac{2(p+1)\theta}{p}},$$

$$\leq C\left(\frac{p^2}{4}\int_\Omega u^{p-2}|\nabla u|^2\right)^{\frac{(p+1)\theta}{p}} (K_q)^{\frac{(p+1)(1-\theta)}{q}}$$

$$+C(K_q)^{\frac{(p+1)\theta}{q}}$$

$$\leq \varepsilon \int_\Omega u^{p-2}|\nabla u|^2 + C\big(p,q,\varepsilon,\theta,K_q,|\Omega|\big) \tag{13}$$

for all $t \in (0, T_{\max})$, where

$$\theta = \frac{\frac{p}{2q} - \frac{p}{2(p+1)}}{1 + \frac{p}{2q} - \frac{1}{2}} = \frac{p}{p+1}\cdot\frac{p+1-q}{p+q} \in (0,1)$$

and $\frac{(p+1)\theta}{p} < 1$.

Third, multiplying the equation (1) by u^{p-1} with $p > 1$ and integration by parts over the set Ω, we get that

$$\frac{1}{p}\cdot\frac{d}{dt}\int_\Omega u^p = -(p-1)\int_\Omega u^{p-2}|\nabla u|^2$$

$$+\chi(p-1)\int_\Omega \frac{u^{p-1}}{v^\lambda}\nabla u \cdot \nabla v$$

$$+h\int_\Omega u^p - k\int_\Omega u^{p+1},$$

for all $t \in (0, T_{\max})$. Letting $m = p$ in Lemma 4 yields

$$\chi(p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v \leq \frac{\chi(p-1)}{p}\int_\Omega u^{p+1}, \tag{14}$$

for all $t \in (0, T_{\max})$. Moreover, by Young's inequality, we have

$$\left(h + \frac{1}{p}\right)\int_\Omega u^p \leq k\int_\Omega u^{p+1} + C(h,k,|\Omega|)$$

for all $t \in (0, T_{\max})$. This together (13) and (14) yields that

$$\chi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v + h\int_\Omega u^p - k\int_\Omega u^{p+1}$$
$$\leq \frac{\chi(p-1)}{p}\int_\Omega u^{p+1} + C(h,k,|\Omega|)$$

$$\leq (p-1)\int_\Omega u^{p-2}|\nabla u|^2 - \frac{1}{p}\int_\Omega u^p + C(p,\chi,h,k,|\Omega|),$$

for all $t \in (0, T_{\max})$. We then arrive at

$$\frac{1}{p}\cdot\frac{d}{dt}\int_\Omega u^p \leq -(p-1)\int_\Omega u^{p-2}|\nabla u|^2$$

$$+(p-1)\int_\Omega u^{p-2}|\nabla u|^2$$

$$-\frac{1}{p}\int_\Omega u^p + C(p,\chi,h,k,|\Omega|),$$

which implies

$$\frac{d}{dt}\int_\Omega u^p \leq -\int_\Omega u^p + C(p,\chi,h,k,|\Omega|)$$

for all $t \in (0, T_{\max})$. Let $y(t) := \int_\Omega u^p(t,x)dx$ such that

$$y'(t) + y(t) \leq C(p,\chi,h,k,|\Omega|)$$

for all $t \in (0, T_{\max})$. Then the ODE's comparison principle implies that

$$y(t) \leq \max\left\{\int_\Omega u_0^p(x)dx, 2C(p,\chi,h,k,|\Omega|)\right\}$$

for all $t \in (0, T_{\max})$. The proof is completed.

3.2. Global existence and boundedness

We now prove Theorem 2.

