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Porosity Supremum-Infimum and Porosity Convergence

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Abstract

In this paper, by using porosity notion for subsets of natural numbers at infinity porosity lower and porosity upper bound of real valued sequences will be defined. By using these new notions, definitions of porosity infimum and porosity supremum will be given, respectively. For a given sequence, the equivalence of porosity infimum and porosity supremum is necessary and sufficient condition for to existence of porosity convergence but it is necessary, not sufficient, condition for to existence of usual convergence.

Keywords: *Porosity, Supremum, Infimum, Convergence.*
2010 Mathematics Subject Classification: *40F99, 97I30*

1. Introduction

The concept of porosity for sets was given by Denjoy [4] and Khintchine [7] under different terminologies in 1920 and 1924, respectively. Then, in 1967 porosity reappeared Dolzenko's work on the concept of cluster sets [5]. Denjoy was dealed with classification of complements of the perfect sets [4]. Also, Khintchine gave some new definitions by using density [7].
Let $A \subset \mathbb{R}^+ = [0,\infty)$, then, the right upper porosity of A at the point 0 is defined as

$$p^+(A) := \limsup_{h\to 0^+} \frac{\lambda(A,h)}{h}$$

where $\lambda(A,h)$ denotes the length of the largest open subinterval of $(0,h)$ that contains no point of A.
The notion of the right lower porosity of A at the point 0 is defined similarly.
By using the right upper porosity of a set at the point 0, the definitions of right upper porosity and right lower porosity for the subsets of natural numbers at infinity was given in [1].
Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a strictly decreasing function such that $\lim\limits_{n\to\infty} \mu(n) = 0$, and let E be a subset of $\mathbb{N}$. Right upper porosity and right lower porosity of E at infinity

$$\overline{p}_\mu(E) := \limsup_{n\to\infty} \frac{\lambda_\mu(E,n)}{\mu(n)}$$

and

$$\underline{p}_\mu(E) := \liminf_{n\to\infty} \frac{\lambda_\mu(E,n)}{\mu(n)}$$

where

$$\lambda_\mu(E,n) := sup\{|\mu(n^{(1)}) - \mu(n^{(2)})| \ : \ n \le n^{(1)} < n^{(2)}, \ (n^{(1)},n^{(2)}) \cap E = \emptyset\}.$$

Throughout this paper, we will consider only the right upper porosity for subsets of $\mathbb{N}$ and use following terminology: A set $E \subseteq \mathbb{N}$ is called

- Porous at infinity if $\overline{p}_\mu(E) > 0$;
- Strongly porous at infinity if $\overline{p}_\mu(E) = 1$;
- Nonporous at infinity if $\overline{p}_\mu(E) = 0$.

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By using right upper porosity of a subset of natural numbers at infinity, porosity convergence was defined and some basic properties was given in [2].

Definition 1.1. Let $x=(x_n)_{n\in\mathbb{N}}$ be a real valued sequence. $x=(x_n)$ is $\overline{p}_\mu$-convergent to l if for every $\varepsilon>0$,

$$\overline{p}_\mu(A_\varepsilon)>0 \text{ and } \overline{p}_\mu(A_\varepsilon^c)=0$$

where $A_\varepsilon := \{n \ : \ |x_n-l|\geq\varepsilon\}$ and A_ε^c is the complement of A_ε. It is denoted by $x_n \to l \ (\overline{p}_\mu)$ or $\overline{p}_\mu - \lim_{n\to\infty} x_n = l$.

Let $x'=(x_{n_k})$ be a subsequence of $x=(x_n)$ and $K:=\{n_k : k\in\mathbb{N}\}$, then, we abbreviate $x'=(x_{n_k})$ by $(x)_K$. Some classes of subsequences of a given sequence given in [3] as follows:

Definition 1.2. Let $x=(x_n)$ be a sequence and $(x)_K$ be a subsequence of $x=(x_n)$.
(*i*) If $\overline{p}_\mu(K)>0$, then, $(x)_K$ is called $\overline{p}_\mu$-thin subsequence of $x=(x_n)$,
(*ii*) If $\overline{p}_\mu(K)=1$, then, $(x)_K$ is called a strongly $\overline{p}_\mu$-thin subsequence of $x=(x_n)$,
(*iii*) If $\overline{p}_\mu(K)=0$, then, $(x)_K$ is a $\overline{p}_\mu$-nonthin subsequence of $x=(x_n)$.

Now, let us recall that the definition of peak point for real valued sequences we will use later.

Definition 1.3. (Peak Point [6]) The point x_l is called upper(or lower) peak point of the sequence $x=(x_n)$ if the inequality $x_l \geq x_n$ (or $x_l \leq x_n$) holds for all $n \geq l$.

