A Note on Commensurate-Order Characteristic Root Equivalency Class of Linear Time Invariant Systems

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Abstract—**This study investigates characteristic root equivalency relations between commensurate order and integer order Linear Time Invariant (LTI) systems. Author introduces some useful properties of a special class of commensurate order systems, which is called characteristic root equivalency class of LTI systems. These properties present potential to facilitate design and analysis efforts of this class of commensurate order systems. In this sense, straightforward stability checking procedures and design approaches for commensurate order root equivalent systems of the first and second order LTI systems are demonstrated. Findings of the study are validated by illustrative examples.**

Index Terms— **Fractional order systems, characteristic root equivalency, stability, design, analysis.**

I. Introduction

THERE has been a growing interest for utilization of Fractional Calculus (FC) in engineering and science problems because of its promises of better describing real world objects and phenomenon [1-5]. It was suggested that real world objects can be more accurately modeled and analyzed by using FC because real world objects do not have to precisely comply with integer order system models: Majority of them may exhibit fractionalty even at a low degree [1,2,5-10]. The fractional order derivative and integration were shown to act upon solutions of many problems in physics [6,7,8], thermodynamics [9], electrical circuits theory and fractances [10,11,12], mechatronics systems [13], chaos theory [14], control systems [15,16,17] etc.

Linear Time Invariant (LTI) systems have been used as a fundamental and substantial modeling tool for theoretical analyses of real systems [18]. Extensive researches on LTI systems offered well established mathematical background, simplified solutions, experimentally proven methods for the characterization and analysis of real systems. In these analyses, LTI system models can be expressed in many forms such as differential equations, state space models and transfer function forms [18]. LTI system modeling techniques, system stability, observability and controllability issues have been studied extensively and applied in engineering problems.

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In recent years, fractional order LTI systems have become the most underpinning topic of researchers and there are extensive researches ongoing for problems of fractional order system stability [19-33], identification, implementation and computation methods [20]. Nonetheless, rather deepened investigations on fractional order LTI system properties are still required for better understanding fractional dynamics of real world systems.

Commensurate order LTI systems are indeed a class of fractional order systems, which provides simplification for analysis and design efforts because it has a proper expression of fractional orders in the form of $k\alpha$, $k = 0,1,2,3...$ Thus, the order fractionality of commensurate order LTI systems can be expressed by a single order parameter $\alpha \in R$. For the integer values of order $\alpha = 1,2,3...$, the system turns into conventional integer order LTI systems. As known, system behavior and model structure strongly depends on root locus of LTI system. Investigating effects of fractional order α on root placement of characteristic equations on complex plane can be helpful to explain connections between fractional order LTI system model and integer order LTI system model.

This study investigates properties of a special class of commeasure order systems. This class of system models has the same characteristic root set for different commeasure order ($\alpha \in R$). These systems were referred to as characteristic root equivalent systems and the characteristic root equivalency was used to establish arithmetical relation between fractional order systems and integer order systems [24]. In the current study, author addresses some fundamental aspects of characteristic root equivalency families of the first and second order LTI systems. It is observed that the root equivalency formed by α order shifts the roots of the characteristic equation in complex plane and thus draws a root trajectory with respect to the commensurate order α . This trajectory represents root locus of root equivalency class of commensurate order systems in complex domain and properties of these trajectories can provide useful information for system analysis and design. On the other hand, characteristic root equivalent systems have complex coefficients and this causes implementation complications for real-valued systems. In order to transform the complex coefficient root equivalent systems into real coefficient commensurate order systems, we use complex conjugate root addition method.

II. METHODOLOGY

A. Basic definitions

Characteristic root equivalence family of fractional order systems were introduced in [24]. Here, we consider root equivalency class of commensurate order LTI systems.

Fractional order characteristic root equivalency: Let's assume characteristic polynomial of a first order LTI system be expressed as $s-\nu=0$ in s-domain. For a given fractional order $\alpha \in R$, fractional order root equivalent of this root can be defined by $s^{\alpha}-\nu^{\alpha}=0$.

Proof: The root of characteristic equation $s-\nu=0$ can be solved as $s=\nu$. By applying α power to the both sides of the equation $s=\nu$, one can obtain $s^{\alpha}=\nu^{\alpha}$. In this case, α -order root equivalent characteristic equation can be written in the form of $s^{\alpha}-\nu^{\alpha}=0$. The transformation of $s=\nu$ into $s^{\alpha}=\nu^{\alpha}$ is called as α -order root equivalency.

Characteristic root equivalent commensurate order systems: Let's consider commensurate systems in its simple state space form $X^{(\alpha)}=AX$. For a given fractional order $\alpha>0$ and integer order characteristic polynomial $\Delta(s)=\prod_{i=1}^{n}(s-\nu_i)$, where $\nu_1,\nu_2,\nu_3\ldots,\nu_n$, $n\in Z^{+}$ are complex roots of system, after applying $s^{\alpha}=\nu_i^{\alpha}$ transformation to all roots of $\Delta(s)$, α -order root equivalent commensurate order polynomial family of the integer order characteristic polynomial is defined as $\Delta_C=\left\{\det(s^{\alpha}I-A)=\prod_{i=1}^{n}(s^{\alpha}-\nu_i^{\alpha}):\alpha\in(0,1)\cup(1,2)\vee n\in Z^{+}\right\}$.

B. Effects of fractional order root equivalency

In this section, we study root equivalency of the first order LTI systems. In further sections, we extent our analyses on root equivalency to second order systems. The first order LTI systems establish the most basic form of root equivalent fractional order systems.

Let consider a commensurate order LTI system, defined in form of $x^{(\alpha)}=\nu x(t)+Ku(t)$, where $u(t)$ and $x(t)$ are input and state of the system, respectively. The system output was assumed as $y(t)=x(t)$ and a zero initial state ($x(0)=0$) is used for the sake of computational simplicity. The transfer function of the first order LTI system for $\alpha=1$ is expressed in general form as,

$$T(s)=\frac{K}{s-\nu}. \tag{1}$$

Characteristic polynomial of the system $T(s)$ can be obtained as $\Delta(s)=(s-\nu)$. By applying α order root equivalency mapping to the root by $s^{\alpha}=\nu^{\alpha}$, the α -order root equivalence system family of $T(s)$ can be written as,

$$T_{\alpha}(s)=\frac{K}{s^{\alpha}-\nu^{\alpha}}=\frac{K}{s^{\alpha}-M^{\alpha}e^{j\alpha\theta}}. \tag{2}$$

In equation (1), the root of $\Delta(s)$ is written in the complex form as $\nu=Me^{j\theta}$, where M is magnitude and θ is angle of complex root, respectively. For the real coefficient first order LTI systems, root angle θ takes two values: $\theta=\pi$ radian for stable systems and $\theta=0$ radian for unstable systems. One can write the α -order root equivalent of ν in the complex form as $\nu^{\alpha}=M^{\alpha}e^{j\alpha\theta}$. This reveals the following fundamental property of the α -order root equivalency in complex plane:

(i) *Nonlinear scaling of root magnitude*: Due to $s^{\alpha}=\nu^{\alpha}$ transformation, root amplitude is scaled by the power of $\alpha\in R$. In other word, the root magnitude goes from M to M^{α}.($M\rightarrow M^{\alpha}$). (i) If $\alpha<1$ then $M^{\alpha}<M$, and (ii) if $\alpha>1$ then $M^{\alpha}>M$ as illustrated in Fig. 1.