Proof of Theorem 2. We will show $T_{\max} = \infty$ by contradiction. First, let us suppose that $T_{\max} < \infty$. Then by the constant formula, we have

$$u(t,\cdot) = e^{-(aI-\Delta)t}u_0$$

$$-\chi\int_0^t e^{-(aI-\Delta)t}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))ds$$

$$+\int_0^t e^{-(aI-\Delta)t}u(\cdot,s)(a+h-ku(\cdot,s))ds$$

$$=: I_1 + I_2 + I_3 \text{ for all } t \in (0, T_{\max}). \tag{15}$$

We now give the following estimates for I_1, I_2 and I_3. First,

$$\parallel I_1 \parallel_{L^\infty(\Omega)} = \parallel e^{-(aI-\Delta)t}u_0 \parallel_{L^\infty(\Omega)}$$

$$\leq \parallel u_0 \parallel_{L^\infty} \text{ for all } \quad t > 0.$$

Second,

$$\parallel I_3 \parallel_{L^\infty(\Omega)}$$

$$= \int_0^t e^{-(aI-\Delta)t}u(\cdot,s)\big(a+h-ku(\cdot,s)\big)ds$$

$$\leq \frac{(a+h)^2}{4k} \text{ for all } \quad t \in [0, T_{\max}).$$

Third, let $p \geq 1$ and assume that $\frac{N}{2} < p < N < q$ and $\lambda \in (1, \infty)$ such that

$$\frac{1}{p} - \frac{1}{N} < \frac{1}{q} \text{ and } \frac{1}{\lambda} < 1 - q\left(\frac{1}{p} - \frac{1}{N}\right). \tag{16}$$

Hence, employing the Gagliardo-Nirenberg inequality and Theorem 1.1, we have

$$\| \nabla v \|_{L^{\frac{q\lambda}{\lambda-1}}(\Omega)} \leq C \| \nabla v \|_{L^{\frac{Np}{N-p}}(\Omega)} \leq C \| u \|_{L^p(\Omega)} \leq M_p \tag{17}$$

for all $t \in (0, T_{\max})$, thanks to

$$\frac{q\lambda}{\lambda - 1} = \frac{q}{1 - \frac{1}{\lambda}} < \frac{q}{(\frac{1}{p} - \frac{1}{N})q} = \frac{1}{\frac{1}{p} - \frac{1}{N}} = \frac{Np}{N-p}.$$

Then, by Lemma 3, Hölder inequality, (16), and (17), we get

$$\| u\nabla v \|_{L^q(\Omega)} \leq \| u \|_{L^{q\lambda}(\Omega)} \cdot \| \nabla v \|_{L^{\frac{q\lambda}{\lambda-1}}(\Omega)}$$

$$\leq \| u \|_{L^1(\Omega)}^{\frac{1}{q\lambda}} \cdot \| u \|_{L^\infty(\Omega)}^{1-\frac{1}{q\lambda}} \cdot \| \nabla v \|_{L^{\frac{Np}{N-p}}(\Omega)}$$

$$\leq (m_0)^{\frac{1}{q\lambda}} M_p \cdot \| u \|_{L^\infty(\Omega)}^{1-\frac{1}{q\lambda}} \tag{18}$$

for all $t \in (0, T_{\max})$. Now let us fix $\beta \in (\frac{N}{2p}, \frac{1}{2})$ and $\zeta \in (0, \frac{1}{2} - \beta)$ as well as $T \in (0, T_{\max})$. Hence, by Lemma 2, Theorem 1 and (18), we obtain

$$\| I_2 \|_{L^\infty(\Omega)} = \| \chi \int_0^t e^{-(aI-\Delta)t} \nabla \cdot u\nabla v \|_{L^\infty(\Omega)}$$

$$\leq C_1 \chi \int_0^t \| e^{-(aI-\Delta)t} \nabla \cdot \mathrm{u}\nabla v \|_{X_p^\beta(\Omega)}$$

$$= C_1 \chi \int_0^t \| (aI - \Delta)^\beta e^{-(aI-\Delta)t} \nabla \cdot \mathrm{u}\nabla v \|_{L^p(\Omega)}$$

$$\leq C_2 \chi \int_0^t (t-s)^{-\beta} (1 + (t-s)^{-\frac{1}{2}}) e^{-\zeta(t-s)} \times \| u(\cdot, s) \nabla v(\cdot, s) \|_{L^p(\Omega)} \, ds$$