2. New Results

In [8] statistical supremum and statistical infimum have been defined for real valued sequences by using natural density. Here porosity infimum and porosity supremum for real valued sequences will be defined and some related results will be given.

Definition 2.1. (Porosity Lower Bound) It is said that $l\in\mathbb{R}$ is a porosity lower bound of a sequence $x=(x_n)$, if the following

$$\overline{p}_\mu(\{n : x_n<l\})>0 \text{ and } \overline{p}_\mu(\{n : x_n\geq l\})=0 \tag{2.1}$$

hold.

The set of all porosity lower bound of the sequence $x=(x_n)$ is denoted by $L_{\overline{p}_\mu}(x)$.
Let us also denote the set of all usual lower bound of the sequence $x=(x_n)$ by $L(x)$, i.e.,

$$L(x) := \{l\in\mathbb{R} : l\leq x_n \text{ for all } n\in\mathbb{N}\}.$$

Definition 2.2. (Porosity Upper Bound) It is said that $L\in\mathbb{R}$ is a porosity upper bound of given sequence $x=(x_n)$, if the following

$$\overline{p}_\mu(\{n : x_n>L\})>0 \text{ and } \overline{p}_\mu(\{n : x_n\leq L\})=0 \tag{2.2}$$

hold.

The set of all porosity upper bound of given sequence $x=(x_n)$ is denoted by $U_{\overline{p}_\mu}(x)$.
Let us denote the set of all usual upper bounds of the sequence $x=(x_n)$ by $U(x)$, i.e.,

$$U(x) := \{L\in\mathbb{R} : x_n\leq L \text{ for all } n\in\mathbb{N}\}.$$

Hence, we have following simple results:

Theorem 2.1. *If $l\in\mathbb{R}$ is an usual lower bound of the sequence $x=(x_n)$, then, $l\in\mathbb{R}$ is a porosity lower bound of the sequence $x=(x_n)$.*

Proof. From the definition of usual lower bound, we have $l\leq x_n$ for all $n\in\mathbb{N}$. So, the set $\{n : x_n<l\}$ is empty. Therefore,

$$\overline{p}_\mu(\{n : x_n<l\})=1>0 \text{ and } \overline{p}_\mu(\{n : x_n\geq l\})=\overline{p}_\mu(\mathbb{N})=0$$

hold. Therefore, $L(x)\subset L_{\overline{p}_\mu}(x)$ holds for any $x=(x_n)$. □

Remark 2.1. The converse of Theorem 2.1 is not true in general.
Let us consider the sequence $x=(x_n)=(-\frac{1}{n})$ and take $l=-\frac{1}{2}\in\mathbb{R}$. Since $\overline{p}_\mu(\{n \ : \ x_n<-\frac{1}{2}\})=\overline{p}_\mu(\{1\})=1$ and $\overline{p}_\mu(\{n \ : \ x_n\geq-\frac{1}{2}\})=0$, then, $l=-\frac{1}{2}$ is a porosity lower bound but it is clear that $l=-\frac{1}{2}$ is not usual lower bound of this sequence.

Theorem 2.2. *If $L\in\mathbb{R}$ is an usual upper bound of the sequence $x=(x_n)$, then, $L\in\mathbb{R}$ is a porosity upper bound of the sequence $x=(x_n)$.*

Proof. Since $L\in\mathbb{R}$ is an usual upper bound of the sequence $x=(x_n)$, then, $x_n\leq L$ holds for all $n\in\mathbb{N}$. So,

$$\{n : x_n>L\}=\emptyset.$$

Therefore,

$$\overline{p}_\mu(\{n : x_n>L\})=1>0 \text{ and } \overline{p}_\mu(\{n : x_n\leq L\})=\overline{p}_\mu(\mathbb{N})=0$$

hold. This means that $U(x)\subset U_{\overline{p}_\mu}(x)$. □

Remark 2.2. The converse of the Theorem 2.2 is not true, in general.

Let us consider the sequence $x=(x_n)=(\frac{1}{n})$ and take $L=\frac{1}{2}\in\mathbb{R}$. It is clear that $L=\frac{1}{2}$ is a porosity upper bound because $\overline{p}_\mu(\{n : x_n > \frac{1}{2}\})=\overline{p}_\mu(\{1\})=1$ and $\overline{p}_\mu(\{n : x_n \le \frac{1}{2}\})=0$, but it is not usual upper bound for the sequence.
From Definition 2.1 and Definition 2.2, we have also following simple result:

Theorem 2.3. *i) If $l\in\mathbb{R}$ is a porosity lower bound and $l'<l$, then, $l'\in\mathbb{R}$ is also porosity lower bound of the sequence $x=(x_n)$.*
ii) If $L\in\mathbb{R}$ is a porosity upper bound and $L<L'$, then, $L'\in\mathbb{R}$ is also porosity upper bound of the sequence $x=(x_n)$.