(ii) *Linear scaling of root angle*: Due to $s^{\alpha}=\nu^{\alpha}$ transformation, root angle is scaled proportional to the order α. In other words, the root angle goes from θ to $\alpha\theta$. ($\theta\rightarrow\alpha\theta$). (i) If $\alpha<1$ then $\alpha\theta<\theta$ and (ii) if $\alpha>1$ then $\alpha\theta>\theta$ as illustrated in Fig. 1.

TABLE I.
EFFECT OF α -ORDER ROOT EQUIVALENCY ON THE COMPLEX ROOTS

	Integer order	α *-order*	*Effect*
Magnitude	M	M^{α}	Nonlinear scaling of root magnitude
Angle	θ	$\alpha\theta$	Linear scaling of root angle

Table 1 summarizes effects of α -order root equivalency on the complex roots. These effects in the root locus is referred as to root equivalency shifting with the α -order. Fig. 1 illustrates this root shifting effect for a stable first order LTI system, which occurs in clockwise direction for $\alpha<1$ and counterclockwise direction for $\alpha>1$. The root angle takes discrete angle values, $\theta=\{0,\pi\}$ for a first order LTI system. The root of first order LTI ($\alpha=1$) is reel for real coefficient first order transfer functions. However, for root equivalent fractional order systems, root angle takes continuous angle values, defined as $\theta=\{0,\alpha\pi\}$ for an $\alpha\in R$ and the trace of shifting root, which is $\alpha\pi$, draws a root equivalency trajectory, which depicts root locus of root equivalency family in the complex plane.

Effects of fractional order on system stability and the first Riemann sheet were discussed in detail in Ref [1,21-24]. By applying $s=u^{m}$ transformation to characteristic polynomial of fractional order systems, the stability analysis of a fractional order system based on the root locus was confined in the first Riemann sheet that is defined as the portion of complex plane with angle range of $-\pi/m\le\phi\le\pi/m$.

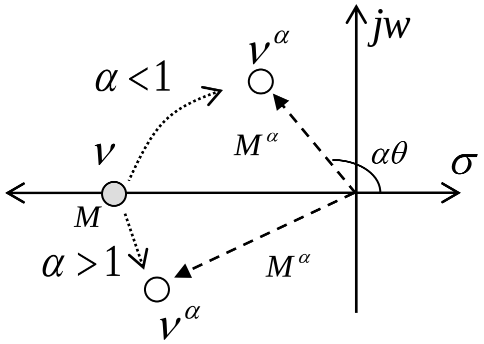

Fig. 1. A representation of root locus shifting effect for a stable first order LTI system

Fig. 2 shows the possible trajectories of shifting roots for $\alpha \in [0, 3.1]$ and different root magnitudes (M). These trajectories are calculated by $\nu^{\alpha} = M^{\alpha} e^{j\alpha\theta}$ to indicate family of root equivalent systems, graphically. As seen in the figure, the magnitude M yields three trajectory types: For $M > 1$ (expanding trajectory), it follows an expanding circular route and the root magnitude increases depending on α. For $M = 1$ (circulating trajectory), it follows a steady circular route and the order α do not change the root magnitude. In the case of $M < 1$ (narrowing trajectory), it follows a narrowing circular route and the root magnitude decrease depending on the α. It is already known that root magnitude takes effect on the response time of a stable system. As magnitude decreases, system response is getting slower.

By considering $e^{j\alpha\theta} = \cos(\alpha\theta) - j\sin(\alpha\theta)$ in equation (2), one can express the root equivalent family of first order LTI systems in the form of

$$T_{\alpha}(s) = \frac{K}{s^{\alpha} - M^{\alpha}\cos(\alpha\theta) - jM^{\alpha}\sin(\alpha\theta)} . \qquad (3)$$

The α order root equivalency mostly generates complex coefficient system models from real coefficient first order LTI system models due to the fact that first order system has a single real root without any conjugate pair with respect to real axis and this root moves in a round trajectory without conjugate trajectory as illustrated in Fig. 1.

Example: Lets figure out $\alpha = 1.2$ and $\alpha = 0.8$ root equivalence of the first order stable LTI system $T(s) = \dfrac{1}{s+2}$.

By considering $M = -2$ and $\theta = \pi$ radian due to the root $\nu = -2 = 2e^{j\pi}$, the $\alpha = 1.2$ order root equivalent system of $T(s)$ is obtained according to equation (3) as,

$$T_{1.2}(s) = \frac{1}{s^{1.2} - (-2)^{1.2}} = \frac{1}{s^{1.2} + 1.8586 + j1.3504} . \qquad (4)$$

Here, the complex root emerges as $\nu^{1.2} = (-2)^{1.2} = (2e^{j\pi})^{1.2} = 2^{1.2} e^{j1.2\pi} = -1.8586 - j1.3504$. The magnitude of root is changed from 2 to 2.2974, ($2 \to 2^{1.2}$), and the angle of root changes from π to 1.2π radian.

