MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

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# On a Subclass of $(k,\ \mu)$-Contact Metric Manifolds

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**Abstract**
The object of this paper is to characterize $(k,\ \mu)$-contact metric manifolds satisfying certain curvature conditions on the contact conformal curvature tensor.

***Keywords:*** *$(k,\ \mu)$-contact metric manifolds, N$(k)$-contact metric manifolds, contact conformal curvature tensor, extended contact conformal curvature tensor, $\eta$-Einstein manifolds.*

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## 1. INTRODUCTION

In [1] Blair, Koufogiorgos and Papantoniou introduced $(k, \mu)$-contact metric manifolds. A class of contact metric manifolds with contact metric structure $(\phi, \xi, \eta, g)$ in which the curvature tensor $R$ satisfies the condition

$$R(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y)$$

for all $X, Y \in \chi(M)$ is called $(k,\ \mu)$-contact metric manifolds.

In [11] Kitahara, Matsuo and Pak defined a new tensor on a Hermitian manifold which is conformally invariant.They called this new tensor the conformal invariant curvature tensor. The contact conformal curvature tensor was constructed from the conformal invariant curvature tensor by using the Boothby-Wang's fibration [4]). It seems to play an important role to study the spectral geometry ([9], [14]).
In [8] the authors defined the contact conformal curvature tensor $C_0$ in a (2n+1)-dimensional contact metric

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manifold$(M, \phi, \xi, \eta, g)$ by

$$\begin{aligned} C_0(X,Y)Z &= R(X,Y)Z \\ &+\frac{1}{2n}[-g(QY,Z)\phi^2 X + g(QX,Z)\phi^2 Y \\ &+g(\phi Y,\phi Z)QX - g(\phi X,\phi Z)QY \\ &g(Q\phi X,Z)\phi Y - g(Q\phi Y,Z)\phi X \\ &2g(\phi X,Y)QZ] + \frac{1}{2n(2n-1)}(2n^2-n-2+\frac{(n+2)r}{2n}) \\ &[g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z] \\ &+\frac{1}{2n(n+1)}(n+2-\frac{(3n+2)r}{2n})[g(Y,Z)X \\ &-g(X,Z)Y] - \frac{1}{2n(n+1)}(4n^2+5n+2-\frac{(3n+2)r}{2n}) \\ &[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + \\ &\eta(X)g(Y,Z)\xi - \eta(Y)g(X,Z)\xi \end{aligned} \tag{1}$$

where R, Q, r are the curvature tensor, the Ricci operator and the scalar curvature respectively.

**Definition 1.1.** *A Riemannian manifold is said to be* ***semisymmetric*** *if its curvature tensor $R$ satisfies the condition* $R(X,Y)\circ R = 0$
*for all $X, Y \in \chi(M)$ where $R(X,Y)$ acts on $R$ as a derivation.*

In [15] the authors studied contact metric manifolds with vanishing contact conformal curvature tensor. Again some properties of contact conformal curvature tensor on $N(k)$-contact metric manifolds were studied by Kim, Choi, Ozgur and Tripathi [10]. They also proved that an $N(k)$-contact metric manifold with vanishing extended contact conformal curvature tensor is a Sasakian manifold, but they did not verify the condition on $(k, \mu)$-contact metric manifolds. For this reason we study extended contact conformal curvature tensor on $(k, \mu)$-contact metric manifolds and prove that a $(2n+1)$ dimensional $(k,\ \mu)$-contact metric manifold with vanishing extended contact conformal curvature tensor is an $N(k)$-contact metric manifold which generalizes the main theorem of [10]. We also verify some interesting curvature properties of contact conformal curvature tensor on $(k, \mu)$-contact metric manifolds.

In the present paper we generalize the results of [10].
The paper is organized as follows:

In section 2 we give necessary details about $(k,\ \mu)$-contact metric manifolds. Section 4 deals with $(k,\ \mu)$-contact metric manifolds with vanishing extended contact conformal curvature tensor. In section 5 $(k,\ \mu)$-contact metric manifolds satisfying $R(\xi, X)\circ C_0 = 0$ have been studied. Section 6 deals with the study of $(k,\ \mu)$-contact metric manifolds satisfying $C_0(\xi, X)\circ S = 0$. Finally we study $(k,\ \mu)$-contact metric manifolds satisfying $C_0(\xi, X)\circ C_0 = 0$.