Before giving the proof, we need following simple lemma:

Lemma 2.1. *[2] Let $A,B\subset\mathbb{N}$ and $A\subset B$. If $\overline{p}_\mu(B)>0$, then, $\overline{p}_\mu(A)>0$. Also, $\overline{p}_\mu(B)<\overline{p}_\mu(A)$ holds.*

Proof of Theorem 2.3. i) Assume that $l\in\mathbb{R}$ is a porosity lower bound of the sequence $x=(x_n)$. That is, the set $\{n : x_n < l\}$ is porous and the set $\{n : x_n \ge l\}$ is nonporous. Since $l'<l$, then, the following inclusion

$$\left\{n : x_n < l'\right\}\subseteq\{n : x_n < l\} \text{ and } \{n : x_n \ge l\}\subseteq\left\{n : x_n \ge l'\right\}$$

hold. Since $\overline{p}_\mu(\{n : x_n < l\})>0$, then, Lemma 2.1 gives that $\overline{p}_\mu(\{n : x_n < l'\})>0$ and $\overline{p}_\mu(\{n : x_n \ge l'\})=0$. So l' is also a porosity lower bound.
ii) Since $L\in\mathbb{R}$ is a porosity upper bound of the sequence $x=(x_n)$, then, the set $\{n : x_n > L\}$ is porous and the set $\{n : x_n \le L\}$ is nonporous. Since $L<L'$, then, the following inclusion

$$\left\{n : x_n > L'\right\}\subset\{n : x_n > L\} \text{ and } \{n : x_n \le L\}\subseteq\left\{n : x_n \le L'\right\}$$

hold. Since $\overline{p}_\mu(\{n : x_n > L\})>0$, then, from Lemma 2.1 $\overline{p}_\mu(\{n : x_n > L'\})>0$ and $\overline{p}_\mu(\{n : x_n \le L'\})=0$. So L' is also a porosity upper bound. □

Corollary 2.1. *If a sequence $x=(x_n)$ has a porosity lower (or porosity upper) bound, then, it has infinitely many porosity lower (or porosity upper) bounds.*

Definition 2.3. (Porosity Infimum ($\inf_{\overline{p}_\mu}$)) A number $l\in\mathbb{R}$ is called porosity infimum of the sequence $x=(x_n)$ if it is supremum of $L_{\overline{p}_\mu}(x)$. That is, $\inf_{\overline{p}_\mu} x_n := \sup L_{\overline{p}_\mu}(x)$.

Definition 2.4. (Porosity Supremum ($\sup_{\overline{p}_\mu}$)) A number $L\in\mathbb{R}$ is called porosity supremum of the sequence $x=(x_n)$ if it is infimum of $U_{\overline{p}_\mu}(x)$. That is, $\sup_{\overline{p}_\mu} x_n := \inf U_{\overline{p}_\mu}(x)$.

Theorem 2.4. *Let $x=(x_n)$ be a sequence of real numbers. Then, following inequalities*

$$\inf x_n \le \inf_{\overline{p}_\mu} x_n \le \sup_{\overline{p}_\mu} x_n \le \sup x_n \tag{2.3}$$

hold.

Proof. From the definition of usual infimum we have

$$\overline{p}_\mu(\{n : x_n < \inf x_n\})=\overline{p}_\mu(\emptyset)=1>0 \text{ and } \overline{p}_\mu(\{n : x_n \ge \inf x_n\})=\overline{p}_\mu(\mathbb{N})=0$$

This gives that $\inf x_n\in L_{\overline{p}_\mu}(x)$. Since $\inf_{\overline{p}_\mu} x_n=\sup L_{\overline{p}_\mu}(x)$, then, we have

$$\inf_{\overline{p}_\mu} x_n \ge \inf x_n.$$

From the definition of usual supremum we have

$$\overline{p}_\mu(\{n : x_n > \sup x_n\})=\overline{p}_\mu(\emptyset)=1>0 \text{ and } \overline{p}_\mu(\{n : x_n \le \sup x_n\})=\overline{p}_\mu(\mathbb{N})=0$$