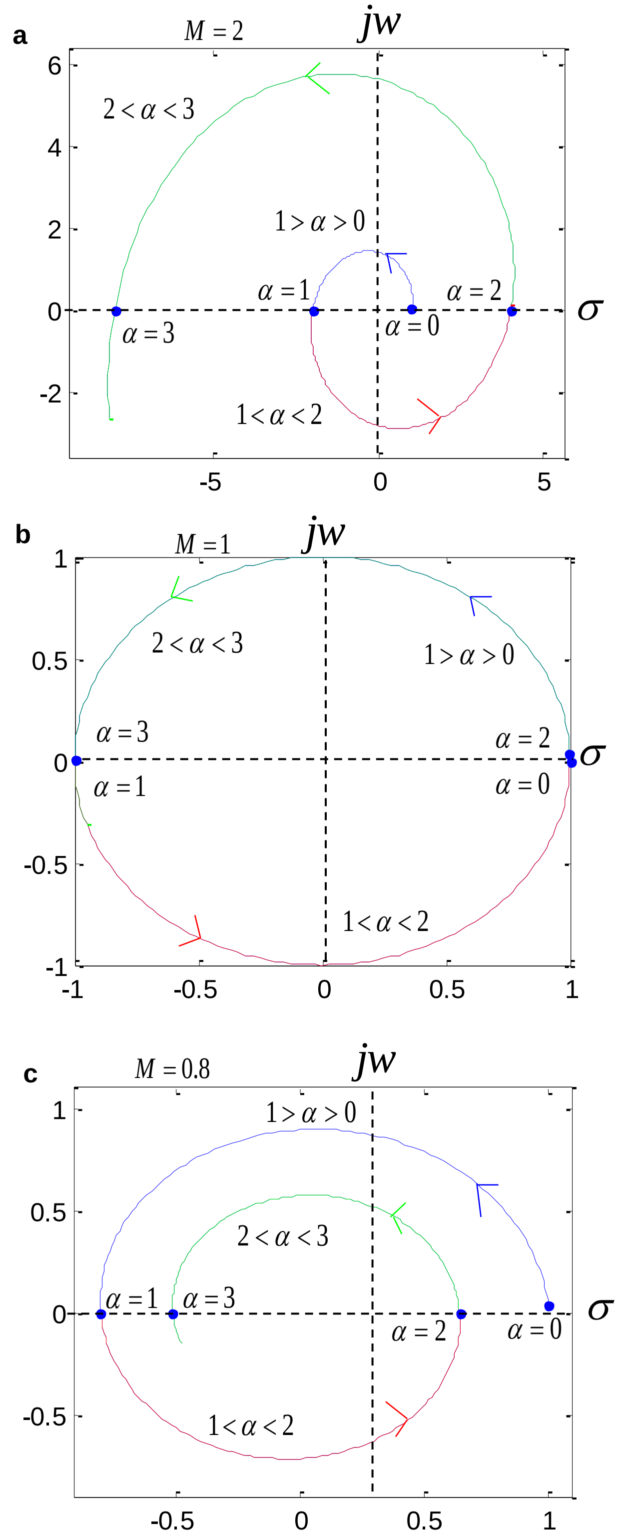

Fig. 2. The root locus shifting in the range of $0 < \alpha < 3.1$ for root magnitudes; $M = 2$ in (a), $M = 1$ in (b) and $M = 0.8$ in (c)

Both magnitude and angle of root increase because of $\alpha > 1$ and $M > 1$. Fig. 3 confirms this effect, graphically.

For $\alpha = 0.8$, the root equivalency of $T(s)$ is obtained as follows,

$$T_{0.8}(s) = \frac{1}{s^{0.8} - (-2)^{0.8}} = \frac{1}{s^{0.8} + 1.4086 - j1.0234} \tag{5}$$

Here, $v^{0.8} = (-2)^{0.8} = (2e^{j\pi})^{0.8} = 2^{0.8} e^{j0.8\pi} = -1.4086 + j1.0234$. The magnitude of root is changed from 2 to 1.7411 ($2 \to 2^{0.8}$) and the angle of root changes from π to 0.8π radian. For $\alpha < 1$, magnitude and angle of the root decrease. Fig. 3 shows this effect, graphically.

Fig. 3 illustrates possible root equivalency trajectories of the first order systems $T(s) = 1/(s+2)$ and $T(s) = 1/(s+0.5)$ in the range of $0 < \alpha < 2$. The root trajectories in the figure demonstrate that the system model has single complex root without a complex conjugate. Practically, it is difficult to implement transfer functions containing complex roots without its conjugate pairs. One see that, for $\alpha = 0$, all trajectories converge to one. For $\alpha > 0$, type of root equivalency trajectory of the system (narrowing, circling or expending trajectories) depends on the magnitude of the root (M).

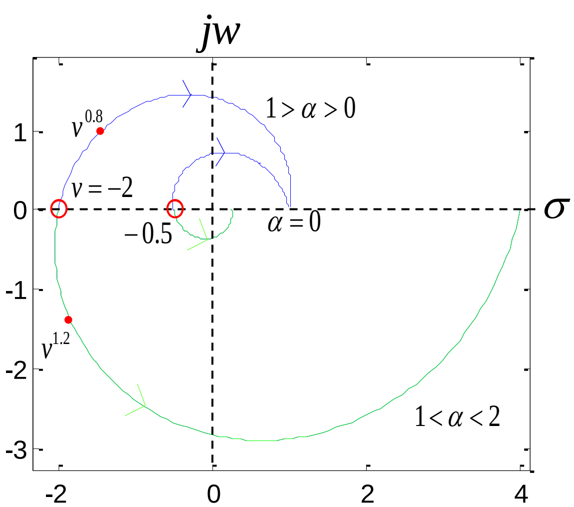

Fig. 3. The root locus shifting of stable first order LTI systems in the range of $0 < \alpha < 2$. The outer trajectory is for $T(s) = 1/(s+2)$ and the inner trajectory is for $T(s) = 1/(s+0.5)$

C. Root addition for complex root conjugating of commensurate order equivalence of first order LTI Systems

System designers may question whether it is possible to derive a real coefficient commensurate order system from the first order LTI systems by using α -order root equivalency so that realization of complex coefficient systems is a complicated problem for practice. Previously, to carry out implementation of complex-order systems, several studies have addressed the implementation problems of complex coefficient transfer functions. To obtain real valued outputs from complex-order systems, conjugated-order system concept were introduced [34,35].

In this section, we benefit from a similar perspective, which involves complex conjugate root addition to system model in order to transform a complex coefficient root equivalent system into a real coefficient commensurate order systems. This method increases root counts of characteristic equation from one to two because of $\Delta_\alpha(s) = (s^\alpha - v^\alpha)(s^\alpha - v^{\alpha*})$, where $v^{\alpha*} = M^\alpha e^{-j\alpha\theta}$ represents the complex conjugate pair of $v^\alpha = M^\alpha e^{j\alpha\theta}$. Thus, this additional conjugate root allows real valued outputs from the system model. With the complex root conjugating, equation (3) expand to the model of real coefficient α commensurate order system as follows.

$$T_\alpha(s) = \frac{K}{(s^\alpha - M^\alpha \cos(\alpha\theta) - jM^\alpha \sin(\alpha\theta))(s^\alpha - M^\alpha \cos(\alpha\theta) + jM^\alpha \sin(\alpha\theta))} \tag{6}$$

When this equation is arranged, one obtains the real coefficient commensurate order root equivalent system with complex conjugate as follows,

$$T_\alpha(s) = \frac{K}{s^{2\alpha} - 2M^\alpha \cos(\alpha\theta) s^\alpha + M^{2\alpha}} \tag{7}$$

Here, the root angle θ can take values of $\{0, \pi\}$ so that a real coefficient first order LTI system $T(s)$ has a characteristic root only on the real axis. If $T(s)$ is a stable system, the root resides on the left half plane and $\theta = \pi$. The real coefficient $T_\alpha(s)$ function expresses a special class for the second order commensurate order system that is based on root equivalency of first order LTI systems. Previously, some analyses addressing the stability of second order conjugate systems were addressed in detail by Radwan et al [23].