## 2. PRELIMINARIES

A $(2n+1)$ dimensional differentiable manifold $M$ is called an almost contact manifold if there is an almost contact structure $(\phi, \xi, \eta)$ consisting of a $(1,1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ satisfying

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta\circ\phi = 0. \tag{2}$$

Let g be a compatible Riemannian metric with $(\phi,\xi,\eta)$, that is,

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{3}$$

or equivalently,

$$g(X,\xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y) \tag{4}$$

for all $X, Y \in \chi(M)$.
If $g$ is such a metric, the quadruple $(\phi, \xi, \eta, g)$ is called an almost contact metric structure on $M$ and $M$ is an almost contact metric manifold.

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\mathrm{M}\times\mathbb{R}$ defined by
$\mathrm{J}(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable where X is tangent to $M$, t is the coordinate of R and f is a smooth function on $\mathrm{M}\times\mathbb{R}$.
The condition for being normal is equivalent to vanishing of the torsion tensor $[\phi,\phi] + 2d\eta\otimes\xi$ where $[\phi,\phi]$ is the Nijenhuis tensor of $\phi$.
An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y) \tag{5}$$

for all $X, Y \in \chi(M)$.
Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, we define a $(1,1)$ tensor field $h$ by $h = \frac{1}{2}\mathrm{L}_\xi\phi$ where L denotes the Lie differentiation. Then $h$ is symmetric and satisfies

$$h\xi = 0,\ h\phi + \phi h = 0, \tag{6}$$

$$\nabla\xi = -\phi - \phi h,\ trace(h) = trace(\phi h) = 0 \tag{7}$$

where $\nabla$ is the Levi-Civita connection.
A contact metric manifold is said to be an $\eta$-Einstein manifold if

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{8}$$

where $a, b$ are smooth functions on M and S is the Ricci tensor.
A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X. \tag{9}$$

On a Sasakian manifold the following relation holds

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{10}$$

for all $X, Y \in \chi(M)$.
Blair, Koufogiorgos and Papantoniou [1] considered the $(k,\ \mu)$-nullity condition on contact metric manifolds and gave several reasons for studying it. The $(k,\ \mu)$-nullity distribution N$(k,\ \mu)$ ([1], [13]) of a contact metric manifold $M$ is defined by

$$N(k,\mu) : p \mapsto N_p(k,\mu) = [U \in T_pM \mid R(X,Y)U = (kI + \mu h)(g(Y,U)X - g(X,U)Y)]$$

for all $X, Y \in \chi(M)$ where $(k, \mu) \in \mathbb{R}^2$.
A contact metric manifold $M^{2n+1}$ with $\xi \in N(k,\mu)$ is called a $(k,\ \mu)$- contact metric manifold. Then we have

$$\begin{aligned} R(X,Y)\xi &= k[\eta(Y)X - \eta(X)Y] + \\ &\quad \mu[\eta(Y)hX - \eta(X)hY] \end{aligned} \tag{11}$$

for all $X, Y \in \chi(M)$. For $(k,\ \mu)$-contact metric manifolds it follows that $h^2 = (k-1)\phi^2$. This class contains Sasakian manifolds for $k = 1$ and $h = 0$. In fact, for a $(k,\ \mu)$-metric manifold the condition of being Sasakian manifold, $k$-contact manifold, $k = 1$ and $h = 0$ are equivalent. If $\mu = 0$, the $(k,\ \mu)$-nullity distribution N$(k,\ \mu)$ is reduced to $k$-nullity distribution N$(k)$ [16]. If $\xi \in N(k)$, we call contact metric manifold M an N$(k)$- contact metric manifold.
$(k,\ \mu)$-contact metric manifolds have been studied by several authors ( [5], [6], [7]) and many authors.