This gives that $\sup x_n\in U_{\overline{p}_\mu}(x)$. Since $\sup_{\overline{p}_\mu} x_n=\inf U_{\overline{p}_\mu}(x)$, then, we have

$$\sup_{\overline{p}_\mu} x_n \le \sup x_n.$$

For to complete the proof, it is enough to show that the inequality

$$l\le L \tag{2.4}$$

holds for an arbitrary $l\in L_{\overline{p}_\mu}(x)$ and $L\in U_{\overline{p}_\mu}(x)$.
Let us assume (2.4) is not true. That is, there exist $l'\in L_{\overline{p}_\mu}(x)$ and $L'\in U_{\overline{p}_\mu}(x)$ such that $L'<l'$ holds. Since L' is a porosity upper bound, then, from Theorem 2.3 (ii), l' is also porosity upper bound of the sequence. This is a contradiction to the assumption on l'. So, (2.4) is true. □

It is possible to find a sequence such that some inequalities in (2.3) turn to equality:

Remark 2.3. *i*) Let $x=(x_n)$ be a constant sequence. Then,

$\inf x_n = \inf_{\overline{p}_\mu} x_n = \sup_{\overline{p}_\mu} x_n = \sup x_n.$

ii) Let $x=(x_n)$ deine as follows:

$$x_n = \begin{cases} x_n, & n \le n_0,\ n_0 \in \mathbb{N} \\ a, & n > n_0, \end{cases}$$

such that $x_n \le a$ for all $n \in \{1,2,3,...,n_0\}$. Then,

$\sup_{\overline{p}_\mu} x_n = \sup x_n.$

iii) Let $x=(x_n)$ deine as follows:

$$x_n = \begin{cases} x_n, & n \le n_0,\ n_0 \in \mathbb{N} \\ a, & n > n_0, \end{cases}$$

such that $x_n \ge a$ for all $n \in \{1,2,3,...,n_0\}$. Then ,

$\inf x_n = \inf_{\overline{p}_\mu} x_n$

In Theorem 2.5 and Theorem 2.6, necessary and sufficient conditions will be given for a number to be a porosity infimum and porosity supremum of a real valued sequence, respectively.

Theorem 2.5. *Let $x=(x_n)$ be a real valued sequence and $L \in \mathbb{R}$. Then, $\inf_{\overline{p}_\mu} x_n = L$ if and only if for every $\varepsilon > 0$*

(i) $\overline{p}_\mu(\{n : x_n < L-\varepsilon\}) > 0$ and $\overline{p}_\mu(\{n : x_n \ge L-\varepsilon\}) = 0$

and

(ii) $\overline{p}_\mu(\{n : x_n < L+\varepsilon\}) = 0$ and $\overline{p}_\mu(\{n : x_n \ge L+\varepsilon\}) > 0$

hold.

Proof. "⇒" Assume that $\inf_{\overline{p}_\mu} x_n = L$. i.e., $\sup L_{\overline{p}_\mu}(x) = L$. So, we have

$(a)\ s \le L, \forall s \in L_{\overline{p}_\mu}(x),$

and

$(b)\ \forall \varepsilon > 0\ \exists s' \in L_{\overline{p}_\mu}(x)$

such that $L-\varepsilon < s'$ holds.
Since from (b) and Theorem 2.3, $L-\varepsilon$ is a porosity lower bound. So, (i) holds.
Now, assume that (ii) is not hold for all $\varepsilon > 0$. That is, there exists ε_0 such that

$\overline{p}_\mu(\{n : x_n < L+\varepsilon_0\}) > 0.$

This mean that $\overline{p}_\mu(\{n : x_n \ge L+\varepsilon_0\}) = 0$ and $L+\varepsilon_0 \in L_{\overline{p}_\mu}(x)$. Since $L < L+\varepsilon_0$, this is contradiction to assumption on L.
"⇐" Now, assume that, (i) and (ii) hold for all positive $\varepsilon > 0$. It is clear that $L-\varepsilon \in L_{\overline{p}_\mu}(x)$ and $L+\varepsilon \notin L_{\overline{p}_\mu}(x)$. Therefore, $L_{\overline{p}_\mu}(x) = (-\infty, L-\varepsilon]$, for all $\varepsilon > 0$. So, we have $\sup L_{\overline{p}_\mu}(x) = L$. □

Theorem 2.6. *Let $x=(x_n)$ be a real valued sequence and $l \in \mathbb{R}$. Then, $\sup_{\overline{p}_\mu} x_n = l$ if and only if for every $\varepsilon > 0$*

(i) $\overline{p}_\mu(\{n : x_n > l+\varepsilon\}) > 0$ and $\overline{p}_\mu(\{n : x_n \le l+\varepsilon\}) = 0$

and

(ii) $\overline{p}_\mu(\{n : x_n > l-\varepsilon\}) = 0$ and $\overline{p}_\mu(\{n : x_n \le l-\varepsilon\}) > 0$

hold.