Let's investigate effects of α order on the system behavior. A discussion on system behavior depending on system pole location was previously given by Monje et al. [32]. We performed simulations and observed the following remarks for root equivalencies of the first order stable LTI systems given by equation (7):

(i) For $\alpha = 0.5$, it yields the original first order LTI system that is expressed by equation (1).
(ii) In the case of $\alpha < 0.5$, simulation results indicate that the commensurate order root equivalent system with complex conjugate can exhibit low rise time step response.
(iii) In the case of $0.5 < \alpha < 4/3$, simulation results showed that the commensurate order root equivalent system with complex conjugate yields a slower step response than the original first order LTI system.
(iv) In the case of $\alpha = 4/3 \cong 1.33..$, the root equivalent system exhibits oscillating behavior.
(v) In the case of $\alpha > 4/3$, the root equivalent system is unstable.

The value of $\alpha = 4/3$ defines an upper boundary for asymptotical stability of commensurate order root equivalent systems that are expressed in the form of equation (7). In order

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to obtain an asymptotically stable commensurate order system, a sufficient condition of $0<\alpha<4/3$ should be satisfied in system design. (See Property 1 in Appendix for the extraction of $\alpha=4/3$ boundary condition). This property allows a straightforward solution for testing of the stability of a special class of commensurate order systems expressed in the form of equation (7). A stability check procedure for a given commensurate order system complying with the equation (7) can be summarized as follows:

Step 1: By considering characteristic polynomial in the form of $\Delta_\alpha(s)=s^{2\alpha}+a_1s^\alpha+a_0$,find commensurate order α by considering orders as $k\alpha$ for $k=0,1,2$.

Step 2: By considering the term a_0, calculate M by $M=a_0^{1/(2\alpha)}$.

Step 3: If $2M^\alpha\cos(\alpha\theta)=-a_1$ for $\theta=\pi$, the system is a root equivalency of a first order stable LTI system. In this case, commensurate order root equivalent system is said to be stable for $\alpha<4/3$ in accordance with the remark (iii).

Since the root angle is zero ($\theta=0$ rad), the α-order root equivalent systems of the first order unstable LTI systems can be expressed $T_\alpha(s)=\dfrac{K}{s^\alpha-M^\alpha e^{j\theta\alpha}}=\dfrac{K}{s^\alpha-M^\alpha}$. This states that α-order root equivalency of the unstable first order LTI systems is always real coefficient and unstable for $\forall\ \alpha\in R$. (See Proper 2 in Appendix)

D. An extension of root equivalency analysis for the second order LTI systems

In this section, we extent our investigation for root equivalence of the second order LTI systems. The transfer function of the second order LTI system can be written in general form as,

$$T(s)=\frac{K}{(s-v_1)(s-v_2)} \tag{8}$$

The root equivalent characteristic polynomial of the equation (8) is written as $\Delta_\alpha(s)=(s^\alpha-v_1^\alpha)(s^\alpha-v_2^\alpha)$. The transfer function of α-order root equivalent commensurate system with two roots can be written in the form of,

$$T(s)=\frac{K}{(s^\alpha-v_1^\alpha)(s^\alpha-v_2^\alpha)} \tag{9}$$

By applying root equivalent transformation to the each root as $s^\alpha=v_1^\alpha=M_1^\alpha e^{j\alpha\theta_1}$ and $s^\alpha=v_2^\alpha=M_2^\alpha e^{j\alpha\theta_2}$, one can rearrange the α-order root equivalent commensurate system as,

$$T_\alpha(s)=\frac{K}{(s^\alpha-M_1^\alpha e^{j\alpha\theta_1})(s^\alpha-M_2^\alpha e^{j\alpha\theta_2})} \tag{10}$$

When the roots of second order system are $v_2^\alpha=v_1^{\alpha*}$, it yields the real coefficient characteristic polynomial. So, it can be possible to find out a α value that makes v_1^α and v_2^α complex conjugate pairs. One can obtain the real coefficient characteristic polynomial, when conditions of $\alpha\theta_1=-\alpha\theta_2$ and $M_1\sin(\alpha\theta_1)=M_2\sin(\alpha\theta_2)$ are satisfied. A solution to satisfy these two conditions can be written as $\alpha\theta_1=-\alpha\theta_2=\eta\pi$, $\eta\in Z^+$. Here, the first α value making equation (10) a real coefficient system model can be found as $\alpha=\pi/\theta_1$ for $\eta=1$. Accordingly, real coefficient commensurate order root equivalent system can be expressed for $\alpha=\pi/\theta_1$ and $\theta_1=-\theta_2$ in the form of,

$$T_\alpha(s)=\frac{K}{s^{2\alpha}-(M_1^\alpha\cos(\alpha\theta_1)+M_2^\alpha\cos(\alpha\theta_2)s^\alpha+M_1^\alpha M_2^\alpha} \tag{11}$$

Example: Let's find α values that makes $T_1(s)=\dfrac{1}{s^2+4s+3}$ and $T_2(s)=\dfrac{1}{s^2+0.5s+1}$ a real coefficient commensurate order root equivalent systems.

For $T_1(s)$, roots are -1 and -3, which is not complex conjugate roots and $\theta_1=\theta_2=\pi$ radian. So, $\alpha=\pi/\theta_1=\pi/\pi=1$. There is not any real coefficient commensurate order root equivalent system. One should add new conjugate roots in a similar way applied for the first order LTI systems.

For $T_2(s)$, roots are complex conjugate pairs as $-0.2500\pm0.9682j$. In this case, $\theta_1=1.8235$ radian and $\alpha=\pi/1.8235=1.7229$. When it is used in equation (11), the real coefficient commensurate order root equivalent systems is found as,

$$T_{1.7229}(s)=\frac{1}{s^{3.4457}+2s^{1.7229}+1} \tag{12}$$

An important remark to notice when the second order LTI system is stable, the real coefficient commensurate order system with $\alpha=\pi/\theta_1$ is also stable. (See Property 3 in Appendix) This allows checking the stability of the commensurate order systems in the form of equation (11) by using root equivalence of second order systems as follows,

Step 1: By considering characteristic polynomial in the form of $\Delta_\alpha(s)=s^{2\alpha}+a_1s^\alpha+a_0$, find α by considering the commensurate order in the form of $k\alpha$ for $k=0,1,2$.

Step 2: Assume $v_2^\alpha=v_1^{\alpha*}$ for real coefficient characteristic polynomial. Therefore, suppose $M_1^\alpha=M_2^\alpha=M^\alpha$, $\theta_1=\pi/\alpha$ and $\theta_2=-\theta_1$.