In a $(k,\ \mu)$-contact metric manifold the following relations hold [1]:

$$h^2 = (k-1)\phi^2, \tag{12}$$

$$(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{13}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{14}$$

$$R(\xi, X)\xi = k[\eta(X)\xi - X] - \mu hX, \tag{15}$$

$$\begin{aligned} S(X,Y) &= [(2n-2)-n\mu]g(X,Y)+ \\ & [(2n-2)+\mu]g(hX,Y)+[(2-2n)+n \\ & (2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{16}$$

$$S(X,\xi) = 2nk\eta(X), \tag{17}$$

$$\begin{aligned} S(X,hY) &= [(2n-2)-n\mu] \\ & g(X,hY)-(k-1)[(2n-2)+\mu]g(X,Y) \\ & +(k-1)[(2n-2)+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{18}$$

$$Q\phi - \phi Q = 2[(2n-2)+\mu]h\phi \tag{19}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.
Let us recall the following result:

**Lemma 2.1.** *[3] A contact metric manifold $M^{2n+1}$ satisfying $R(X,Y)\xi = 0$ is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$ for $n > 1$.*

From (1) we have

$$C_0(X,Y)\xi = 2(k-1)[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{20}$$

$$\begin{aligned} & C_0(\xi,X)Y = 2(k-1)[g(X,Y)\xi - \eta(Y)X] + \\ & \mu[g(hX,Y)\xi - \eta(Y)hX], \end{aligned} \tag{21}$$

$$C_0(\xi,X)\xi = 2(k-1)[\eta(X)\xi - X] - \mu hX, \tag{22}$$

## 3. $\eta$-EINSTEIN ($k,\ \mu$)-CONTACT METRIC MANIFOLDS

In general, in a ($k$, $\mu$)-contact metric manifold the Ricci operator Q does not commute with $\phi$. However, Yildiz and De [18] proved the following:

**Proposition 3.1.** *In a non-Sasakian ($k,\ \mu$)-contact metric manifold the following conditions are equivalent:*
*(a) $\eta$-Einstein manifold,*
*(b) $Q\phi = \phi Q$.*

For $n = 1$, from (19) and Proposition 3.1 we can state the following:

**Corollary 3.1.** *A 3-dimensional non-Sasakian $\eta$-Einstein ($k,\ \mu$)-contact metric manifold is an $N(k)$-contact metric manifold.*

## 4. ($k,\ \mu$)-CONTACT METRIC MANIFOLDS WITH VANISHING EXTENDED CONTACT CONFORMAL CURVATURE TENSOR

In [10] the authors define the extended contact conformal curvature tensor $C_e$ in a contact metric manifold as follows

$$\begin{aligned} C_e(X,Y)Z &= C_0(X,Y)Z - \eta(X)C_0(\xi,Y)Z - \\ & \eta(Y)C_0(X,\xi)Z - \eta(Z)C_0(X,Y)\xi. \end{aligned} \tag{23}$$

Let M be a (2n+1)-dimensional ($k,\ \mu$)-contact metric manifold with vanishing extended contact conformal curvature tensor. From (23) we obtain

$$\begin{aligned} & C_0(X,Y)Z - \eta(X)C_0(\xi,Y)Z - \\ & \eta(Y)C_0(X,\xi)Z - \eta(Z)C_0(X,Y)\xi = 0. \end{aligned} \tag{24}$$

Putting $Y=Z=\xi$ and using (22) we have

$$(2k-2)[X-\eta(X)\xi]+\mu hX=0. \tag{25}$$

Putting $X=hX$ we have

$$(2k-2)hX+\mu h^2X=0. \tag{26}$$

Applying trace in both sides and using trace $h=0$ we obtain

$$\mu=0$$

Thus the following theorem is obtained:

**Theorem 4.1.** *A (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold with vanishing extended contact conformal curvature tensor is an $N(k)$-contact metric manifold.*

## 5. $(k,\ \mu)$-CONTACT METRIC MANIFOLDS SATISFYING $R(\xi,X)\circ C_0=0$

Let M be a (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold satisfying $R(\xi,X)\circ C_0=0$. Then

$$\begin{aligned}&R(\xi,U)C_0(X,Y)Z-C_0(R(\xi,U)X,Y)Z-\\&C_0(X,R(\xi,U)Y)Z-C_0(X,Y)R(\xi,U)Z=0.\end{aligned} \tag{27}$$