$$\leq C_3(p, q, \lambda, m_0, M_p) \chi \times \int_0^t (t-s)^{-\beta-\frac{1}{2}} e^{-\zeta(t-s)} \| u(\cdot, s) \|_{L^\infty(\Omega)}^{1-\frac{1}{q\lambda}} \, ds$$

$$\leq C_4(p, q, \lambda, m_0, M_p, \chi) \times \int_0^\infty (t-s)^{-\beta-\frac{1}{2}} e^{-\zeta(t-s)} ds \times \sup_{s \in [0,T]} \| u(\cdot, s) \|_{L^\infty(\Omega)}^{1-\frac{1}{q\lambda}}$$

$$\leq \tilde{C}_p \sup_{s \in [0,T]} \| u(\cdot, s) \|_{L^\infty(\Omega)}^{1-\frac{1}{q\lambda}}$$

for all $t \in [0, T]$, where $\tilde{C}_p \in (0, \infty)$. Substituting I_1, I_2 and I_3 into (4.1), we get that

$$\sup_{t \in [0,T]} \| u(t, \cdot) \|_{L^\infty} \leq + \| u_0 \|_{L^\infty(\Omega)} + \frac{(a+h)^2}{4k} + \tilde{C}_p \cdot (\sup_{t \in [0,T]} \| u(\cdot, t) \|_{L^\infty(\Omega)})^{1-\frac{1}{q\lambda}}$$

for all $T \in (0, T_{\max})$, where $0 < 1 - \frac{1}{q\lambda} < 1$ and $\tilde{C}_p > 0$. Hence,

$$\limsup_{t \nearrow T_{\max}} \|u(t, \cdot)\|_{L^\infty(\Omega)} < \infty,$$

which contradicts to Lemma 1. This immidiately yields that $T_{\max} = \infty$ and $\sup\|u(t, \cdot)\|_{L^\infty(\Omega)}$ is bounded for all $t > 0$. The proof is thus complete.

3.3. Mass persistence

This section is dedicated to analysis of the mass persistence of solutions to (1). Note that, by Theorem 2, we obtained that $T_{\max} = \infty$, and $(u(t, x; u_0), v(t, x; u_0))$ is the globally bounded classical solution of (1) on $(0, \infty)$.

We first provide following elementary lemma, which was established in [33, Lemma 2.5].

Lemma 6. Let α, β_1, β_2 be positive and $\theta_1 > 1, \theta_2 > 1$ and $t_0 \in \mathbb{R}$, and $y \in C^1([t_0, \infty))$ is nonnegative and satisfies

$$y'(t) \geq \alpha y(t) - \beta_1 y^{\theta_1}(t) - \beta_2 y^{\theta_2}(t)$$

for all $t > t_0$. Then

$$y(t) \geq min\left\{ y(t_0), \left(\frac{\alpha}{2\beta_1}\right)^{\frac{1}{\theta_1 - 1}}, \left(\frac{\alpha}{2\beta_2}\right)^{\frac{1}{\theta_2 - 1}} \right\}$$

for all $t > t_0$.

We next provide an estimate for $u(t, x)$ from below.

Lemma 7. Let $r \in (0,1)$. Then

$$\int_\Omega u^r(t, x; u_0) dx \geq \delta, \tag{19}$$

for all $t > t_0 > 0$.

Proof. First, multiply the equation (1) by u^{r-1} and integrate over Ω to obtain

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r = (1-r)\int_\Omega u^{r-2}|\nabla u|^2$$
$$-(1-r)\chi\int_\Omega u^{r-1}\nabla u\cdot\nabla v$$
$$+h\int_\Omega u^r - k\int_\Omega u^{r+1}, \tag{20}$$

for all $t>0$. Second, letting $m=r$ in Lemma 3.1 gives

$$-(1-r)\chi\int_\Omega u^{r-1}\nabla u\cdot\nabla v = \frac{(1-r)\chi r}{r}\int_\Omega vu^r$$
$$-\frac{(1-r)\chi b}{r}\int_\Omega u^{r+1}, \tag{21}$$

for all $t>0$.