Step 3: By considering the term a_0, calculate M by $M=a_0^{1/(2\alpha)}$.

Step 4: If $2M^\alpha\cos(\alpha\theta_1)=-a_1$, the system is a root equivalent of second order LTI system, and if $\dfrac{\pi}{2}<\theta_1<\dfrac{3\pi}{2}$, the second order LTI system is stable and therefore the commensurate order system are also stable.

III. ILLUSTRATIVE EXAMPLES

Example 1: Let's find real coefficient commensurate order root equivalent of the first order stable LTI system $T(s)=\dfrac{1}{s+2}$ by applying complex root conjugation addition approach for $\alpha=1.2$ and $\alpha=0.8$.

If the characteristic root of system is considered in the complex form as $s=-2=2e^{j\pi}=Me^{j\theta}$, one obtains $M=2$ and $\theta=\pi$. For $\alpha=1.2$, by using equation (7), one obtains the real coefficient commensurate order root equivalent systems of $T(s)$ as,

$$T_{1.2}(s)=\frac{1}{s^{2.4}+3.7173s^{1.2}+5.278} \tag{13}$$

and, similarly, for $\alpha=0.8$, one obtains the real coefficient commensurate order root equivalent systems of $T(s)$ as,

$$T_{0.8}(s)=\frac{1}{s^{1.6}+2.8172s^{0.8}+3.0314} \tag{14}$$

Example 2: Lets investigate behavior of complex conjugated commensurate order root equivalent systems derived from the first order stable LTI system $T(s)=\dfrac{1}{s+3}$ for various α.

When the root of characteristic polynomial of $T(s)$ is considered in complex form, $s=-3=3e^{j\pi}=Me^{j\theta}$, one obtains $M=3$ and $\theta=\pi$. By applying equation (7), we can obtain complex conjugated commensurate order root equivalent systems of $T(s)=\dfrac{1}{s+3}$ as listed in Table 2. Step responses of these systems are shown in Fig. 4. The step responses in the figure confirm system behavior corresponding to values of α, which are explained in the section 2.C.

TABLE II.
COMPLEX CONJUGATED COMMENSURATE ORDER ROOT EQUIVALENT SYSTEMS FOR VARIOUS α

α	*Commensurate Order Root Equivalent Systems*	*Remarked Item*
0.3	$T_{0.3}(s)=\dfrac{1}{s^{0.6}-1.6354s^{0.3}+1.9332}$	(ii) $\alpha<0.5$
0.5	$T_{0.5}(s)=T(s)=\dfrac{1}{s+3}$	(i) $\alpha=0.5$
0.9	$T_{0.9}(s)=\dfrac{1}{s^{1.8}+5.1126s^{0.9}+7.2247}$	(iii) $0.5<\alpha<4/3$
1.333	$T_{1..3509}(s)=\dfrac{1}{s^{2.7018}+4.3267s^{1.3509}+18.7208}$	(iv) $\alpha=4/3$
1.4	$T_{0.9}(s)=\dfrac{1}{s^{2.8}+2.8773s^{1.4}+21.674}$	(v) $\alpha>4/3$

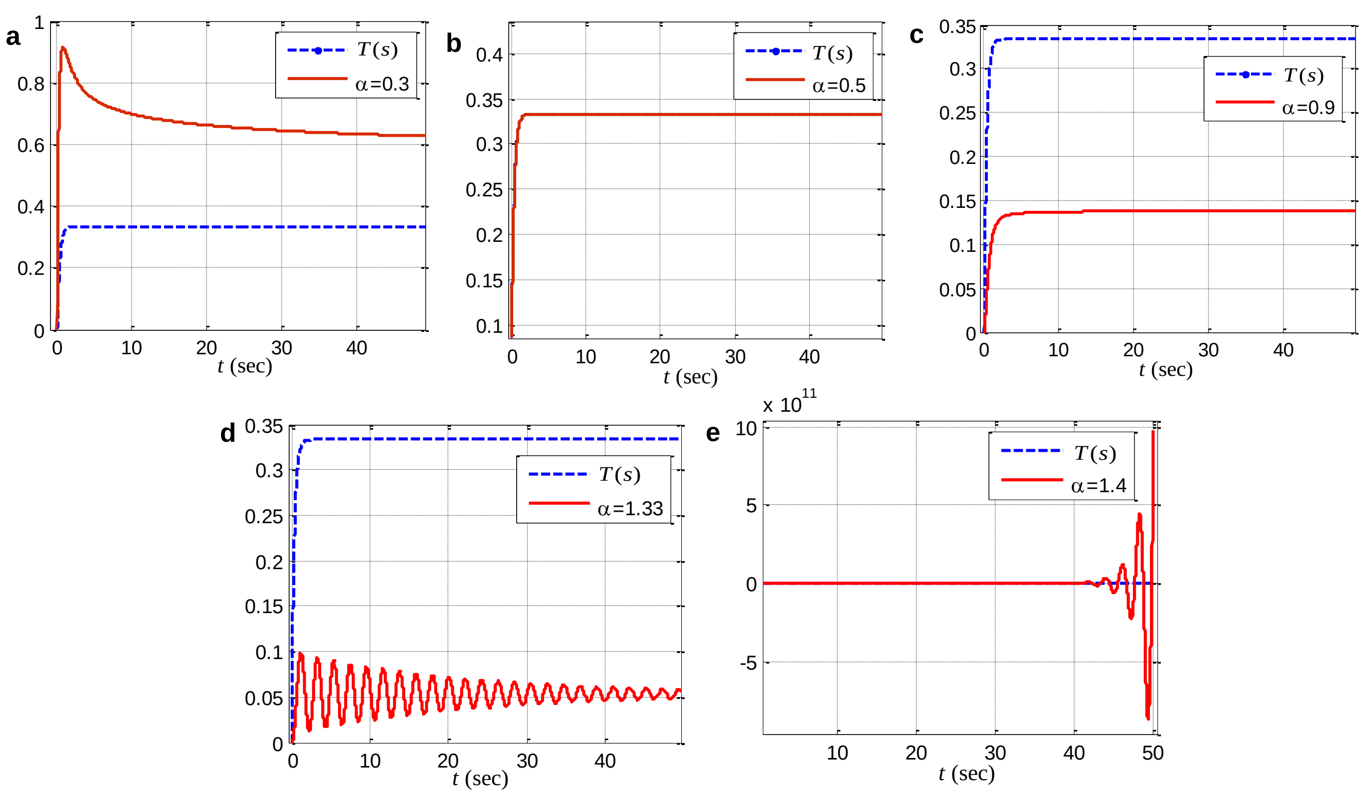

Fig. 4. Step responses obtained for commensurate order root equivalent systems, which are listed in Table 2

We observed from step response simulation results that for $\alpha<0.5$, the step response of commensurate order root equivalent systems is faster than the first order integer order LTI system. For $\alpha > 0.5$, the step response of commensurate order root equivalent system is slower than the first order integer order LTI system. Another noteworthy point should be emphasized that, $\alpha \cong 1.33..$ yields oscillations due to root placement on the stability boundary. More comprehensive discussions on sinusoidal oscillation condition of fractional and integer order systems was given in [22,23].