Proof. "⇒" Since $\sup_{\overline{p}_\mu} x_n = l$, then, $l = \inf U_{\overline{p}_\mu}(x)$. Therefore, we have

$(a)\ l \le s, \forall\ s \in U_{\overline{p}_\mu}(x),$

and

$(b)\ \forall \varepsilon > 0\ \exists s' \in U_{\overline{p}_\mu}(x)$

such that $s' < l+\varepsilon$ holds.
From Theorem 2.3 and (b), $l+\varepsilon$ is a porosity upper bound of the sequence. So, (i) holds.
Now, assume that, (ii) is not true. That is, there exists an $\varepsilon_0 > 0$ such that $\overline{p}_\mu(\{n : x_n > l-\varepsilon_0\}) > 0$. It means that $\overline{p}_\mu(\{n : x_n \le l-\varepsilon_0\}) = 0$ and $l-\varepsilon_0 \in U_{\overline{p}_\mu}(x)$. But this is contradiction to $l = \inf U_{\overline{p}_\mu}(x)$.
"⇐" Now, assume that, for every $\varepsilon > 0$, (i) and (ii) hold. From (i) and (ii) we have $l+\varepsilon \in U_{\overline{p}_\mu}(x)$ and $l-\varepsilon \notin U_{\overline{p}_\mu}(x)$, respectively. Therefore, $U_{\overline{p}_\mu}(x) = [l+\varepsilon, \infty)$ and $\inf U_{\overline{p}_\mu}(x) = l$. □

Theorem 2.7. *Let $x=(x_n)$ be a real valued sequence. The following statements are true:*
(i) If the sequence $x=(x_n)$ is monotone increasing, then, $\inf_{\overline{p}_\mu} x_n = \sup x_n$,
(ii) If the sequence $x=(x_n)$ is monotone decreasing, then, $\sup_{\overline{p}_\mu} x_n = \inf x_n$.

Proof. We shall give only the proof of (i). The case (ii) can be obtained by doing suitable changes in the proof of (i).
Now, assume that $x=(x_n)$ is a monotone increasing sequence such that

$$\sup x_n < \infty,$$

holds. So, we have for all $n \in \mathbb{N}$,

$$x_n \leq \sup x_n, \tag{2.5}$$

and for every $\varepsilon > 0$ there exist an $n_0 \in \mathbb{N}$ such that

$$\sup x_n - \varepsilon < x_{n_0}. \tag{2.6}$$

From (2.5) we have, $\sup x_n \notin L_{\overline{p}_\mu}(x)$. Also, from (2.6) we have

$$\{n : x_n < \sup x_n - \varepsilon\} \subset \{1,2,3,...,n_0\}.$$

Hence, Lemma 2.1 gives that

$$\sup x_n - \varepsilon \in L_{\overline{p}_\mu}(x).$$

Therefore, from Remark 2.3

$$L_{\overline{p}_\mu}(x) = (-\infty, \sup x_n - \varepsilon),$$

holds for all $\varepsilon > 0$. So,

$$\inf_{\overline{p}_\mu} x_n = \sup L_{\overline{p}_\mu}(x) = \sup x_n.$$

Now, assume that

$$\sup x_n = \infty.$$

It means that for all $l \in \mathbb{R}$ there is an $n_0 = n_0(x) \in \mathbb{N}$ such that $l \leq x_{n_0}$ and for every $n \geq n_0$ the inequality $x_{n_0} \leq x_n$ holds. So, we have following inclusion

$$\{n : x_n < l\} \subseteq \{1,2,3,...,n_0\}.$$

From last inclusion and Lemma 2.1, we have

$$\overline{p}_\mu(\{n : x_n < l\}) = 1 > 0 \text{ and } \overline{p}_\mu(\{n : x_n \geq l\}) = 0.$$

So, $l \in L_{\overline{p}_\mu}(x)$ for any $l \in \mathbb{R}$. Therefore,

$$L_{\overline{p}_\mu}(x) = (-\infty, \infty) \text{ and } \sup L_{\overline{p}_\mu}(x) = \infty.$$

This gives the proof. □

Corollary 2.2. *Assume $x=(x_n)$ real valued bounded sequence. If the sequence $x=(x_n)$ is monotone decreasing (or increasing), then,*

$$\lim_{n\to\infty} x_n = \sup_{\overline{p}_\mu} x_n \ (or = \inf_{\overline{p}_\mu} x_n).$$

Theorem 2.8. *Let $x=(x_n)$ be a real valued sequence. If the element x_{n_0} is an upper(or lower) peak point of (x_n), then, the element x_{n_0} is a porosity upper (or porosity lower) bound.*