Third, let us define $\varepsilon>0$ such that

$$0<\frac{q(n-2r)}{n(q-r)}<\varepsilon<1<q,$$

where $r\in(0,1)$, and $q\in(1,2)$ as in Lemma 5.

Then, by Lemma 5, and Hölder's inequality we get that

$$\int_\Omega u^{r+1} = \int_\Omega u^\varepsilon\cdot u^{r+1-\varepsilon}$$
$$\le\left(\int_\Omega u^q\right)^{\frac{\varepsilon}{q}}\left(\int_\Omega u^{\frac{(r+1-\varepsilon)q}{q-\varepsilon}}\right)^{\frac{q-\varepsilon}{q}}$$
$$\le (C_q)^{\frac{\varepsilon}{q}}\left(\int_\Omega u^{\frac{(r+1-\varepsilon)q}{q-\varepsilon}}\right)^{\frac{q-\varepsilon}{q}}$$

for all $t>t_0$. Note that by the Gagliardo-Nirenberg inequality and Young's inequality, we also obtain that

$$\left(\int_\Omega u^{\frac{(r+1-\varepsilon)q}{q-\varepsilon}}\right)^{\frac{q-\varepsilon}{q}} = \left\|u^{\frac{r}{2}}\right\|_{L^{\frac{2q(r+1-\varepsilon)}{r(q-\varepsilon)}}(\Omega)}^{\frac{2(r+1-\varepsilon)}{r}}$$
$$\le C\left\|\nabla u^{\frac{r}{2}}\right\|_{L^2(\Omega)}^{\frac{2(r+1-\varepsilon)\vartheta}{r}}\left\|u^{\frac{r}{2}}\right\|_{L^2(\Omega)}^{\frac{2(r+1-\varepsilon)(1-\vartheta)}{r}}$$
$$+C\left\|u^{\frac{r}{2}}\right\|_{L^2(\Omega)}^{\frac{2(r+1-\varepsilon)}{r}}$$
$$\le(1-r)(C_q)^{-\frac{\varepsilon}{q}}\left(k+\frac{(1-r)\chi b}{r}\right)^{-1}\int_\Omega u^{r-2}|\nabla u|^2$$
$$+C_1\left(\int_\Omega u^{\frac{r}{2}}\right)^{\frac{(r+1-\varepsilon)(1-\vartheta)}{r-\vartheta(r+1-\varepsilon)}}+C_2\left(\int_\Omega u^{\frac{r}{2}}\right)^{\frac{r+1-\varepsilon}{r}},$$

where

$$\vartheta=\frac{\frac{1}{2}-\frac{r(q-\varepsilon)}{2q(r+1-\varepsilon)\vartheta}}{\frac{1}{n}+\frac{1}{2}-\frac{1}{2}}=\frac{n}{2q}\cdot\frac{q-q\varepsilon+r\varepsilon}{r+1-\varepsilon}\in(0,1),$$

and

$$\frac{(r+1-\varepsilon)\vartheta}{r}=\frac{n(q-q\varepsilon+r\varepsilon)}{2qr}\in(0,1),$$

and

$$\frac{(r+1-\varepsilon)(1-\vartheta)}{r-\vartheta(r+1-\varepsilon)}>1,$$

and

$$\frac{r+1-\varepsilon}{r}>1.$$

It then follows that

$$\left(k+\frac{(1-r)\chi b}{r}\right)\int_\Omega u^{r+1}\le(1-r)\int_\Omega u^{r-2}|\nabla u|^2$$
$$+\beta_1\left(\int_\Omega u^{\frac{r}{2}}\right)^{\theta_1}+\beta_2\left(\int_\Omega u^{\frac{r}{2}}\right)^{\theta_2} \tag{22}$$

for all $t>t_0$, where β_1,β_2 are certain positive constants and $\theta_1>1$ and $\theta_2>1$.