Example 3: Let us check whether the commensurate order LTI system $G(s) = \dfrac{1}{s^{2.2} + 11.171 s^{1.1} + 34.4932}$ is stable?

Lets apply the steps of stability check procedure given in previous section.

Step 1: Due to terms of $s^{2.2}$, $s^{1.1}$ and s^{0}, one finds $\alpha = 1.1$.

Step 2: Considering $M = a_0^{1/(2\alpha)}$, then, one calculate $M = (34.4932)^{1/2.2} \cong 5$.

Step 3: Considering $2M^{\alpha}\cos(\alpha\theta) = a_1$ for $\theta = \pi$ radian, one calculates $2.5^{1.1}\cos(1.1\pi) = -11.171 = -a_1$. This confirms that $G(s)$ is the root equivalency of the stable LTI $1/(s+5)$.

Step 4: In this case, since $\alpha = 1.1 < 4/3$, the system $G(s)$ is stable.

In order to validate this result, we demonstrate the step response of $G(s)$ in Fig. 5(a) and root locus in the in the first Riemann sheet in Fig. 5(b) to apply Radwan stability test procedure [23]. Radwan stability test procedure is essentially based on Matignon's theorem [21]. This procedure applies $s = u^m$ transformation to characteristic polynomial and investigates root angle of expanded degree integer order polynomial on the first Riemann sheet. After applying $s = u^{10}$ to characteristic equation $\Delta_{1.1}(s) = s^{2.2} + 11.171\, s^{1.1} + 34.49$, one obtains $\Delta_{1.1}(u^{10}) = u^{22} + 11.17u^{11} + 34.49$. If any root angle ($\phi$) of satisfies $-\pi/20 \le \phi \le \pi/20$ in the first Riemann sheet, Radwan procedure suggests us that the system is unstable. Fig. 5(b) shows root locus of $\Delta_{1.1}(u^{10})$ and it confirms stability of the system so that roots are out of instability region defined by $-\pi/m \le \phi \le \pi/m$. The roots are indicated by blue stars in Fig 5(b). In some cases, this test procedure introduces high computational complexity. For instance, if $\alpha = 1.111$, one need to apply $s = u^{1000}$ and calculate root angles of the polynomial $\Delta_{1.111}(u^{1000})$. Accordingly, amount of digits in fractional part increases computational complexity, significantly. One of the advantages of root equivalency analyses may appear in such cases. If commensurate order LTI system is a member of root equivalency class, it can significantly reduce computational complexity in stability analyses.

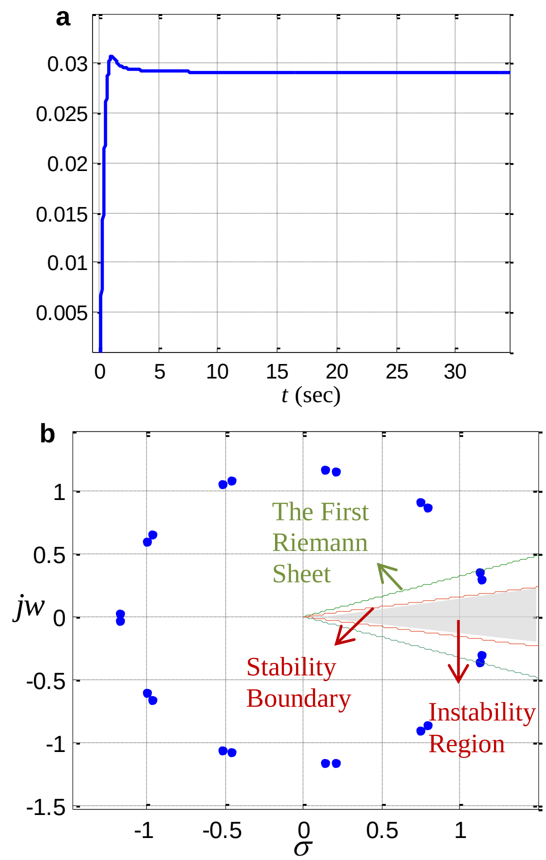

Fig. 5. (a) Step response of $G(s)$ and (b) the root locus showing that $G(s)$ system is stable according to Radwan stability test procedure

Example 4: Lets design an oscillating response from the root equivalency family of the commensurate order system that was analyzed in the previous example.

In order to provide oscillation of $G(s) = \dfrac{1}{s^{2.2} + 11.171\, s^{1.1} + 34.4932}$, which is a member of the root equivalency class of the function $1/(s+5)$ for $M = 5$ and $\theta = \pi$ radian as shown in previous example. Here, one configures $\alpha = 4/3 = 1.33...$ for oscillating behavior. By using equation (7), one obtains transfer function of oscillating commensurate order root equivalent system as,

$$G(s) = \frac{1}{s^{2.6812} + 8.3063 s^{1.3406} + 74.8295} \tag{15}$$

We used step function proposed in [33] to numerically calculate step responses in this study. Fig. 6 shows oscillating behavior of the system. As $\alpha = 4/3 = 1.33...$ is an infinite decimal number, it cannot be exactly implemented and calculated in digital systems. Therefore, the finite decimal implementation of $\alpha = 1.33...$ can result in an unbalanced oscillation as shown in Fig. 6.

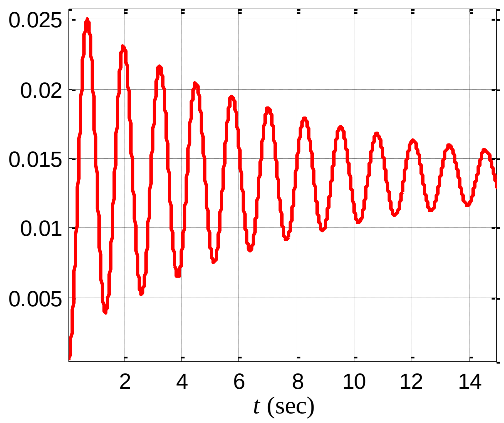

Fig. 6. Step response showing oscillating system behavior

Example 5: Let's find the x coefficient that makes $G(s)=\dfrac{1}{s^{1.82}+xs^{0.91}+9.777}$ function a root equivalence class of a first order LTI system.