Putting $Z=\xi$ and using (14), (15), (20) we obtain

$$\begin{aligned}&2(k-1)\eta(R(\xi,U)X)Y+\mu\eta(R(\xi,U)X)hY-\\&2(k-1)\eta(R(\xi,U)Y)X-k\eta(U)C_0(X,Y)\xi+\\&kC_0(X,Y)U+\mu C_0(X,Y)hU=0.\end{aligned} \tag{28}$$

If we put $X=\xi$ and (15), (21) and (22), the equation (28) turns into

$$\mu[kg(hY,U)\xi+\mu g(h^2Y,U)\xi]=0. \tag{29}$$

Then either

$$\mu=0, \tag{30}$$

or,

$$kg(hY,U)\xi+\mu g(h^2Y,U)\xi=0, \tag{31}$$

which implies that

$$khY+\mu h^2Y=0. \tag{32}$$

Taking trace in both sides and using trace $h=0$ we have

$$\mu=0. \tag{33}$$

From equations (30) and (33) we can conclude the following

**Theorem 5.1.** *Let M be a (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold satisfying $R(\xi,X)\circ C_0=0$. Then M is an $N(k)$-contact metric manifold.*

## 6. $(k,\ \mu)$-CONTACT METRIC MANIFOLDS SATISFYING $C_0(\xi,X)\circ S=0$

Let M be a (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold satisfying $C_0(\xi,X)\circ S=0$. Hence we have

$$S(C_0(\xi,X)Y,Z)+S(Y,C_0(\xi,X)Z)=0. \tag{34}$$

Putting $Z=\xi$ and using (17), (21) and (22) we obtain

$$4nk(k-1)g(X,Y)+2nk\mu g(hX,Y)+ \\ 2(k-1)S(Y,X)-\mu S(Y,hX)=0. \tag{35}$$

Using (16) and (18) in (35) yields

$$g(hX,Y)=ag(X,Y)+b\eta(X)\eta(Y) \tag{36}$$

where

$$a=\frac{2(k-1)[(2n-2)-n\mu]-4nk-\mu(k-1)[(2n-2)+\mu]}{2nk+2(k-1)[(2n-2)+\mu]-\mu[(2n-2)-n\mu]}$$

and

$$b=\frac{\mu(k-1)[(2n-2)+\mu]-2(k-1)[(2-2n)+n(2k+\mu)]}{2nk+2(k-1)[(2n-2)+\mu]-\mu[(2n-2)-n\mu]}.$$

Using (36) in (16) we get

$$S(X,Y)=a_1g(X,Y)+b_1\eta(X)\eta(Y), \tag{37}$$

where

$$a_1=[(2n-2)-n\mu]+a[(2n-2)+\mu]$$

and

$$b_1=[(2-2n)+n(2k+\mu)]+b[(2n-2)+\mu].$$

From equation (37) we can state the following

**Theorem 6.1.** *Let M be a (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold satisfying $C_0(\xi,X)\circ S=0$. Then M is an $\eta$-Einstein manifold.*

From Proposition 3.1 and Theorem 6.1 we can state that

**Corollary 6.1.** *Let M be a (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold satisfying $C_0(\xi,X)\circ S=0$. Then the Ricci operator Q commutes with $\phi$.*

## 7. $(k,\ \mu)$-CONTACT METRIC MANIFOLDS SATISFYING $C_0(\xi,U)\circ C_0=0$

Let M be a (2n+1)-dimensional $(k,\ \mu)$-contact metric manifold satisfying $C_0(\xi,U)\circ C_0=0$.
Then we have

$$C_0(\xi,U)C_0(X,Y)Z-C_0(C_0(\xi,U)X,Y)Z- \\ C_0(X,C_0(\xi,U)Y)Z-C_0(X,Y)C_0(\xi,U)Z=0. \tag{38}$$