Substituting (21) and (22) into (20) yields that

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r=(1-r)\int_\Omega u^{r-2}|\nabla u|^2$$
$$-(1-r)\chi\int_\Omega u^{r-1}\nabla u\cdot\nabla v$$
$$+h\int_\Omega u^r-k\int_\Omega u^{r+1}$$
$$=(1-r)\int_\Omega u^{r-2}|\nabla u|^2$$
$$+\frac{(1-r)\chi b}{r}\int_\Omega vu^r+h\int_\Omega u^r$$
$$-\left(k+\frac{(1-r)\chi b}{r}\right)\int_\Omega u^{r+1}$$
$$\ge h\int_\Omega u^r-\beta_1\left(\int_\Omega u^{\frac{r}{2}}\right)^{\theta_1}-\beta_2\left(\int_\Omega u^{\frac{r}{2}}\right)^{\theta_2}$$

for every $t>t_0$, due to nonnegativity of the term $\int_\Omega vu^r$. Now let us denote

$$y(t):=\int_\Omega u^r\quad\text{for all } t>t_0.$$

Then we get

$$y'(t) \geq hy(t) - \beta_1 y^{\theta_1}(t) - \beta_2 y^{\theta_2}(t)$$

for all $t > t_0$. Then by Lemma 6, we get

$$\int_\Omega u^r \geq \delta := min\left\{y(t_0), \left(\frac{h}{2\beta_1}\right)^{\frac{1}{\theta_1 - 1}}, \left(\frac{h}{2\beta_2}\right)^{\frac{1}{\theta_2 - 1}}\right\}$$

for all $t > t_0$, where $h > 0$, $\theta_1 = \frac{(r+1-\varepsilon)(1-\vartheta)}{r-\vartheta(r+1-\varepsilon)} > 1$, $\theta_2 = \frac{r+1-\varepsilon}{r} > 1$, $\beta_1 = C_1(C_q)^{-\frac{\varepsilon}{q}}\left(k + \frac{(1-r)\chi b}{r}\right) > 0$, as well as $\beta_2 = C_2(C_q)^{-\frac{\varepsilon}{q}}\left(k + \frac{(1-r)\chi b}{r}\right) > 0$. The lemma is thus complete.

Proof of Theorem 3. Note that, by Hölder inequality, for any given $r \in (0,1)$, we have

$$\int_\Omega u(t,x)dx \geq |\Omega|^{\frac{r-1}{r}}\left(\int_\Omega u^r(t,x)dx\right)^{\frac{1}{r}} \qquad (23)$$

for all $t > 0$. Note also that, by Lemma 7, for any given $r \in (0,1)$, there is $\delta > 0$ such that

$$\int_\Omega u^r(t,x;u_0)dx \geq \delta \quad \text{for all } t > t_0.$$

This together with (23) follows that

$$\int_\Omega u(t,x;u_0)dx \geq |\Omega|^{\frac{r-1}{r}}\left\{\int_\Omega u^r(t,x;u_0)dx\right\}^{\frac{1}{r}}$$

$$\geq |\Omega|^{\frac{r-1}{r}}\delta^{\frac{1}{r}} =: m^*,$$

for every all $t > t_0$. The theorem then follows.

4. Discussion and Conclusion

We now discuss our results that have been obtained the theorems 1, 2, and 3.

First, we highlight that the technique applied in Theorem 3 is completely different from the approach utilized in [24, Theorem 1.1]. We also note that our approach to prove the L^p-norm of u at the critical condition in Theorem 1 is literally different from approach the given in [12, Lemma 2.5].

Next, observe that the mass persistence of classical solutions for model (1) was gained in [24] provided that $\Omega \subset \mathbb{R}^N$ is convex. However, Theorem 3 ruled out this condition completely.

Third, it is important to remark that the assumption (7) presented in this paper recovers the assumption (4) established in the previous works [10, 24].

Fourth, L^p - norm of u yields global existence and global boundedness as well as mass persistence of classical solutions to model (1).

Finally, the results of Theorems 1, 2 and 3 are considerably beyond the findings obtained in [10, 12, 24]. We also point out that Theorem 1 recovers both [12, Lemma 2.5] and [24, Lemma 2.3]. Moreover, Theorem 2 recovers both [10, Lemma 2.4] and [12, Theorem 1.1]. Finally, Theorem 3 recovers [24, Theorem 1.1].

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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