It is apparent from $s^{1.82}$ and $s^{0.91}$ that the α order of the system is 0.91 and we calculate the magnitude as $M=(9.777)^{1/1.82}\cong 3.5$ by considering $M^{2\alpha}=9.777$ according to Equation (7). In this case, the function $G(s)$ can be the root equivalence of the first order LTI system $1/(s+3.5)$. We can calculate $x=-2M^{\alpha}\cos(\alpha\theta)=-2.(3.5)^{0.91}\cos(0.91\pi)\cong 6.0053$ for $\alpha=0.91$ and $\theta=\pi$.

Example 6: Let's check stability of $G_1(s)=\dfrac{1}{s^{4.5508}+18.3022s^{2.2754}+83.7423}$ and $G_2(s)=\dfrac{1}{s^{3.0884}+6.9296s^{1.5442}+12.0047}$ by using root equivalent class of second order LTI systems.

By applying stability check procedure for $G_1(s)$,

Step 1: Due to terms $s^{4.5508}$, $s^{2.2754}$ and s^{0}, one finds $\alpha=2.2754$.

Step 2: The root angle is $\theta_1=\pi/\alpha=\pi/2.2754=1.3807$

Step 3: Considering $M=a_0^{1/(2\alpha)}$, then, one calculate $M=(83.7423)^{1/2.2754}\cong 2.6458$.

Step 4: Considering $2M^{\alpha}\cos(\alpha\theta_1)=-a_1$, one calculates $2.2.6458^{2.2754}\cos(2.2754.1.3807)=-18.3022=-a_1$. This confirms that the system is root equivalency of the second order stable LTI system. However, the root angle $\theta_1=1.3807$ does not satisfy the condition $\dfrac{\pi}{2}<\theta_1<\dfrac{3\pi}{2}$, $G_1(s)$ system is unstable system.

By applying stability check procedure for $G_2(s)$,

Step 1: Due to terms $s^{3.0884}$, $s^{1.5442}$ and s^{0}, one finds $\alpha=1.5442$.

Step 2: The root angle is $\theta_1=\pi/2.2754=2.0344$

Step 3: Considering $M=a_0^{1/(2\alpha)}$, then, one calculate $M=(12.0047)^{1/(1.5442)}\cong 2.2361$.

Step 4: Considering $2M^{\alpha}\cos(\alpha\theta_1)=-a_1$, one calculates $2.2.2361^{1.5442}\cos(1.5442.2.0344)=-6.9296=-a_1$. This confirms that the system is a root equivalency of the second order stable LTI system. Since, the root angle $\theta_1=2.0344$ does satisfy the condition $\dfrac{\pi}{2}<\theta_1<\dfrac{3\pi}{2}$, $G_2(s)$ system is stable.

Fig. 7 demonstrates step responses of $G_1(s)$ and $G_2(s)$ functions, which confirm results of the stability analyses. Fig. 7(a) indicates step response of an unstable system and Fig. 7(b) indicates the step response of a stable system.

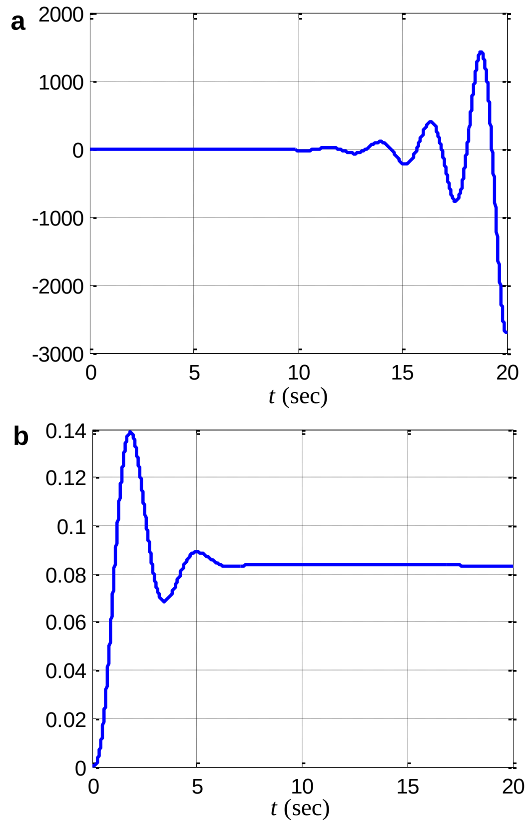

Fig. 7.Step responses of $G_1(s)$ in (a) and $G_2(s)$ in (b)

IV. Conclusions

This study shows that one can establish analytical relations between fractional order LTI systems and integer order LTI systems via characteristic root equivalency. These relations can facilitate design and analysis efforts of commensurate order systems on bases of root equivalency of integer order LTI systems and help comprehension of effects of fractional orders on the system behavior.

In summary, α-order commensurate root equivalence classes of the first and second order stable LTI systems are investigated in the paper. The first order LTI system produces complex coefficient commensurate order root equivalent systems with single root. To transform them to a real coefficient commensurate order root equivalent systems, root conjugation by a complex root addition was used. Then, we extent our investigation to root equivalency of second order LTI systems. We derived a sufficient condition that transforms a second order LTI system to a real coefficient root equivalent commensurate order LTI system without any conjugate root addition. Numerical examples were presented to demonstrate straightforward analytical solutions for stability analysis and design problems for commensurate order systems. Some important remarks can be summarized as follows:

(i) The characteristic root equivalency transformation $s^{\alpha} = v^{\alpha}$ for $\alpha \in R$ results in shifting of roots on the trajectory in complex plane according to nonlinear magnitude scaling (M^{α}) and linear angle scaling ($\alpha\theta$). These properties of this trajectories can be used to establish analytical relations between fractional order systems and integer order LTI systems. This can provide simplifications in design and analyses of this class of fractional order systems.

(ii) First order LTI system yields real coefficient root equivalent commensurate order systems by addition of a complex conjugate root. This class of root equivalent commensurate order systems of first order stable LTI systems establishes a constant stability boundary at $\alpha = 4/3$, which yields oscillating systems. For $0 < \alpha < 4/3$, the system is stable [23]. In the stable region, we observed that effect of order α on time responses of this class exhibit consistently.

(iii) Second order LTI system can yield real coefficient root equivalent commensurate order systems at $\alpha = \pi/\theta_1$ without addition of any complex conjugate root. If the second order LTI system is stable, the real coefficient root equivalent commensurate order system is also stable.

Computational complexity of stability analyses for this type commensurate order systems can be reduced, significantly. In future studies, fractional order root equivalency relations can be extended higher order LTI systems.

V. Appendix

Properties for the first order LTI systems:

Property 1 (Stability Boundary at $\alpha = 4/3$) : The real coefficient commensurate order root equivalent systems of the stable first order LTI systems, which is expressed in the form of equation (7), is (i) stable for $0 < \alpha < 4/3$, (ii) oscillating for $\alpha = 4/3$ and (iii) unstable for $\alpha > 4/3$.