Proof. Assume that the point x_{n_0} is an upper (or lower) peak point of the sequence $x=(x_n)$ such that $x_n \leq x_{n_0}$ (or $x_{n_0} \leq x_n$) holds for all $n \geq n_0$. So, the inclusion

$$\{n : x_n > x_{n_0}\} \subseteq \{1,2,...,n_0\}, \text{ (or } \{n : x_n < x_{n_0}\} \subset \{1,2,...,n_0\})$$

holds. From Lemma 2.1, we have

$$\overline{p}_\mu(\{n : x_n > x_{n_0}\}) = 1 > 0 \text{ and } \overline{p}_\mu(\{n : x_n \leq x_{n_0}\}) = 0 \text{ (or } \overline{p}_\mu(\{n : x_n < x_{n_0}\}) = 1 > 0 \text{ and } \overline{p}_\mu(\{n : x_n \geq x_{n_0}\}) = 0).$$

This give us the point x_{n_0} is a porosity upper (or lower) bound of the sequence $x=(x_n)$. □

Definition 2.5. Any two real valued sequences $x=(x_n)$ and $y=(y_n)$ are called porosity equivalent if the set $A = \{n : x_n \neq y_n\}$ is porous. It is denoted by $x \asymp y(\overline{p}_\mu)$.

Theorem 2.9. *If the sequence $x=(x_n)$ and $y=(y_n)$ are porosity equivalent, then,*

$$\inf_{\overline{p}_\mu} x_n = \inf_{\overline{p}_\mu} y_n \ and \ \sup_{\overline{p}_\mu} x_n = \sup_{\overline{p}_\mu} y_n.$$

Proof. Since the sequences $x=(x_n)$ and $y=(y_n)$ are porosity equivalent, then, the set $A=\{n : x_n \neq y_n\}$ is porous. Let us consider an arbitrary element $l \in L_{\overline{p}_\mu}(x)$. The element $l \in \mathbb{R}$ is a porosity lower bound of the sequence $x=(x_n)$, then, we have

$$\overline{p}_\mu(\{n : x_n < l\}) > 0 \text{ and } \overline{p}_\mu(\{n : x_n \geq l\}) = 0.$$

From the following inclusions

$$\begin{aligned} \{n : x_n < l\} &= \{n : x_n \neq y_n < l\} \cup \{n : x_n = y_n < l\} \\ &\subset A \cup \{n : x_n = y_n < l\}, \end{aligned}$$

and $\{n : y_n \geq l\} = \{n : x_n \neq y_n \geq l\} \cup \{n : x_n = y_n \geq l\}$ we obtain

$$\overline{p}_\mu(\{n : y_n < l\}) > 0 \text{ and } \overline{p}_\mu(\{n : y_n \geq l\}) = 0. \tag{2.7}$$

From (2.7), the element $l \in \mathbb{R}$ is a porosity lower bound of the sequence $y=(y_n)$. That is, $L_{\overline{p}_\mu}(x) \subset L_{\overline{p}_\mu}(y)$. If we consider arbitrary point $l \in L_{\overline{p}_\mu}(y)$, it can obtain easily $l \in L_{\overline{p}_\mu}(x)$ such that $L_{\overline{p}_\mu}(y) \subset L_{\overline{p}_\mu}(x)$. Therefore,

$$L_{\overline{p}_\mu}(y) = L_{\overline{p}_\mu}(x) \tag{2.8}$$

hold. Since $\sup L_{\overline{p}_\mu}(y) = \sup L_{\overline{p}_\mu}(x)$, then, $\inf_{\overline{p}_\mu} x_n = \inf_{\overline{p}_\mu} y_n$ is obtained.
By using the same idea as above it can be obtained $\sup_{\overline{p}_\mu} x_n = \sup_{\overline{p}_\mu} y_n$. □

Remark 2.4. The converse of Theorem 2.9 is not true.

Let us consider the sequences $x=(x_n)$ and $y=(y_n)$ as follows:

$$x_n := 1 - \frac{1}{n} \text{ and } y_n := 1 + \frac{1}{n}$$

for all $n \in \mathbb{N}$. Then, it is clear that

$$\inf_{\overline{p}_\mu} x_n = \inf_{\overline{p}_\mu} y_n = 1 \text{ and } \sup_{\overline{p}_\mu} x_n = \sup_{\overline{p}_\mu} y_n = 1.$$

But, $A=\{n : x_n \neq y_n\} = \mathbb{N}$ and $\mathbb{N}$ is nonporous. So, $x=(x_n)$ and $y=(y_n)$ are not porosity equivalent.