Using (21) in (38) we obtain

$$\begin{aligned}&2(k-1)[g(U,C_0(X,Y)Z)\xi-\eta(C_0(X,Y)Z)U-\\&g(U,X)C_0(\xi,Y)Z+\eta(X)C_0(U,Y)Z-\\&g(U,Y)C_0(X,\xi)Z+\eta(Y)C_0(X,U)Z-\\&g(U,Z)C_0(X,Y)\xi+\eta(Z)C_0(X,Y)U]\\&+\mu[g(hU,C_0(X,Y)Z)\xi-\eta(C_0(X,Y)Z)hU-\\&g(hU,X)C_0(\xi,Y)Z+\eta(X)C_0(hU,Y)Z-\\&g(hU,Y)C_0(X,\xi)Z+\eta(Y)C_0(X,hU)Z\\&-g(hU,Z)C_0(X,Y)\xi+\eta(Z)C_0(X,Y)hU]=0.\end{aligned} \tag{39}$$

Let us consider the following cases:
**CASE 1.** $k=0=\mu$.
**CASE 2.** $k\neq 0$, $\mu=0$.
**CASE 3.** $k=0$, $\mu\neq 0$.

**CASE 4.** $k \neq 0$, $\mu \neq 0$
For Case 1 we observe that $R(X,Y)\xi = 0$ for all X,Y. Hence, by Lemma 2.1 $M$ is locally the Riemannian product $E^{n+1} \times S^n(4)$.
For Case 2 from (39) we have either $k = 1$, or,

$$[g(U, C_0(X,Y)Z)\xi - \eta(C_0(X,Y)Z)U \\ -g(U,X)C_0(\xi,Y)Z + \eta(X)C_0(U,Y)Z - g(U,Y)C_0(X,\xi)Z \\ +\eta(Y)C_0(X,U)Z - g(U,Z)C_0(X,Y)\xi + \eta(Z)C_0(X,Y)U] = 0 \tag{40}$$

. Putting $Y = Z = \xi$ in (40) and using (20) and (22) we get

$$(k-1)[g(X,U)\xi - \eta(U)X] = 0. \tag{41}$$

Taking inner product with $\xi$ in equation (41) we have

$$(k-1)[g(X,U) - \eta(U)\eta(X)] = 0. \tag{42}$$

As $g(U,X) - \eta(U)\eta(X) \neq 0$, $k = 1$.
Thus M is a Sasakian manifold.
For Case 3 from (39) we obtain

$$(-2)[g(U, C_0(X,Y)Z)\xi - \\ \eta(C_0(X,Y)Z)U - g(U,X)C_0(\xi,Y)Z + \eta(X) \\ C_0(U,Y)Z - g(U,Y)C_0(X,\xi)Z \\ +\eta(Y)C_0(X,U)Z - g(U,Z)C_0(X,Y)\xi + \\ \eta(Z)C_0(X,Y)U] + \mu[g(hU, C_0(X,Y)Z)\xi - \\ \eta(C_0(X,Y)Z)hU - g(hU,X)C_0(\xi,Y)Z \\ +\eta(X)C_0(hU,Y)Z - g(hU,Y)C_0(X,\xi)Z + \eta(Y)C_0(X,hU)Z \\ -g(hU,Z)C_0(X,Y)\xi + \eta(Z)C_0(X,Y)hU] = 0. \tag{43}$$

Putting $Y = Z = \xi$ in (43) and using (20) and (22) we get

$$-2[-4g(X,U)\xi - 4\eta(U)X + 2\mu g(hX,U)\xi - \\ 2\mu\eta(U)hX] + \mu[-4g(X,hU)\xi + 2\mu g(X,U)\xi - \\ 2\mu\eta(X)\eta(U)\xi] = 0. \tag{44}$$

Taking inner product with $\xi$ in equation (44) we have

$$-2[-4g(X,U) - 4\eta(U)\eta(X) + 2\mu g(hX,U)] \\ +\mu[-4g(X,hU) + 2\mu g(X,U)\xi - 2\mu\eta(X)\eta(U)] = 0. \tag{45}$$