Proof:

Let consider a stable first order system. This infers the condition of $\theta = \pi$. Now, we can treat separately two cases of $\alpha > 0$, which are $0 < \alpha \le 1$ and $\alpha > 1$. According to value of α , the stability region of commensurate order systems change according to the root angle ranges $-\pi < \alpha\theta < -\alpha\frac{\pi}{2}$ and $\alpha\frac{\pi}{2} < \alpha\theta \le \pi$ as seen in Fig. 8 [1].

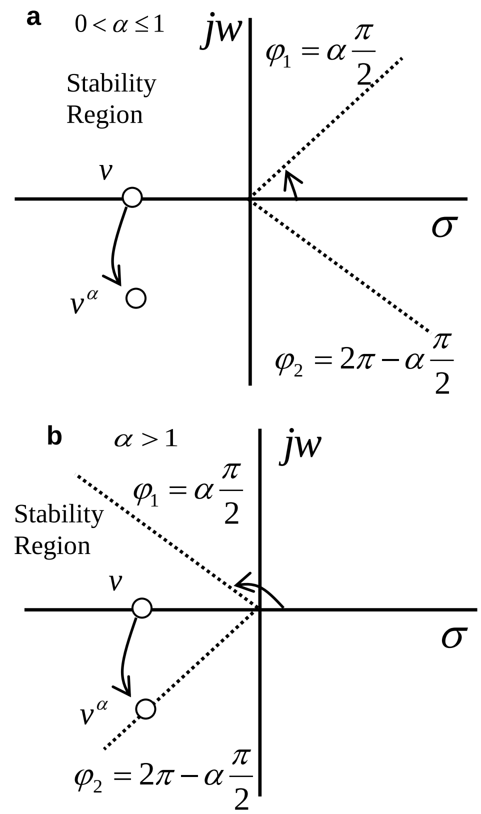

Fig. 8.Root shifting analysis for the first order stable LTI systems in the stability region for $0 < \alpha \le 1$ in (a) and $\alpha > 1$ in (b) [1]

(i) In the case of $0 < \alpha \le 1$, lower stability boundary $\varphi_1 = \alpha\frac{\pi}{2}$ moves toward zero angle and this expand stability region as shown Fig. 8(a). Due to $\alpha\theta = \alpha\pi$ for commensurate order root equivalency of a first order LTI system, the condition $\alpha\theta > \alpha\frac{\pi}{2}$ is always satisfied for every $0 < \alpha \le 1$ and therefore the roots of equivalent systems always stay in the stability region.

(ii) In the case of $\alpha > 1$, due to narrowing stability region as in Fig. 8(b), the root of the first order equivalent system comes closer to the boundary $\varphi_2 = 2\pi - \alpha\frac{\pi}{2}$. The α -order, which shifts the root of the first order LTI system on the boundary line φ_2, results in sinusoidal oscillation of commensurate order systems. So, to figure out α -order to shift the root $v^{\alpha} = M^{\alpha}e^{j\alpha\pi}$ to the stability boundary $\varphi_2 = 2\pi - \alpha\frac{\pi}{2}$, one can solve the equation $\alpha\theta = \alpha\pi = 2\pi - \alpha\frac{\pi}{2}$ and calculate the stability boundary of $\alpha = 4/3$. Here, $\alpha = 4/3$ results in sinusoidal oscillation of commensurate order root equivalent system of the first order LTI systems. For unstable commensurate order root equivalent systems, the root should move beyond the stability boundary. So, $\alpha\theta = \alpha\pi > 2\pi - \alpha\frac{\pi}{2}$

For $0 < \alpha < 4/3$, the system response stable because of placement of root inside the stability region. For $\alpha = 4/3$, the system oscillates because of placement of root on the stability boundary, and for $\alpha > 4/3$, the system becomes unstable because of placement of root outside the stability region.

Property 2: The real coefficient commensurate order root equivalent systems of the unstable first order LTI systems is unstable for $\forall\ \alpha \in R$.

Proof:
For the first order unstable systems, one can take the root angle $\theta = 0$. Fig. 9 clearly shows that the root angle of α -order equivalent root is $\alpha\theta = \alpha 0 = 0$ for $\alpha \in R$ and therefore it always stays out of the stability region that expressed the angle range $-\pi < \alpha\theta < -\alpha\frac{\pi}{2}$ and $\alpha\frac{\pi}{2} < \alpha\theta \le \pi$.

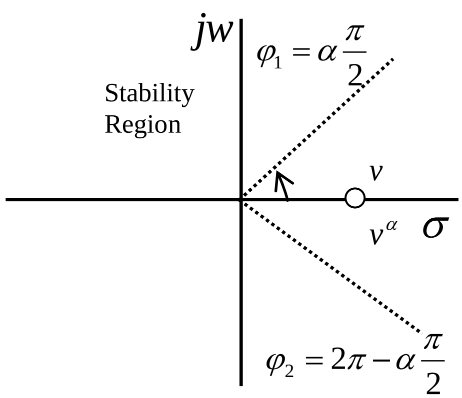

Fig. 9.Root shifting analysis for the first order unstable LTI systems [1]

Properties for the second order LTI systems:
Property 3 (Stability Boundary $\theta_1 = \pi/2$) if integer order LTI system is stable, the real coefficient root equivalent commensurate order system in the form of equation (11) for $\alpha = \pi/\theta_1$ is stable.

Proof:
Let assume a stable second order system is given. This infers the condition of $\frac{\pi}{2} < \theta_1 < \frac{3\pi}{2}$. Now, let's treat two cases of α, which are $0 < \alpha \le 1$ and $\alpha > 1$.

(i) In the case of $0 < \alpha \le 1$, roots shifts towards the stability boundary $\varphi_1 = \alpha\frac{\pi}{2}$. If the second order system is stable, the real coefficient commensurate order root equivalent systems obtained for $\alpha = \pi/\theta_1$ are always stable because the closest root is always stays in stability region because $\alpha\theta_1 = \alpha\frac{\pi}{\alpha} = \pi$ and therefore $\alpha\frac{\pi}{2} < \alpha\theta_1$ is always satisfied.

(ii) In the case of $\alpha > 1$, the complex root can move toward the upper stability boundary $\varphi_2 = 2\pi - \alpha\frac{\pi}{2}$. When it place on this boundary, the α order commensurate order system oscillates. If $\alpha\theta_1 = 2\pi - \alpha\frac{\pi}{2}$ equation is solved for real coefficient root equivalent with $\alpha = \pi/\theta_1$, one obtains $\theta_1 = \pi/2$. If root angle θ_1 is equal or greater than $\frac{\pi}{2}$, the commensurate order system are stable. As a consequence, if integer order LTI system is stable, the root equivalent commensurate order system with $\alpha = \pi/\theta_1$ is also stable.

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