Theorem 2.10. *Let $x=(x_n)$ be a real valued sequence and $(x)_K$ be a $\overline{p}_\mu$-nonthin subsequence of $x=(x_n)$.*
(i) If $\inf_{\overline{p}_\mu} x_n = m$, then, m is porosity infimum of $(x)_K$.
(ii) If $\sup_{\overline{p}_\mu} x_n = l$, then, l is porosity supremum of $(x)_K$.

Proof. We will prove only (i). The other can be obtained by the same way.
(i) Let $\inf_{\overline{p}_\mu} x_n = m$. Then, from Theorem 2.5

$$\overline{p}_\mu(\{n : x_n < m-\varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n : x_n \geq m-\varepsilon\}) = 0$$

holds for every $\varepsilon > 0$. It is clear that $\{n_k : x_{n_k} < m-\varepsilon\} \subseteq \{n : x_n < m-\varepsilon\}$. This inclusion and Lemma 2.1 give that

$$\overline{p}_\mu(\{n : x_{n_k} < m-\varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n_k : x_{n_k} \geq m-\varepsilon\}) = 0.$$

Hence, m is porosity infimum of $(x)_K$. □

Remark 2.5. The converse of Theorem 2.10 is not true, in general.
Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a scaling function such that $\mu(n) = \frac{1}{n}, n \in \mathbb{N}$. Let a sequence $x=(x_n)$ and its subsequence (x_{2m}) as follows:

$$x_n = \begin{cases} -2^k, & n = 2^k, \\ -3^k, & n = 2k-1, \\ 0, & otherwise, \end{cases}$$

and

$$x_{2m} = \begin{cases} -2^k, & m = 2^{k-1}, \\ 0, & otherwise. \end{cases}$$

It is clear that (x_{2m}) is a $\overline{p}_\mu$-nonthin subsequence of $x=(x_n)$. So the condition of Theorem 2.10 holds. Also $\inf_{\overline{p}_\mu}(x_{2m}) = 0$. But $\inf_{\overline{p}_\mu}(x_n) \neq 0$.

Remark 2.6. $\overline{p}_\mu$-nonthiness of subsequence can not be omitted in Theorem 2.10.

Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a scaling function such that $\mu(n) = \frac{1}{n}, n \in \mathbb{N}$. Let a sequence $x=(x_n)$ as follows:

$$x_n := \begin{cases} -n, & n = 2^m, \ m \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

Let take the subsequence (x_{2^m}) of $x=(x_n)$. It is clear that (x_{2^m}) is not $\overline{p}_\mu$-nonthin subsequence of (x_n). Also, $\inf_{\overline{p}_\mu}(x_n) = 0$ but $\inf_{\overline{p}_\mu}(x_{2^m}) \neq 0$.

Theorem 2.11. *If $\lim_{n\to\infty} x_n = l$, then, $\sup_{\overline{p}_\mu} x_n = \inf_{\overline{p}_\mu} x_n = l$.*

Proof. Assume $\lim_{n\to\infty} x_n = l$. That is, for any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon, \tag{2.9}$$

holds for all $n \geq n_0$. Hence, following inclusions deduced from (2.9) easily

$$\{n : x_n < l - \varepsilon\} \subset \{1, 2, ..., n_0\}, \ \{n : x_n > l + \varepsilon\} \subset \{1, 2, ..., n_0\}. \tag{2.10}$$

So, by using (2.10) and Lemma 2.1, we have

$$\overline{p}_\mu(\{n : x_n < l - \varepsilon\}) = 1 \text{ and } \overline{p}_\mu(\{n : x_n \geq l - \varepsilon\}) = 0$$

and

$$\overline{p}_\mu(\{n : x_n > l + \varepsilon\}) = 1 \text{ and } \overline{p}_\mu(\{n : x_n \leq l + \varepsilon\}) = 0.$$

This gives that

$$l - \varepsilon \in L_{\overline{p}_\mu}(x), \ l + \varepsilon \in U_{\overline{p}_\mu}(x)$$

for all $\varepsilon > 0$. Also, Theorem 2.3 gives that

$$L_{\overline{p}_\mu}(x) = (-\infty, l] \ \text{ and } \ U_{\overline{p}_\mu}(x) = [l, \infty).$$

Therefore,

$$\inf_{\overline{p}_\mu} x_n = \sup(-\infty, l] = l$$

and

$$\sup_{\overline{p}_\mu} x_n = \inf[l, \infty) = l$$

hold. This give the proof of theorem. □

Remark 2.7. The converse of the Theorem 2.11 is not true, in general.
Let us consider the sequence $x = (x_n)$ as

$$x_n = \begin{cases} 1, & n = 2^k, k = 1, 2, ..., \\ 0, & \textit{otherwise}. \end{cases}$$

It is clear that $\inf_{\overline{p}_\mu} x_n = \sup_{\overline{p}_\mu} x_n = 0$ but the sequence is not convergent to 0. On the other hand, $x = (x_n)$ is porosity convergent to 0.
In the following theorem, on the contrary of Theorem 2.11, it is proved that the equality of these two numbers is necessary and sufficient for existence of porosity limit.