Thus

$$g(hX,U) = a_0 g(X,U) + b_0\eta(X)\eta(U), \tag{46}$$

where

$$a_0 = \frac{4+\mu^2}{4\mu}$$

and

$$b_0 = \frac{4-\mu^2}{4\mu}$$

From (16) and (46) we have

$$S(X,U) = a_1 g(X,U) + b_1\eta(X)\eta(U) \tag{47}$$

where

$$a_1 = [(2n-2) - n\mu] + [(2n-2) + \mu]\frac{4+\mu^2}{4\mu}$$

and

$$b_1 = [(2-2n) + n(2k+\mu)]\frac{4-\mu}{4\mu}$$

Thus for Case $3$ $M$ is an $\eta$-Einstein manifold.
For Case $4$ from (39) we obtain

$$\begin{aligned}&2(k-1)[g(U, C_0(X,Y)Z)\xi - \\ &\eta(C_0(X,Y)Z)U - g(U,X)C_0(\xi,Y)Z + \\ &\eta(X)C_0(U,Y)Z - g(U,Y)C_0(X,\xi)Z + \\ &\eta(Y)C_0(X,U)Z - g(U,Z)C_0(X,Y)\xi + \\ &\eta(Z)C_0(X,Y)U] + \mu[g(hU, C_0(X,Y)Z)\xi - \\ &\eta(C_0(X,Y)Z)hU - g(hU,X)C_0(\xi,Y)Z \\ &+\eta(X)C_0(hU,Y)Z - g(hU,Y)C_0(X,\xi)Z + \\ &\eta(Y)C_0(X,hU)Z - g(hU,Z)C_0(X,Y)\xi + \\ &\eta(Z)C_0(X,Y)hU] = 0.\end{aligned} \tag{48}$$

Putting $Y = Z = \xi$ in (48) and using (20) and (22) we get

$$\begin{aligned}&2(k-1)\{[g(X,U)\xi + 4\eta(U)X + 2\mu g(hx,U)\xi - \\ &2\mu\eta(U)hX] + \mu[2g(X,hU)\xi - \mu g(X,U)\xi + \\ &\mu\eta(X)\eta(U)\xi]\} = 0.\end{aligned}$$

Thus either $k = 1$, or,

$$\begin{aligned}&[4(k-1)g(X,U)\xi + 4(k-1)\eta(U)X + 2\mu g(hx,U)\xi - \\ &2\mu\eta(U)hX] + \mu[2g(X,hU)\xi - \mu g(X,U)\xi + \\ &\mu\eta(X)\eta(U)\xi] = 0.\end{aligned} \tag{49}$$

Taking inner product with $\xi$ in equation (49) we have

$$\begin{aligned}&[4(k-1)g(X,U) + 4(k-1)\eta(U)\eta(X) + 2\mu g(hx,U)] \\ &+\mu[2g(X,hU) - \mu g(X,U) + \mu\eta(X)\eta(U)] = 0.\end{aligned} \tag{50}$$

Thus

$$g(hX,U) = a_2 g(X,U) + b_2\eta(X)\eta(U), \tag{51}$$

where

$$a_2 = \frac{4(k-1)+\mu^2}{\mu}$$

and

$$b_2 = \frac{4(k-1)-\mu^2}{4\mu}$$

From (16) and (51) we have

$$S(X,U) = a_3 g(X,U) + b_3\eta(X)\eta(U,) \tag{52}$$

where

$$a_3 = [(2n-2) - n\mu] + [(2n-2)+\mu]\frac{4(k-1)+\mu^2}{6\mu}$$

and

$$b_3 = [(2-2n) + n(2k+\mu)]\frac{4(k-1)-\mu^2}{6\mu}$$

Thus, for Case $3$ $M$ is either a Sasakian manifold or an $\eta$-Einstein manifold.
Summing up we can state the following:

**Theorem 7.1.** *Let $M$ be an $(2n+1)$-dimensional $(k,\,\mu)$-contact metric manifold satisfying $C_0(\xi,U)\circ C_0=0$.Then we have one of the following:*
*(a) $M$ is locally isomorphic to the Riemannian product $E^{n+1}\times S^n(4)$.*
*(b) $M$ is a Sasakian manifold.*
*(c) $M$ is an $\eta$-Einstein manifold.*

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