Theorem 2.12. *$\overline{p}_\mu - \lim_{n\to\infty} x_n = l$ if and only if $\sup_{\overline{p}_\mu} x_n = \inf_{\overline{p}_\mu} x_n = l$.*

Proof. "$\Longrightarrow$" Assume that $\overline{p}_\mu - \lim_{n\to\infty} x_n = l$ holds. From the assumption, we have for any $\varepsilon > 0$,

$$\overline{p}_\mu(\{n : |x_n - l| \geq \varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n : |x_n - l| < \varepsilon\}) = 0. \tag{2.11}$$

Also, we have

$$\{n : \ |x_n - l| \geq \varepsilon\} = \{n : \ x_n \geq l + \varepsilon\} \cup \{n : \ x_n \leq l - \varepsilon\} \text{ and } \{n : \ |x_n - l| < \varepsilon\} = \{n : \ x_n < l + \varepsilon\} \cup \{n : \ x_n > l - \varepsilon\}$$

From the equation (2.11), and Lemma 2.1 we obtain

$$\overline{p}_\mu(\{n : x_n \geq l + \varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n : x_n < l + \varepsilon\}) = 0 \tag{2.12}$$

and

$$\overline{p}_\mu(\{n : x_n \leq l - \varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n : x_n > l - \varepsilon\}) = 0. \tag{2.13}$$

So, (2.12) and (2.13) gives that for every $\varepsilon > 0$, the number $l + \varepsilon$ is a porosity upper bound, $l - \varepsilon$ is a porosity lower bound, respectively. Therefore,

$$L_{\overline{p}_\mu}(x) = (-\infty, l) \text{ and } U_{\overline{p}_\mu}(x) = (l, \infty)$$

for all $\varepsilon > 0$. So, we have

$$\sup L_{\overline{p}_\mu}(x) = l, \ \ \inf U_{\overline{p}_\mu}(x) = l.$$

"$\Longleftarrow$" Assume that

$$\sup_{\overline{p}_\mu} x_n = \inf_{\overline{p}_\mu} x_n = l.$$

That is,

$\inf U_{\overline{p}_\mu}(x) = \sup L_{\overline{p}_\mu}(x) = l.$

From the definition of usual supremum and infimum, for all $\varepsilon > 0$, there exists at least one element $l' \in L_{\overline{p}_\mu}(x)$ and $l'' \in U_{\overline{p}_\mu}(x)$ such that the inequalities

$l - \varepsilon < l'$ and $l'' < l + \varepsilon$

hold.
Since l'' is a porosity upper bound, then, the following inclusion

$\{n : x_n > l + \varepsilon\} \subset \{n : x_n > l''\}$ and $\{n : x_n \le l''\} \subset \{n : x_n \le l + \varepsilon\}$

hold. So, we have

$$\overline{p}_\mu(\{n : x_n > l + \varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n : x_n \le l + \varepsilon\}) = 0. \tag{2.14}$$

Since l' is an porosity lower bound, then, the following inclusion

$\{n : x_n < l - \varepsilon\} \subset \{n : x_n < l'\}$ and $\{n : x_n \ge l'\} \subset \{n : x_n \ge l - \varepsilon\}$

hold. So, we have

$$\overline{p}_\mu(\{n : x_n < l - \varepsilon\}) > 0 \text{ and } \overline{p}_\mu(\{n : x_n \ge l - \varepsilon\}) = 0. \tag{2.15}$$

From (2.14), (2.15) and

$\{n : |x_n - l| \ge \varepsilon\} = \{n : x_n \ge l + \varepsilon\} \cup \{n : x_n \le l - \varepsilon\}$ and $\{n : |x_n - l| < \varepsilon\} = \{n : x_n < l + \varepsilon\} \cup \{n : x_n > l - \varepsilon\}$

we have

$\overline{p}_\mu(\{n : |x_n - l| \ge \varepsilon\}) > 0$ and $\overline{p}_\mu(\{n : |x_n - l| < \varepsilon\}) = 0.$

Therefore, the sequence $x = (x_n)$ is porosity convergent to $l \in \mathbb{R}$. □

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